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Exact WKB analysis and connection problems for differential equations

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1 Introduction

For these ten years, together with T. Kawai (RIMS, Kyoto University) and T. Aoki (Kinki University), I have been studying ordinary differential equations in the complex domain from the viewpoint of the exact WKB analysis. The purpose of this report is to give an expository survey of our research on this subject.

One of the most important problems in quantum mechanics is the eigenvalue problem of Schrödinger equations and to attack this problem asymptotic solutions called "WKB solutions" have been used by many physicists. Exact WKB analysis is, in a word, a new treatment of WKB solutions of 1-dimensional Schrödinger equations based upon the Borel resummation technique. This new analysis using the Borel resummation allows us to handle exponentially small terms ("terms beyond all orders") in an exact manner and consequently becomes a powerful tool in analyzing various connection problems (such as eigenvalue problems, monodromy problems etc.) of differntial equations.

Historically speaking, the exact WKB analysis was initiated by Voros ([V]) and then has been developed by Pham and his collaborators (cf. [CNP], [DDP1], [DDP2], [DP]) in the framework of Ecalle's theory of resurgent functions ([E1]–[E3]).

In this report we explain an outline of the exact WKB analysis of 1-dimensional Schrödinger equations according to our way of understanding. For details we refer the reader to our monograph [KT3]. Recently an analogous analysis has been (almost) established for Painlevé equations, typical nonlinear ordinary differential equations of second order. We will briefly discuss this generalization of the theory to the nonlinear equations also.

2 WKB solutions & Borel resummation

In the case of a 1-dimensional Schrödinger equation

(1) $$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi(x,\eta) = 0 \qquad (\eta\text{: large parameter}),$$

WKB solutions can be easily constructed in the following manner:

Assume that ψ is of the form

(2) $$\psi = \exp \int_{x_0}^{x} S dx$$

with an arbitrarily fixed point x_0, then we find that S should satisfy the so-called Riccati equation

(3) $$S^2 + \frac{\partial S}{\partial x} = \eta^2 Q.$$

Here we further assume that S has an expansion $S = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots$ for η^{-1}, then we obtain the following recursive relations:

(4) $$S_{-1}^2 = Q,$$

(5) $$2S_{-1}S_j = -\left(\sum_{k=0}^{j-1} S_k S_{j-1-k} + \frac{dS_{j-1}}{dx}\right) \qquad (j \geq 0).$$

That is, $S_{-1} = \pm\sqrt{Q(x)}$ and, once (the sign of) S_{-1} is fixed, the other terms S_j $(j \geq 0)$ are uniquely determined by the relation (5). In this way we have two (formal power series) solutions $S_\pm$ of the Riccati equation and, substituting these solutions into (2), we obtain two linearly independent solutions of the Schrödinger equation. Note that, if we decompose $S_\pm$ into the sum of the odd order part and the even order part with respect to the power of η, *i.e.*, $S_\pm = \pm S_{\text{odd}} + S_{\text{even}}$, then S_{odd} and S_{even} must satisfy the relation $S_{\text{even}} = -\frac{1}{2}\frac{\partial}{\partial x}\log S_{\text{odd}}$. Hence

(6) $$\psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}}\exp\left(\pm\int_{x_0}^{x} S_{\text{odd}}dx\right)$$

also become solutions of the Schrödinger equation (1). In this report we call (6) *WKB solutions* of (1).

Although the construction of WKB solutions is easy, they experience a divergence problem in return for it. (This is a common feature to equations of singular perturbations.) To give an analytic meaning to WKB solutions we employ the Borel resummation technique. Let us recall here the definition of the Borel resummation; for an infinite series

(7) $$\psi = e^{\eta y_0}\sum_{j=0}^{\infty}\psi_j\eta^{-(\alpha+j)} \qquad (\alpha \in \mathbb{R}\setminus\mathbb{Z})$$

of η^{-1}, its *Borel transform* is, by definition,

$$\psi_B(y) = \sum_{j=0}^{\infty} \frac{\psi_j}{\Gamma(\alpha+j)}(y+y_0)^{\alpha+j-1}. \tag{8}$$

If $\psi_B(y)$ defines an analytic function in an appropriate domain so that the Laplace integral

$$\int_{-y_0}^{\infty} e^{-y\eta}\psi_B(y)dy \tag{9}$$

(where the path of integration is taken to be parallel to the positive real axis) may become well-defined, we say ψ is Borel summable and call the integral (9) its *Borel sum*.

WKB solutions (6) can actually be expanded into the form (7) with $\alpha = 1/2$ and $y_0 = \pm\int_{x_0}^{x}\sqrt{Q(x)}dx$ and we apply this technique to (6), considering them as infinite series of η^{-1} and regarding x as just a parameter. Then Borel resummed WKB solutions thus obtained enjoy some very interesting properties. In the subsequent section we discuss two basic examples to see the interesting properties of WKB solutions.

3 Basic examples

Example 1 [Airy equation] (cf. [AKT1]) Let us first consider the Airy equation

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0. \tag{10}$$

We normalize WKB solutions as well as their exponential part (phase factor) denoted by $y_0(x)$ in the following way:

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_0^x S_{\mathrm{odd}}dx\right), \qquad y_0(x) = \int_0^x \sqrt{x}dx = \frac{2}{3}x^{3/2}. \tag{11}$$

In this case the WKB solutions $\psi_\pm$ and their Borel transform $\psi_{\pm,B}(x,y)$ have special homogeneity; $x\psi_{\pm,B}(x,y)$ are functions of one variable $t = y/x^{3/2}$. Using this homogeneity together with the fact that $\psi_{\pm,B}(x,y)$ satisfy a partial differential equation (Borel transformed equation of (1)) $\left(\frac{\partial^2}{\partial x^2} - x\frac{\partial^2}{\partial y^2}\right)\psi_{\pm,B}(x,y) = 0$, we obtain the following explicit description of $\psi_{\pm,B}(x,y)$ in terms of Gauss' hypergeometric functions:

$$\begin{cases} \psi_{+,B} &= \sqrt{\dfrac{3}{4\pi}\dfrac{1}{x}}\left[s^{-1/2}F(\frac{1}{6},\frac{5}{6},\frac{1}{2};s)\right]\Bigg|_{s=3y/4x^{3/2}+1/2}, \\ \psi_{-,B} &= \sqrt{\dfrac{3}{4\pi}\dfrac{1}{x}}\left[(s-1)^{-1/2}F(\frac{1}{6},\frac{5}{6},\frac{1}{2};1-s)\right]\Bigg|_{s=3y/4x^{3/2}+1/2}. \end{cases} \tag{12}$$

This expression (12) tells us that $\psi_{+,B}(x,y)$ (as well as $\psi_{-,B}(x,y)$) has singularities both at $y = \pm y_0(x)$. Since a hypergeometric function has at most polynomial growth at each singular point, this implies that the Borel sum of $\psi_+(x,\eta)$

$$\int_{-y_0(x)}^{\infty} e^{-y\eta}\psi_{+,B}(x,y)dy \tag{13}$$

is well-defined as far as the other singular point $y = y_0(x)$ does not meet the path of integration. In other words, $\psi_+(x,\eta)$ (and $\psi_-(x,\eta)$ also) is Borel summable when $\Im x^{3/2} \neq 0$ (cf. the left of Figure 1).

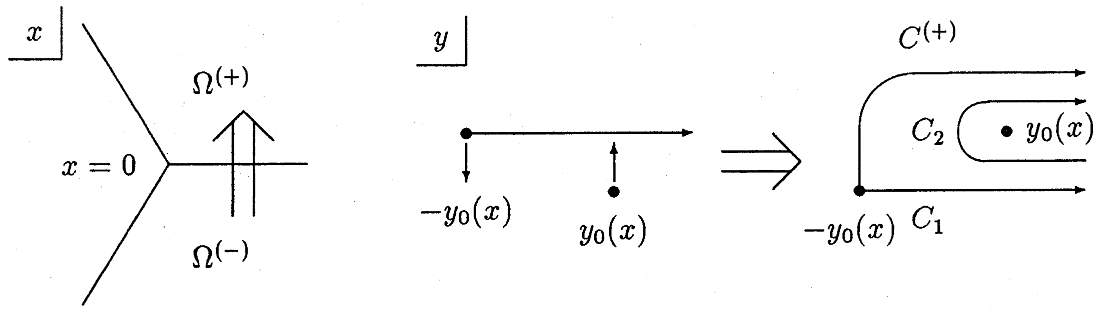

Figure 1: Change of the integration path of (13) when crossing the positive real axis.

On the other hand, on the lines $\Im x^{3/2} = 0$ the Borel resummed WKB solutions show the following "Stokes phenomena": Let us consider, for example, the analytic continuation across the positive real axis of the Borel sum (13) of $\psi_+(x,\eta)$ in the lower region $\Omega^{(-)} = \{-2\pi/3 < \arg x < 0\}$. If we vary x across the positive real axis from below, the singular point $y = y_0(x)$ crosses the integration path of (13) (see Figure 1). As a consequence, the analytic continuation of (13) to the upper region $\Omega^{(+)} = \{0 < \arg x < 2\pi/3\}$ is given by the integral along the path $C^{(+)}$ in Figure 1, which is the sum of the integral along C_1 and that along C_2. The former integral is the Borel sum of $\psi_+(x,\eta)$ in $\Omega^{(+)}$ by the definition, while it follows from the following relation

$$\Delta_{y=y_0(x)}\psi_{+,B}(x,y) = i\psi_{-,B}(x,y), \tag{14}$$

where $\Delta_{y=y_0(x)}$ denotes the discontinuity at $y = y_0(x)$, that the latter one coincides with $i = \sqrt{-1}$ times the Borel sum of $\psi_-(x,\eta)$. Note that the relation (14) is verified by using the classical connection formula of Gauss for hypergeometric functions. We have thus obtained the following connection formula for Borel resummed WKB solutions on the positive real axis:

$$\psi_+^{(-)} = \psi_+^{(+)} + i\psi_-^{(+)} \tag{15}$$

where $\psi_\pm^{(-)}$ and $\psi_\pm^{(+)}$ respectively denote the Borel resummed WKB solutions in $\Omega^{(-)}$ and in $\Omega^{(+)}$.

Example 2 [Weber equation] (cf. [T2]) Let us next consider the Weber equation

$$\left(\frac{d^2}{dx^2}-\eta^2\left(\frac{x^2}{4}-\eta^{-1}\kappa\right)\right)\psi=0 \tag{16}$$

with a parameter κ. Normalizing WKB solutions and their phase factor as

$$\begin{aligned}\psi_\pm &= \frac{1}{\sqrt{S_{\mathrm{odd}}}}\left(\eta^{1/2}x\right)^{\mp\kappa}\exp\pm\left\{\eta\int_0^x S_{-1}dx+\int_\infty^x\left(S_{\mathrm{odd}}-\eta S_{-1}+\frac{\kappa}{x}\right)dx\right\},\\ y_0(x) &= \int_0^x \frac{x}{2}dx=\frac{x^2}{4},\end{aligned} \tag{17}$$

we obtain the following expression of $\psi_{\pm,B}(x,y)$:

(18)
$$\begin{cases}\psi_{+,B} = \dfrac{x^{-3/2}2^{1-\kappa/2}}{\Gamma((1+\kappa)/2)}\left[s^{(-1+\kappa)/2}F\left(\frac{1}{4}+\frac{\kappa}{2},\frac{3}{4}+\frac{\kappa}{2},\frac{1}{2}+\frac{\kappa}{2};s\right)\right]\Bigg|_{s=2y/x^2+1/2},\\[2ex] \psi_{-,B} = \dfrac{x^{-3/2}2^{1+\kappa/2}}{\Gamma((1-\kappa)/2)}\left[(s-1)^{-(1+\kappa)/2}F\left(\frac{1}{4}-\frac{\kappa}{2},\frac{3}{4}-\frac{\kappa}{2},\frac{1}{2}-\frac{\kappa}{2};1-s\right)\right]\Bigg|_{s=2y/x^2+1/2}.\end{cases}$$

Again $\psi_{\pm,B}(x,y)$ have singularities both at $y=\pm y_0(x)$ and consequently $\psi_\pm(x,\eta)$ are Borel summable when $\Im x^2\neq 0$ (that is, except on the real and imaginary axes). In a similar manner to Example 1 we can prove the following connection formula for Borel resummed WKB solutions of the Weber equation on the positive real axis:

$$\psi_+^{(-)}=\psi_+^{(+)}+\frac{i\sqrt{2\pi}}{\Gamma(\kappa+1/2)}\psi_-^{(+)} \tag{19}$$

where $\psi_\pm^{(-)}$ and $\psi_\pm^{(+)}$ respectively denote the Borel resummed WKB solutions in $\{-\pi/2<\arg x<0\}$ and in $\{0<\arg x<\pi/2\}$.

4 Fundamental properties

In the precedent section we investigated Borel resummed WKB solutions of the Airy and Weber equations. One of the most important properties of them is that they show what is called Stokes phenomenon and it is explicitly described by the connection formula (cf. (15), (19)). Such a phenomenon can be observed not only for these basic examples but also for general equations of the form (1). In this section we state Stokes phenomena for general equations without referring to their detailed proofs.

Definition 1 (i) A *turning point* of (1) is, by definition, a zero of $Q(x)$. A turning point is said to be simple if it is a simple zero of $Q(x)$.
(ii) A *Stokes curve* of (1) is defined by the following relation:

$$\Im\int_a^x\sqrt{Q(x)}dx=0, \tag{20}$$

where a is a turning point of (1).

For the Airy equation (10) the origin $x = 0$ is a unique simple turning point and Stokes curves are given by $\Im x^{3/2} = 0$. For the Weber equation (16) we should regard the leading part $x^2/4$ of the potential as "true potential" $Q(x)$. Obeying this convention, we find $x = 0$ is a unique turning point (which is not simple) of (16) and Stokes curves are given by $\Im x^2 = 0$.

In what follows we discuss only equations of the form (1) and assume further that all turning points are simple for the sake of simplicity. Then, under some generic conditions (whose precise formulation is omitted in this report), Borel resummed WKB solutions of (1) have the following properties:

Fact 1 *WKB solutions (6) of the equation (1) are Borel summable except on Stokes curves.*

Fact 2 *On each Stokes curve emanating from a simple turning point a the following connection formula holds:*

$$\text{(21)} \qquad \begin{cases} \psi_+^{(0)} &= \psi_+^{(1)} \pm i\psi_-^{(1)} \\ \psi_-^{(0)} &= \psi_-^{(1)} \end{cases} \quad \textit{or} \quad \begin{cases} \psi_+^{(0)} &= \psi_+^{(1)} \\ \psi_-^{(0)} &= \psi_-^{(1)} \pm i\psi_+^{(1)}, \end{cases}$$

where WKB solutions $\psi_\pm$ are assumed to have the following normalization

$$\text{(22)} \qquad \psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}} dx\right),$$

and $\psi_\pm^{(j)}$ $(j = 0, 1)$ respectively denote the Borel sum of $\psi_\pm$ in the two adjacent regions having the Stokes curve in question as a common boundary.

The Borel summability of WKB solutions (Fact 1) is due to Ecalle ([E2]) and their connection formula at a simple turning point in all orders (Fact 2) is given first by Voros ([V], who derived (21) from the fact that Borel resummed WKB solutions are single-valued even at a simple turning point). The formula (21) can be verified also by combining the connection formula (15) for the Airy equation and the following local reduction of the equation (1) to the Airy equation near a simple turning point (cf. [AKT1]): For any simple turning point a we can find an infinite series

$$\text{(23)} \qquad \tilde{x}(x, \eta) = \tilde{x}_0(x) + \eta^{-2}\tilde{x}_2(x) + \cdots$$

with holomorphic coefficients $\{\tilde{x}_{2j}(x)\}_{j=0,1,\ldots}$ defined in a neighborhood of a so that the equation (1) may be transformed to the Airy equation

$$\text{(24)} \qquad \left(\frac{d^2}{d\tilde{x}^2} - \eta^2\tilde{x}\right)\tilde{\psi}(\tilde{x}, \eta) = 0$$

by the (formal) coordinate transformation (23) and the transformation of unknown functions of the following form

$$\text{(25)} \qquad \psi(x, \eta) = \left(\frac{\partial \tilde{x}}{\partial x}\right)^{-1/2} \tilde{\psi}(\tilde{x}(x, \eta), \eta).$$

5 An application — Computation of monodromy groups

The connection formula (21) and the configuration of Stokes curves enable us to analyze the global behavior of solutions of the equation (1). In this section, to illustrate how to solve connection problems of (1), we compute the monodromy group of the following example of Fuchsian equations by using the exact WKB analysis.

Example 3

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi(x,\eta) = 0, \quad Q(x) = -\frac{(4x^2-(1+i)^2)(4x^2-(1-3i)^2)}{((4x^2-(1-i)^2)(4x^2-(1+3i)^2))^2}. \tag{26}$$

Let us number the turning points (*i.e.*, zeros of the numerator) and the singular points (*i.e.*, those of the denominator) of (26) as follows:

$$\begin{aligned} &a_0 = (-1+3i)/2, \quad a_1 = (1+i)/2, \quad a_2 = (-1-i)/2, \quad a_3 = (1-3i)/2, \\ &b_0 = (1+3i)/2, \quad b_1 = (-1+i)/2, \quad b_2 = (1-i)/2, \quad b_3 = (-1-3i)/2. \end{aligned}$$

Then the configuration of Stokes curves of (26) becomes the following:

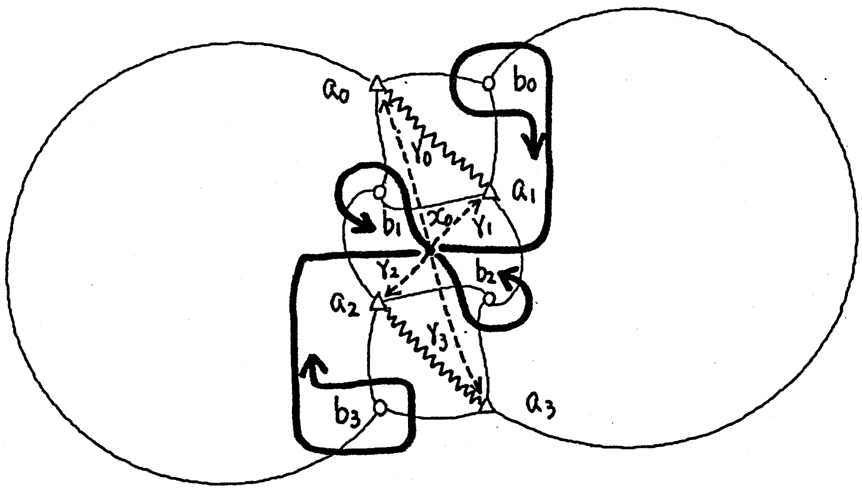

Figure 2: Stokes curves of (26).

(In Figure 2 Stokes curves are designated by lightface lines. A boldface line, a wiggly line and a broken line respectively denote a path of continuation (around a singular point b_k), a cut (to define the Riemann surface of $\sqrt{Q(x)}$) and an oriented path from a base point x_0 to a turning point a_j.)

Since all the singular points $\{b_k\}$ are regular singular points, the equation (26) is a Fuchsian equation. For such a Fuchsian equation the global behavior of solutions can be described by the monodromy group, which is defined in the following way: Take a base point x_0 and a fundamental system of solutions (ψ_0, ψ_1) at x_0, and consider the analytic continuation of (ψ_0, ψ_1) along a closed path C in $P^1(\mathbb{C}) \setminus \{b_0, \cdots, b_3\}$ with the base point x_0. Then, after the analytic continuation, (ψ_0, ψ_1) becomes a linear combination $(\psi_0, \psi_1)A_C$ ($A_C \in GL(2, \mathbb{C})$) of them and we thus obtain a homomorphism ("monodromy representation")

$$\pi_1(P^1(\mathbb{C}) \setminus \{b_0, \cdots, b_3\}) \ni C \longmapsto A_C \in GL(2, \mathbb{C}). \tag{27}$$

The monodromy group is (the conjugacy class of) the image of this homomorphism and each matrix A_C is called a monodromy matrix.

In the case of the equation (26) we take the origin $x = 0$ as base point and the Borel resummed WKB solutions $\psi_\pm = (S_{\mathrm{odd}})^{-1/2} \exp\left(\pm \int_0^x S_{\mathrm{odd}} dx\right)$ as fundamental system of solutions at $x_0 = 0$. Then, looking at Figure 2 and using the connection formula (21), we can compute the monodromy matrix A_k along a path C_k which encircles a singular point b_k once and returns to $x = 0$ (cf. Figure 2) as follows:

$$\begin{aligned}
A_0 &= \begin{pmatrix} \nu_0^+ & 0 \\ 0 & \nu_0^- \end{pmatrix} \begin{pmatrix} 1 & -iu_1^{-1}\dfrac{\nu_0^-}{\nu_0^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1\dfrac{\nu_0^+}{\nu_0^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_0u_{01}^2\dfrac{\nu_0^+}{\nu_0^-} & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 \\ -iu_3\dfrac{\nu_2^-}{\nu_2^+} & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_1^{-1} \\ 0 & 1 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} \nu_1^+ & 0 \\ 0 & \nu_1^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\dfrac{\nu_1^+}{\nu_1^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1 & 1 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} \nu_2^+ & 0 \\ 0 & \nu_2^- \end{pmatrix} \begin{pmatrix} 1 & -iu_1^{-1}\dfrac{\nu_2^-}{\nu_2^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_3^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_2^{-1} \\ 0 & 1 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} \nu_3^+ & 0 \\ 0 & \nu_3^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\dfrac{\nu_3^+}{\nu_3^-} & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_2^{-1}\dfrac{\nu_3^-}{\nu_3^+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -iu_3^{-1}u_{23}^2\dfrac{\nu_3^-}{\nu_3^+} \\ 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & -iu_0^{-1}\dfrac{\nu_1^+}{\nu_1^-} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_2 & 1 \end{pmatrix}.
\end{aligned}$$

Here we have used the notation

$$\nu_k^\pm = \exp(2\pi i \mu_k^\pm) \quad \text{(where } \mu_k^\pm \text{ denote the characteristic exponents at } b_k\text{)}$$

$$u_j = \exp\left(2\int_{\gamma_j} S_{\mathrm{odd}} dx\right), \quad u_{jj'} = u_j^{-1}u_{j'}.$$

Each triangular matrix appearing in the above expression of A_k represents the fact that we have used the formula (21) whenever a path C_k of analytic continuation crosses a Stokes curve. The reason why some complicated constant appears in the off-diagonal components of these triangular matrices is the following:

(i) The normalization of WKB solutions used here is different from (22) used in the description of (21).

(ii) S_{odd} has a simple pole at each regular singular point b_k and its residue is described by a characteristic exponent $\mu_k^{\pm}$.

Note that $u_{jj'}$ is given by a contour integral of S_{odd} around a cut connecting two turning points a_j and $a_{j'}$. The property (ii) mentioned above also explains the necessity of a diagonal matrix with components $\nu_k^{\pm}$ in the expression of A_k.

Now, replacing the fundamental system of solutions $\psi_{\pm}$ at $x_0 = 0$ by

$$\tilde{\psi}_{\pm} = \exp\left(\mp \int_{\gamma_1} S_{\mathrm{odd}}dx\right)\psi_{\pm} \tag{28}$$

and, furthermore, using fundamental relations among contour integrals

$$\nu_k^+\nu_k^- = 1 \quad \text{and} \quad u_{01}u_{32}\nu_0^+\nu_1^+\nu_2^+\nu_3^+ = 1, \tag{29}$$

we obtain the following description of monodromy matrices in terms of $\nu_k^{\pm}$ and $u_{jj'}$:

$$\begin{aligned}
\widetilde{A}_0 &= \begin{pmatrix} -\theta & -i(\nu_0^- + \theta) \\ -i(\nu_0^+ + \theta) & \nu_0^+ + \nu_0^- + \theta \end{pmatrix}, \\
\widetilde{A}_1 &= \begin{pmatrix} \nu_1^+ & 0 \\ -i(\nu_1^- + u_{01}^{-1}\nu_1^- + u_{12}\nu_1^+) & \nu_1^- \end{pmatrix}, \\
\widetilde{A}_2 &= \begin{pmatrix} \nu_2^+ & -i(\nu_2^- + u_{12}^{-1}\nu_2^+ + u_{01}^{-1}u_{12}^{-1}\nu_0^-\nu_1^-\nu_3^-) \\ 0 & \nu_2^- \end{pmatrix}, \\
\widetilde{A}_3 &= \begin{pmatrix} \nu_3^+ + \nu_3^- + \nu_1^+\nu_2^+\theta & -iu_{12}^{-1}(\nu_3^- + \nu_1^+\nu_2^+\theta) \\ -iu_{12}(\nu_3^+ + \nu_1^+\nu_2^+\theta) & -\nu_1^+\nu_2^+\theta \end{pmatrix},
\end{aligned}$$

where $\theta = u_{01}\nu_0^+ + u_{01}u_{12}\nu_1^+\nu_2^-\nu_3^+$.

In general, by using the exact WKB analysis we find that the monodromy group of Fuchsian equations of the form (1) can be described in terms of characteristic exponents at regular singular points and contour integrals $\oint_{\gamma} S_{\mathrm{odd}}dx$ of S_{odd} along closed paths on the Riemann surface of $\sqrt{Q(x)}$.

6 Further development — WKB analysis of Painlevé equations

Recently the exact WKB analysis of Schrödinger equations has been generalized to Painlevé equations, typical second-order nonlinear ordinary differential equations. In this section we briefly explain an outline of the WKB analysis of Painlevé equations. For details we refer the reader to [KT1], [AKT2], [KT2] and [T2].

The Painlevé equations (P_J) $(J = \mathrm{I}, \dots, \mathrm{VI})$ with a large parameter η have the following form:

$$(P_J) \qquad \frac{d^2\lambda}{dt^2} = \eta^2 F_J(\lambda, t) + G_J\left(\lambda, \frac{d\lambda}{dt}, t\right),$$

where F_J and G_J are rational functions. For example, the explicit form of (P_{I}) and that of (P_{II}) are as follows:

$$(P_{\mathrm{I}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$(P_{\mathrm{II}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c).$$

Painlevé equations were first discovered by Painlevé and his student Gambier in their classification of second-order nonlinear ordinary differential equations (of normal type) with what is now called "Painlevé property". The Painlevé property, which means that any solution has no movable branch point, actually guarantees that global connection problems for a differential equation in question are "well-posed". In this sense Painlevé equations are good objects of WKB analysis because, as we have observed so far, the exact WKB analysis is a powerful tool for analyzing global connection problems. Furthermore, Painlevé equations are closely related to connection problems of some linear ordinary differential equations through "isomonodromic deformations" (cf. [JMU], [O]). From this point of view also they are considered to be good equations worth being analyzed by the exact WKB analysis.

Our WKB theory of Painlevé equations (P_J) with a large parameter is constructed in a parallel way to that of Schrödinger equations. First of all, as a substitute for WKB solutions of Schrödinger equations, we have the following 2-parameter family of formal solutions of (P_J) called *instnton-type solutions*:

$$(30) \qquad \lambda_J(t;\alpha,\beta) = \lambda_0(t) + \eta^{-1/2}\lambda_{1/2}(t,\eta) + \eta^{-1}\lambda_1(t,\eta) + \cdots,$$

where $\lambda_0(t)$ is an algebraic function determined by

$$(31) \qquad F_J(\lambda_0(t), t) = 0$$

and $\lambda_{j/2}(t,\eta)$ $(j \geq 1)$ has the following expansion:

$$\begin{aligned} \lambda_{1/2}(t,\eta) &= \mu_J(t)\left\{\alpha(\theta_J(t)\eta^2)^{\alpha\beta}\exp(\eta\phi_J(t)) + \beta(\theta_J(t)\eta^2)^{-\alpha\beta}\exp(-\eta\phi_J(t))\right\}, \\ \lambda_{j/2}(t,\eta) &= \sum_{k=0}^{j} b_{j-2k}^{(j/2)}(t)(\theta_J(t)\eta^2)^{(j-2k)\alpha\beta}\exp\left((j-2k)\eta\phi_J(t)\right) \qquad (j \geq 2). \end{aligned}$$

Here

$$\phi_J(t) = \int^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}dt, \tag{32}$$

$\mu_J(t)$ and $\theta_J(t)$ are some appropriate functions of t, and (α, β) denotes a pair of free parameters. (For the construction of instanton-type solutions see [A] or [T1].) Having the explicit form of instanton-type solutions $\lambda_J(t; \alpha, \beta)$ in mind, we next define turning points and Stokes curves for them in the following way:

Definition 2 (i) A *turning point* of (P_J) is, by definition, a point r which satisfies

$$F_J(\lambda_0(r), r) = \frac{\partial F_J}{\partial \lambda}(\lambda_0(r), r) = 0. \tag{33}$$

A turning point r is said to be simple if $(\partial^2 F_J/\partial \lambda^2)(\lambda_0(r), r) \neq 0$.
(ii) A *Stokes curve* of (P_J) is defined by the following relation:

$$\Im \int_r^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}dt = 0, \tag{34}$$

where r is a turning point of (P_J).

For example, in the case of (P_{I}) $t = 0$ is a unique simple turning point and the Stokes curves are given by the relation $\Im x^{5/4} = 0$ (cf. Figure 3 below).

Then, we can prove the following fundamental properties for $\lambda_J(t; \alpha, \beta)$:

Fact 3 (Local reduction to (P_{I}), [KT2]) *In a neighborhood of a point t_* on a Stokes curve emanating from a simple turning point r of (P_J), the Painlevé equation (P_J) and their instanton-type solutions $\lambda_J(t; \alpha, \beta)$ can be transformed to (P_{I}) and $\tilde{\lambda}_{\mathrm{I}}(\tilde{t}; \tilde{\alpha}, \tilde{\beta})$. To be more precise, for each instanton-type solution $\lambda_J(t; \alpha, \beta)$ of (P_J) we can find an instanton-type solution $\tilde{\lambda}_{\mathrm{I}}(\tilde{t}; \tilde{\alpha}, \tilde{\beta})$ of (P_{I}) for which the following holds:*

There exist formal series

$$\tilde{x}(x, t, \eta) = \sum_{j \geq 0} \eta^{-j/2} \tilde{x}_{j/2}(x, t, \eta) \tag{35}$$

and

$$\tilde{t}(t, \eta) = \sum_{j \geq 0} \eta^{-j/2} \tilde{t}_{j/2}(t, \eta) \tag{36}$$

so that the following relation may be satisfied:

$$\tilde{x}(\lambda_J(t; \alpha, \beta), t, \eta) = \tilde{\lambda}_{\mathrm{I}}(\tilde{t}(t, \eta); \tilde{\alpha}, \tilde{\beta}). \tag{37}$$

Fact 4 (Connection formula for (P_{I}), [T2]) *Let $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ be an instanton-type solution of (P_{I}) in the region $\{-3\pi/5 < \arg t < -\pi/5\}$ and $\lambda_{\mathrm{I}}(t;\tilde{\alpha},\tilde{\beta})$ its analytic continuation to $\{-\pi/5 < \arg t < \pi/5\}$ across a Stokes curve $\arg t = -\pi/5$ (cf. Figure 3). Then the following relations hold between (α,β) and $(\tilde{\alpha},\tilde{\beta})$.*

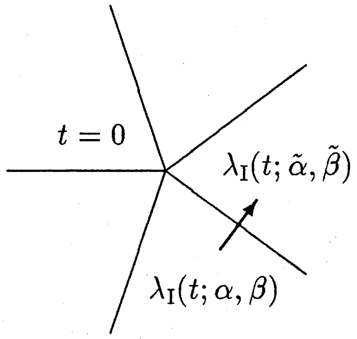

Figure 3:
Stokes curves of (P_{I}).

(38)
$$\begin{cases} \alpha e^{-i\pi E/4}\chi(E) = \tilde{\alpha} e^{-i\pi \tilde{E}/4}\chi(\tilde{E}), \\ e^{i\pi E/2} + \beta e^{i\pi E/2}\chi(-E) = \tilde{\beta} e^{i\pi \tilde{E}/2}\chi(-\tilde{E}), \end{cases}$$

where $\chi(z) = \dfrac{\sqrt{\pi}}{\Gamma(z/4+1)} 2^{z/4+1}$, $E = -8\alpha\beta$, *and* $\tilde{E} = -8\tilde{\alpha}\tilde{\beta}$.

Unfortunately, as it is not known how to give an analytic meaning to instanton-type formal solutions at the present stage, we have not yet succeeded in verifying the analytic version of these fundamental properties. However, in parallel with the Schrödinger case the connection formula at a simple turning point for general (P_J) should, in principle, be derived from combination of Fact 3 and Fact 4 and by using the formula for (P_J) thus obtained (which might probably be of the same form with that for (P_{I})) together with the configuration of Stokes curves we should be able to analyze the global behavior of solutions of (P_J). To exemplify the validity and effectiveness of this approach we briefly explain our discussion about a well-known connection problem for (P_{II}) from this point of view in what follows.

Let us consider a solution $u(z)$ of the equation

$$u'' = zu + 2u^3 \tag{39}$$

with the following asymptotic behavior for $z > 0, z \to \infty$:

$$u(z) \sim a\mathrm{Ai}(z) \sim \frac{a}{2\sqrt{\pi}} z^{-1/4} \exp(-\frac{2}{3} z^{3/2}) \quad (z \to +\infty) \tag{40}$$

where a is a constant satisfying $0 < a < 1$. It is known that, after the analytic continuation along the real axis, $u(z)$ has the following asymptotic expansion for $z \to -\infty$:

$$u(z) \sim d(-z)^{-1/4} \sin\left(\frac{2}{3}(-z)^{3/2} - \frac{3}{4} d^2 \log(-z) + \theta\right) \quad (z \to -\infty), \tag{41}$$

where d and θ are given by

$$(42)\qquad \begin{cases} d^2 &= -\dfrac{1}{\pi}\log(1-a^2), \\ \theta &= \dfrac{\pi}{4} - \dfrac{3}{2}d^2\log 2 - \arg\Gamma\left(1-\dfrac{id^2}{2}\right). \end{cases}$$

The formula (42) was first discovered by Ablowitz and Segur (cf. [AS], [SA]). Our question is the following: Is it possible to derive this formula (42) by using our WKB analysis?

Note that by a simple scaling transformation

$$(43)\qquad u = \eta^{1/3}\lambda, \quad z = \eta^{2/3}t$$

the equation (39) is transformed to the second Painlevé equation with a large parameter

$$(P_{\mathrm{II}}^0)\qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda)$$

where the parameter c is taken to be 0. Furthermore, through the scaling transformation (43) the asymptotic solution (40) [resp. (41)] of (39) corresponds (at least at the leading order level) to an instanton-type solution $\lambda_{\mathrm{II}}^0(t;0,\beta)$ [resp. $\lambda_{\mathrm{II}}^0(t;\alpha,\beta)$] of (P_{II}^0) with the identically vanishing top term $\lambda_0(t) \equiv 0$ for $z \to \infty$ [resp. $z \to -\infty$]. Therefore what we want to discuss is the connection problem between $\lambda_{\mathrm{II}}^0(t;0,\beta)$ for $z \to \infty$ and $\lambda_{\mathrm{II}}^0(t;\alpha,\beta)$ for $z \to -\infty$.

It can be easily observed that

(i) $t = 0$ is a unique (non-simple) turning point of (P_{II}^0),

(ii) the configuration of Stokes curves of (P_{II}^0) is the same with that of the Airy equation, that is, the Stokes curves of (P_{II}^0) are given by $\Im x^{3/2} = 0$.

Although the geometry of Stokes curves of (P_{II}^0) is not complicated, it is difficult to determine the connection formula on each Stokes curve since $t = 0$ is not a simple turning point and we cannot apply the connection formula for (P_{I}) to this equation. To overcome this difficulty we first consider (P_{II}) with a non-zero parameter c and then take a limit $c \to 0$. Figure 4 indicates the configuration of Stokes curves of (P_{II}) with non-zero c. There appear three turning points in Figure 4, all of which are simple. Hence we expect that the connection formula for (P_{I}) should be applicable to each Stokes curve in Figure 4. As a matter of fact, looking at this configuration of Stokes curves (more precisely, its lift to the Riemann surface of the algebraic equation $2\lambda^3 + t\lambda + c = 0$ determining $\lambda_0(t)$) and using the connection formula for (P_{I}) repeatedly, we can obtain the formula (42) for the limit $c \to 0$. (For the details of computation see [T3].) In other words, the Ablowitz-Segur formula can be derived from repeated use of the connection formula for (P_{I}). We think this strongly supports the validity and effectiveness of our WKB theory for Painlevé equations.

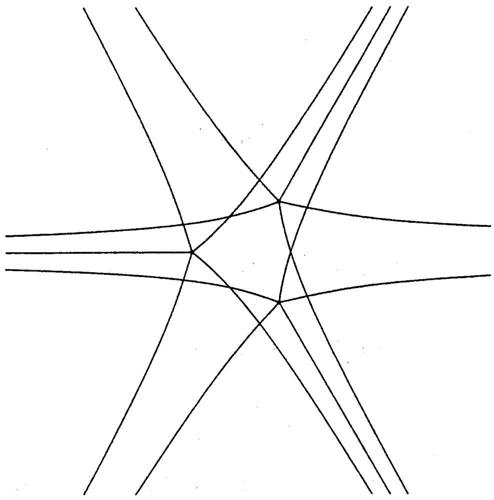

Figure 4 : Stokes curves of (P_{II}).

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