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Analytic Continuation beyond the Ideal Boundary

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§1. Introduction.

In the present article we shall be concerned with the analytic continuation of a meromorphic function or differential "beyond the ideal boundary". To explain the results we start with a noncompact Riemann surface R and a meromorphic differential $\psi = d\Psi$ on R. We try to extend the differential ψ (or the function Ψ) beyond the ideal boundary ∂R of R. Of course this makes no sense in an ordinary way, for there is nothing beyond ∂R. The surface R is a whole world as a Riemann surface, so that we cannot reach even ∂R.

The traditional way to deal with a similar (not the same in any sense) problem is to regard R as a subsurface of another surface R_0 and try to ψ to (the whole or to a portion of) R_0. The Schwarz reflection principle and Painlevé theorem in the classical theory of functions give famous examples of such procedure. They supply powerful techniques provided that the ideal boundary is supposed to be visible or touchable and is actually realized as analytic curves. Although such observation often gives a good device for the study of analytic functions and Riemann surfaces, it essentially yields a kind of tautology. Indeed, what we really want to do should be the construction of R_0! In the following we shall pay more attention to the construction of R_0 than to the extension of ψ to the difinite supersurface of R.

On the other hand, the Riemann mapping theorem or the generalized uniformization theorem due to Koebe serve another kind of analytic continuations, on which our idea is based. We review these theorems to see why we are more interested in them for the study of noncompact Riemann surfaces. For the sake of simplicity we suppose that R is a simply connected plain domain with more than two boundary points. Then the Riemann mapping theorem states that there exists a univalent holomorphic function f on R

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which maps R onto the open unit disk $\mathbb{D}$. By this theorem — or more precisely we have to say "through f" — the (ideal) boundary ∂R is realized as an analytic curve — the unit circle $\partial\mathbb{D}$. The function f is then extended to a meromorphic function on the whole Riemann sphere $\hat{\mathbb{C}}$. Each rational function comes from an analytic functions on R by extending analytically beyond the ideal boundary to $\hat{\mathbb{C}}$. More precisely: for any rational function ρ (on $\hat{\mathbb{C}}$) the composition $(\rho|_{\mathbb{D}}) \circ f$ of its restriction to the open unit disk with f gives a function which can be analytically continued beyond the ideal boundary.

Similarly for the Koebe uniformization theorem. We note that these theorems are, as is well known, valid also for any planar Riemann surface R, where the ideal boundary plays a substantial role. Our aim is thus to find a compact surface R_0 and a conformal embedding of R into R_0.

Now, how about surfaces of positive genus? This is the case where our point of view is more important and we are concerned with the problem in the present paper.

§2. Embedding Theorems.

The following theorem is basic for the subsequent study.

DEFINITION. A (multivalued) meromorphic function f on a general non-compact Riemann surface R is called an S-function, if df is a canonical semiexact differential of Kusunoki (see [4]), or equivalently, if $\operatorname{Im} f$ has the same boundary behavior as a $(Q)L_1$-principal function of Sario (see [1] and [5]).

The name S-function is an abbreviation of a *stream* function which suggests that it describes a steady flow (of an ideal fluid) on the surface. In fact, an S-function is a multivalued meromorphic function (a complex velocity potential function) on R such that

1. the number of poles of f is finite,
2. for some compact set K the Diriclet integral $||df||_{R\setminus K}$ of df is finite, and
3. $\operatorname{Im} f$ is, in a sense to be precisely stated, constant on each ideal boundary component of R.

These mathemnatical conditions can be physically expressed as follows: an S-function f is a complex velocity potential function on R (in the classical sense of Klein (see [7] and [13])) such that

1. the sinks and sources on R are finite in number,
2. the (kinetic) energy of the flow is finite apart from the sinks and sources, and
3. the (ideal) boundary of R is impenetrable.

THEOREM 2.1. Let R be a noncompact Riemann surface of (positive) finite genus, and f an S-function on R. Then there exists a compact surface $\tilde{R}$ of the same genus as R, a conformal mapping $\iota : R \longrightarrow \tilde{R}$, and a meromorphic function $\tilde{f}$ on $\tilde{R}$ such that

1. the set $\tilde{R} \setminus \iota(R)$ is a null set in the sense of Lebesgue,
2. $\tilde{f}$ is holomorphic on $\tilde{R} \setminus \iota(R)$,
3. $\tilde{f} \circ \iota = f$ on R, and
4. on each component of $\tilde{R} \setminus \iota(R)$ the function $\operatorname{Im} \tilde{f}$ is constant.

Rougly speaking, the ideal boundary ∂R of R is realized on $\tilde{R}$ as a set of stream arcs of the extended flow $\tilde{f}$ on $\tilde{R}$ whose total area vanishes.

By a routine method of thinking (that is, via the conformal embedding $\iota : R \to \tilde{R}$), the surface R can be identified with the enbedded subsurface $\iota(R)$ of $\tilde{R}$, so that we can say that f is <u>analytically continued beyond the ideal boundary</u> to the compact Riemann surface $\tilde{R}$ of the same genus.

We sometimes restrict ourselves to the case of singularity-free continuations, that is, we consider only the case where the extended functionis holomorphic (instead of meromorphic) on the extended part $\tilde{R} \setminus \iota(R)$. To show the situation explicitly we use the term <u>holomorphic</u> continuation instead of analytic continuation. Nonholomorphic continuations beyond the ideal boundary are studied in [2], [3] and [12], for example, while one of the examples of the holomorphic continuations beyond the ideal boundary is given in the previous embedding theorem.

Note that when we refer to a holomorphic or meromorphic continuation of a function or differential on a noncompact Riemann surface R we always consider the conformal embeding of the given R into compact surfaces of the same genus.

We use the above theorem to show the following theorem.

THEOREM 2.2. Let R be an noncompact torus (= a noncompact Riemann surface of genus one). Then there is a closed disk $\mathfrak{M}(R)$ in the upper half plane $\mathbb{H}$ which has the following property: There is a meromorphic fnction on R which is meromorphically continued to a torus $\tilde{R}$, if and only if the modulus of $\tilde{R}$ (with respect to a canonical homology basis of $\tilde{R}$) belongs to the disk $M\mathfrak{M}(R)$, where M is a unimodular transformation.

This is just a paraphrase of the main theorem in [8]. Hence we omit the detailed proof.

§3. Uniqueness Theorem.

To see another aspect of our theorem we give an explanation — limited for the case of finite genus — why the uniquness theorem for Kusunoki's canonical semiexact differentials holds. For the original and precise definition of the canonical semiexact differentials, see [4]. Cf. also [1] and [5]. We shall be content with the intuitively stated definition given in section 2. The uniqueness theorem of Kusunoki is

THEOREM 3.1. A canonical semiexact differential on R identically vanishes if it is holomorphic on R.

Kusunoki also showed the existence of the so-called elementary differentials:

THEOREM 3.2. Let R be a noncompact Riemann surface of genus $g(0 < g \leq \infty)$ and $\{a_j, b_j\}_{j=1}^{g}$ a canonical homology basis modulo dividing cycles of R. Then, there is a holomorphic canonical semiexact differential ϕ_j on R such that

$$\int_{a_k} \phi_j = \delta_{jk}, \qquad j, k = 1, 2, \cdots, g,$$

where δ_{jk} is the Kronecker's symbol. For each j the differential ϕ_j is uniquely determined.

REMARK 3.1. In these theorems the finiteness of the genus of R is not required. But we will consider only the case of finite genus in the following.

Now suppose that R is of finite genus. Then, our theorem in §2 explains why the uniqueness theorem holds. Indeed, the differential is holomorphically continued to a compact Riemann surface and it suffices to apply the classical theory to the extended differential. This theorem also proves the uniqueness part of the following theorem:

THEOREM 3.3. Let R be a noncompact Riemann surface of finite genus g and $\{a_j, b_j\}_{j=1}^{g}$ a canonical homology basis modulo dividing cycles. Then, for any g-tuple $(c_1, c_2, \cdots, c_g) \in \mathbb{C}^g$, there is a unique holomorphic canonical semiexact differential ϕ on R with

$$\int_{a_k} \phi = c_k, \qquad k = 1, 2, \cdots, g.$$

To verify the existence it suffices to consider the linear combination

$$c_1\phi_1 + c_2\phi_2 + \cdots + c_g\phi_g.$$

One might think that the theory of canonical semiexact differentials on a Riemann surface of finite genus simply reflects the theory of abelian differentials on a single compact Riemann surface. This is not the case, however. Although we finally have to consider a single differential

$$\hat{\phi} := \phi - (c_1\phi_1 + c_2\phi_2 + \cdots + c_g\phi_g)$$

and a single compact Riemann surface $\tilde{R}$ onto which $\hat{\phi}$ is holomorphically continued, each differential ϕ_j holomorphically continued to a compact Riemann surface $\tilde{R}_j$ of genus g, which may be different from each other. In other words, theory of canonical semiexact differentials (or equivalently, the theory of principal functions) on a single noncompact Riemann surface is, even in the case of finite genus, simultaneously concerned with infinitely many compact Riemann surfaces. The Riemann-Roch and the Abel theorems formulated by them ([4] and [5]) therefore reflect a deep function-theoretic property.

§4. Some Necessary Conditions.

Now we recall the theorem of Behnke-Sommer: There is a holomorphic differential on a R with arbitrarily prescribed periods. To give general criteria for a given meromorphic function (resp. differential) on R to be holomorphically (resp. meromorphically) continued (to a compact torus) is not easy. Here we give one of the simpleset examples, which corresponds to the second theorem in §2.

THEOREM 3.4. Let R and $\mathfrak{M}(R)$ be as in Theorem 2.2. Let df be a meromorphic differential on R and set

$$\alpha := \int_a df, \qquad \text{and} \qquad \beta := \int_b df,$$

where $\{a, b\}$ is a canonical homology basis of R modulo dividing cycles. If df is meromorphically continued to a torus, then

$$\frac{m'\alpha + n'\beta}{m''\alpha + n''\beta} \in \mathfrak{M}(R)$$

for some integers m', n', m'', n'' with $m'n'' - m''n' = \pm 1$.

The observation in this section loses sense for surfaces of infinite genus; it would be interesting and important to make clear the mechanism for the general case. For the case of finite genus (> 1) the results in [9] and [10] will be useful.

A more detailed necessary condition than Theorem 4.1 can be given by using the results in [6]. We have proved there and will prove in a forthcoming paper that the added portion $\tilde{R} \setminus \iota(R)$ can be neither so large nor too small. For the simplicity we restrict ourselves to a so-called normal holomorphic differential on a noncompact Riemann surface of genus one and continuations which preserve the canonical homology bases. We state the following theorem without proof, which will appear elsewhere.

THEOREM 4.2. For any noncompact torus R with a canonical homology basis $\chi = \{a, b\}$ modulo dividing cycles, there exists a positive constant K with the following property. If a holomorphic differential $\varphi = d\Phi$ on R with

$$\int_a \varphi = 1$$

is meromorphically continued to a compact torus $\tilde{R}$ with a canonical homology basis corresponding to χ, then the oscillation of Φ on the ideal boubdary of R is bounded by K.

REMARK 4.1. We have a similar condition as for the lower bound.

REMARK 4.2. Another metrical property of the realized ideal boundary are studied in [11]; the area of the added portion has an interesting property.

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