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Justification of a formal derivation of the Euler-Maclaurin summation formula

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1 Introduction

The Euler-Maclaurin summation formula

$$\sum_{k=0}^{n-1} f(x+kh)$$
$$= \frac{1}{h}\int_0^{nh} f(x+t)\,dt - \frac{1}{2}[f(x+nh)-f(x)]$$
$$+ \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} h^{2r-1}[f^{(2r-1)}(x+nh) - f^{(2r-1)}(x)] \tag{1.1}$$

is very important in many branches of mathematics. Here B_{2r} $(r=1,2,\ldots)$ are the Bernoulli numbers, which are defined by

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} z^{2r}. \tag{1.2}$$

Up to the present several ways have been known of deriving the Euler-Maclaurin summation formula. Among them, a way employing the so-called symbolic calculus is the simplest, although the calculus is merely formal in general ([2], [5], [8], [9]). This situation raises naturally the question

" Can the symbolic calculus employed there be justified
under certain conditions ? "

The main aim of the present paper is to answer the question in the affirmative.

At first sight there seems to be no clue for answering the question. The fact is however pointed out by Hardy [5], that the infinite series appearing in the right-hand side of the Euler-Maclaurin summation formula is convergent if $f(x)$ is an entire function of exponential type less than $2\pi/h$. It makes us foresee the possibility that the symbolic calculus might be justified for a class of entire functions of exponential type. In the paper we prove this possibility. To be more specific, we here show that the symbolic calculus is valid if it is regarded as the operational calculus for the differential operator on a normed linear space of entire functions of exponential type.

This paper is organized as follows. In §2, for completeness, we reproduce the derivation of the Euler-Maclaurin formula based on the symbolic calculus. In §3 we show that the symbolic calculus can be justified by taking three steps: the first step is to introduce a normed linear space of entire functions of exponential type; the second step is to study basic properties of the differential operator on the space introduced in the first step; the final step is to see that the symbolic calculus is valid if it is interpreted as the operational calculus for the differential operator on the space. Finally, in §4 we deal with the Euler-Maclaurin summation formula with the remainder term.

2 Derivation of the Euler-Maclaurin summation formula based on the symbolic calculus

Following [2] and [8], we derive the Euler-Maclaurin summation formula by means of the symbolic calculus.

We define the following operators:

(1) Shift operator $\mathbf{E}_h f(x) = f(x+h)$;
(2) Differential operator $\mathbf{D} f(x) = f'(x)$.

A key to derive the Euler-Maclaurin formula is the relation

$$\mathbf{E}_h = e^{h\mathbf{D}}, \tag{2.1}$$

which is formally proved as follows:

$$\begin{aligned}
\mathbf{E}_h f(x) &= f(x+h) \\
&= f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots \quad (\text{ The Taylor expansion of } f(x+h)\ ) \\
&= (1 + \mathbf{D}h + \frac{\mathbf{D}^2}{2!}h^2 + \cdots)f(x) \\
&= e^{h\mathbf{D}} f(x).
\end{aligned}$$

We first write the left-hand side of the Euler-Maclaurin formula in operator form as

$$\sum_{k=0}^{n-1} f(x+kh) = (1 + \mathbf{E}_h + \cdots + \mathbf{E}_h^{n-1})f(x).$$

And using the relation (2.1) and the formula for the sum of a geometric series, we get

$$\begin{aligned}
\sum_{k=0}^{n-1} f(x+kh) &= (1 + e^{h\mathbf{D}} + \cdots + (e^{h\mathbf{D}})^{n-1})f(x) \\
&= \frac{(e^{h\mathbf{D}})^n - 1}{e^{h\mathbf{D}} - 1} f(x).
\end{aligned}$$

Further by virtue of the addition formula for the exponential function, we have

$$\sum_{k=0}^{n-1} f(x+kh) = \frac{e^{nh\mathbf{D}}-1}{e^{h\mathbf{D}}-1} f(x). \tag{2.2}$$

Now express the right-hand side of (2.2) as

$$\frac{e^{nh\mathbf{D}}-1}{e^{h\mathbf{D}}-1} f(x) = \frac{e^{nh\mathbf{D}}-1}{h\mathbf{D}} \frac{h\mathbf{D}}{e^{h\mathbf{D}}-1} f(x). \tag{2.3}$$

And substituting the Taylor series

$$\frac{h\mathbf{D}}{e^{h\mathbf{D}}-1} = 1 - \frac{h\mathbf{D}}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} (h\mathbf{D})^{2r}$$

(see (1.2), i.e. the generating function of the Bernoulli numbers) into (2.3), we obtain

$$\begin{aligned} \frac{e^{nh\mathbf{D}}-1}{e^{h\mathbf{D}}-1} f(x) &= \frac{e^{nh\mathbf{D}}-1}{h\mathbf{D}} \left(1 - \frac{h\mathbf{D}}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} (h\mathbf{D})^{2r}\right) f(x) \\ &= \frac{E_{nh}-1}{h\mathbf{D}} \left(1 - \frac{h\mathbf{D}}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} (h\mathbf{D})^{2r}\right) f(x) \\ &= \left\{\frac{E_{nh}-1}{h\mathbf{D}} - \frac{E_{nh}-1}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} (E_{nh}-1)(h\mathbf{D})^{2r-1}\right\} f(x). \end{aligned}$$

Interpreting $\mathbf{D}^{-1} f(x)$ as a primitive function of $f(x)$, we reach to

$$\begin{aligned} &\sum_{k=0}^{n-1} f(x+kh) \\ &= \frac{1}{h}\int_0^{nh} f(x+t)\,dt - \frac{1}{2}[f(x+nh) - f(x)] \\ &\qquad + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} h^{2r-1} [f^{(2r-1)}(x+nh) - f^{(2r-1)}(x)], \end{aligned} \tag{2.4}$$

which is the Euler-Maclaurin summation formula to be desired.

3 Justification of the symbolic calculus

3.1 A normed linear space of entire functions of exponential type τ

We introduce a normed linear space of entire functions of exponential type τ. We first recall the definition of " entire functions of exponential type τ ".

Definition ([1]) *The entire function $f(z)$ is of exponential type τ if, for every positive ε but for no negative ε,*

$$\max_{|z|=r} |f(z)| = O(e^{(\tau+\varepsilon)r}) \qquad (r \to \infty).$$

In view of this definition it seems natural to set up the space

$$\mathcal{E}_\tau = \left\{ f(z) \mid f(z) \text{ is entire, and } \sup_{z\in\mathbf{C}} e^{-\tau|z|}|f(z)| < \infty \right\},$$

where the norm of f is defined by

$$||f|| \equiv \sup_{z\in\mathbf{C}} e^{-\tau|z|}|f(z)|. \tag{3.1}$$

Obviously $\mathcal{E}_\tau$ is a Banach space.

We should here note that $\mathcal{E}_\tau$ does not contain all of the entire functions of exponential type τ. In fact, $ze^{\tau z}$ is an entire function of exponential type τ, but $ze^{\tau z}$ does not belong to $\mathcal{E}_\tau$. However $\mathcal{E}_\tau$ contains typical entire functions of exponential type τ, such as $e^{\tau z}$, $\sin\tau z$, $\cos\tau z$, and $J_n(\tau z)$ (J_n is the Bessel function of order n).

3.2 Differential operator on the space $\mathcal{E}_\tau$

We study basic properties of the differential operator on $\mathcal{E}_\tau$. The result is summarized in the following theorem.

Theorem 3.1 *Let D be the differential operator on the space $\mathcal{E}_\tau$, and denote by $||D||$, $r(D)$, and $\sigma(D)$ the norm of D, the spectral radius of D, and the spectrum of D respectively. Then*

$$\tau \le ||D|| \le e\tau, \tag{3.2}$$

$$r(D) = \tau, \tag{3.3}$$

and

$$\sigma(D) = \{\lambda \in \mathbf{C} \mid |\lambda| \le \tau\}. \tag{3.4}$$

(Proof) To establish (3.2) and (3.3), we prove the inequalities

$$\tau^n \le ||D^n|| \le n!\frac{e^n\tau^n}{n^n}. \tag{3.5}$$

The proof of the first inequality is easy. In fact,

$$||D^n|| \equiv \sup_{f\neq 0} \frac{||D^n f||}{||f||} \ge \frac{||D^n e^{\tau z}||}{||e^{\tau z}||} = \tau^n.$$

The proof of the second inequality is a little laborious. We first represent $D^n f(z)$ by Cauchy's formula for derivatives,

$$D^n f(z) = \frac{n!}{2\pi i}\int_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \tag{3.6}$$

where r is the arbitrary positive real number. Taking the absolute value of (3.6), we get

$$\begin{aligned}|D^n f(z)| &\leq \frac{n!}{2\pi}\int_{|\zeta-z|=r}\frac{|f(\zeta)|}{|\zeta-z|^{n+1}}\,d|\zeta|\\ &\leq \frac{n!}{r^n}\sup_{|\zeta-z|=r}|f(\zeta)|.\end{aligned}$$

Then, from the definition of the norm (3.1), we have

$$||D^n f|| \leq \frac{n!}{r^n}\sup_{z\in\mathbf{C}}\Big(\sup_{|\zeta-z|=r} e^{-\tau|z|}|f(\zeta)|\Big). \tag{3.7}$$

Here we write the right-hand side of (3.7) as

$$\frac{n!}{r^n}\sup_{z\in\mathbf{C}}\Big(\sup_{|\zeta-z|=r} e^{-\tau|z|}|f(\zeta)|\Big) = \frac{n!}{r^n}e^{\tau r}\sup_{z\in\mathbf{C}}\Big(\sup_{|\zeta-z|=r} e^{-\tau(|z|+r)}|f(\zeta)|\Big).$$

Using the inequality

$$\sup_{|\zeta-z|=r}|f(\zeta)| \leq \sup_{|\zeta|=|z|+r}|f(\zeta)|,$$

which is obvious from the maximum principle, we find

$$\begin{aligned}\frac{n!}{r^n}e^{\tau r}\sup_{z\in\mathbf{C}}\Big(\sup_{|\zeta-z|=r} e^{-\tau(|z|+r)}|f(\zeta)|\Big) &\leq \frac{n!}{r^n}e^{\tau r}\sup_{z\in\mathbf{C}}\Big(\sup_{|\zeta|=|z|+r} e^{-\tau|\zeta|}|f(\zeta)|\Big)\\ &\leq \frac{n!}{r^n}e^{\tau r}||f||.\end{aligned}$$

Thus we have

$$||D^n f|| \leq n!\frac{e^{\tau r}}{r^n}||f||,$$

which implies

$$||D^n|| \leq n!\frac{e^{\tau r}}{r^n}.$$

Finally, noting that r is arbitrarily positive, we get

$$||D^n|| \leq n!\,\min_{r>0}\frac{e^{\tau r}}{r^n} = n!\frac{e^n\tau^n}{n^n}.$$

This is the second inequality in (3.5).

Now (3.2) is trivial. In fact it is a special case of (3.5) with $n = 1$. To establish (3.3), we utilize the well-known formula

$$r(D) = \lim_{n\to\infty}||D^n||^{1/n}. \tag{3.8}$$

Substituting the inequality (3.5) into the right-hand side of (3.8), we get

$$\tau \leq r(D) \leq \lim_{n\to\infty}\left(n!\frac{e^n\tau^n}{n^n}\right)^{1/n} = \tau,$$

which is to be proved. In the last equality, we used the Stirling formula

$$n! \sim \sqrt{2\pi n}e^{-n}n^n \qquad (n \to \infty).$$

It remains to prove (3.4). Owing to the inclusion relation

$$\sigma(D) \subseteq \{\lambda \in \mathbf{C} \mid |\lambda| \leq r(D)(= \tau)\},$$

it suffices to show that every $\lambda \in \{\lambda \in \mathbf{C} \mid |\lambda| \leq \tau\}$ is an eigenvalue of the operator D. This is however evident from the fact that, for every $\lambda \in \{\lambda \in \mathbf{C} \mid |\lambda| \leq \tau\}$, $e^{\lambda z} \in \mathcal{E}_\tau$ and $De^{\lambda z} = \lambda e^{\lambda z}$. The proof of Theorem 3.1 is now completed. ∎

3.3 Operational calculus for the differential operator D on $\mathcal{E}_\tau$ and justification of the symbolic calculus

As is seen in the preceding theorem, the spectrum of the differential operator D on $\mathcal{E}_\tau$ is the closed disk of radius τ. Therefore, from the general theory of the operational calculus ([4], [10]), we know that the following two theorems hold. We here denote by $\mathcal{A}_\tau$ the family of all functions holomorphic in the closed disk of radius τ.

Theorem 3.2 *For every function $f(z) \in \mathcal{A}_\tau$, let $\sum_{k=0}^{\infty} a_k z^k$ be the Taylor expansion of $f(z)$. Then the series of D, $\sum_{k=0}^{\infty} a_k D^k$, is convergent in the operator norm topology. That is to say, for every function $f(z) \in \mathcal{A}_\tau$, a function $f(D)$ of D can be defined throught the Taylor expansion of $f(z)$.*

Theorem 3.3 *The following operational calculus for the differential operator D on the space $\mathcal{E}_\tau$ holds good.*

1. *If f and g are in $\mathcal{A}_\tau$, and α and β are complex numbers, then*
   (1-1) $\alpha f + \beta g \in \mathcal{A}_\tau$ *and* $\alpha f(D) + \beta g(D) = (\alpha f + \beta g)(D)$,
   and
   (1-2) $f \cdot g \in \mathcal{A}_\tau$ *and* $f(D) \cdot g(D) = (f \cdot g)(D)$.
2. *Let $f_n \in \mathcal{A}_\tau$ $(n = 0, 1, \ldots)$ and $f \in \mathcal{A}_\tau$ be holomorphic in a fixed neighborhood U of the closed disk of radius τ. If $f_n(z)$ converges to $f(z)$ uniformly on U, then $f_n(D)$ converges to $f(D)$ in the operator norm topology.*
3. *If $d(z)$ is an entire function and $f(z)$ is in $\mathcal{A}_\tau$, then $d(z) = e(f(z)) \in \mathcal{A}_\tau$ and $d(D) = e(f(D))$.*

From Theorem 3.2 we obtain the following lemma. Note that the functions in the lemma are those which appear in the process of formally deriving the Euler-Maclaurin formula.

Lemma 3.4 1. *The function* e^{hD} *is well defined and*

$$e^{hD} = E_h, \tag{3.9}$$

where E_h *is the shift operator on* $\mathcal{E}_\tau$, *i.e.*

$$E_h f(x) = f(x+h).$$

2. *The function* $\dfrac{hD}{e^{hD}-1}$ *is well defined on condition that* $\tau < \dfrac{2\pi}{h}$.
3. *The function* $\dfrac{e^{nhD}-1}{hD}$ *is well defined and*

$$\frac{e^{nhD}-1}{hD} = \frac{K_{nh}}{h}, \tag{3.10}$$

where K_{nh} *is the integral operator on* $\mathcal{E}_\tau$, *i.e.*

$$K_{nh} f(x) = \int_0^{nh} f(x+t)\, dt.$$

(Proof) The proof is straightforward. ∎

Now we are ready to attain the goal, that is, to justify the symbolic calculus employed in the process of formally deriving the Euler-Maclaurin formula.

Let $h > 0$ be fixed, and let $\tau > 0$ be less than $2\pi/h$. We look upon the operators $\mathbf{E}_h$ and $\mathbf{D}$, which are involved in the symbolic calculus, as the operators E_h and D on the space $\mathcal{E}_\tau$. Then, to justify the symbolic calculus, we have to show that the functions of the operators appearing there are well defined and that the operations performed are valid. But we have already done this work in Lemma 3.4 and Theorem 3.3. Thus we have completed the justification of the symbolic calculus.

4 The Euler-Maclaurin summation formula with the remainder term

In the previous sections we treated only the Euler-Maclaurin summation formula *without* the remainder term. However the formula *with* the remainder term is rather preferable, especially in applications to numerical analysis, so that to derive it by the operational calculus developed in the preceding section is desired. We here see that this desire is accomplished.

We first recall the Euler-Maclaurin summation formula with the remainder term([7]):

$$\sum_{k=0}^{n-1} f(x+kh)$$

$$
\begin{aligned}
&= \frac{1}{h}\int_0^{nh} f(x+t)dt - \frac{1}{2}[f(x+nh) - f(x)] \\
&\qquad + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} h^{2r-1}[f^{(2r-1)}(x+nh) - f^{(2r-1)}(x)] \\
&\qquad + \int_0^{nh} \frac{B_{2m+1}(t/h - [t/h])}{(2m+1)!} h^{2m} f^{(2m+1)}(x+t)\, dt,
\end{aligned} \tag{4.1}
$$

where $B_p(t)\,(p = 0, 1, \cdots)$ are the Bernoulli polynomials, which are defined by

$$
\frac{ze^{tz}}{e^z - 1} = \sum_{p=0}^{\infty} \frac{B_p(t)}{p!} z^p.
$$

And we prepare the following two formulas:

$$
\frac{B_p(t-[t])}{p!} = -\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{e^{2\pi i j t}}{(2\pi i j)^p} \qquad (p \geq 1), \tag{4.2}
$$

$$
\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} z^{2r} - \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi i j)^{2m}(z + 2\pi i j)} z^{2m+1}. \tag{4.3}
$$

The first formula is well known ([3]), and the second one is easily obtained from the partial fraction expansion of the function $z/(e^z - 1)$([6]):

$$
\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 + (2\pi n)^2}.
$$

Now we are ready to derive the Euler-Maclaurin summation formula with the remainder term. Let $h > 0$ be fixed, and let $\tau > 0$ be less than $2\pi/h$. Based on the operational calculus on the space $\mathcal{E}_\tau$, we have

$$
\begin{aligned}
& f(x) + f(x+h) + \cdots + f(x+(n-1)h) \\
&= (1 + E_h + \cdots + E_h^{n-1}) f(x) \\
&= \left(1 + e^{hD} + \cdots + (e^{hD})^{n-1}\right) f(x) \\
&= \frac{(e^{hD})^n - 1}{e^{hD} - 1} f(x) \\
&= \frac{e^{nhD} - 1}{e^{hD} - 1} f(x) \\
&= \frac{e^{nhD} - 1}{hD} \frac{hD}{e^{hD} - 1} f(x) \\
&= \frac{e^{nhD} - 1}{hD} \left(1 - \frac{hD}{2} + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} (hD)^{2r} \right. \\
&\qquad \left. + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi i j)^{2m}(hD + 2\pi i j)} (hD)^{2m+1} \right) f(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{h}\int_0^{nh} f(x+t)dt - \frac{1}{2}[f(x+nh)-f(x)] \\
&\quad + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} h^{2r-1}[f^{(2r-1)}(x+nh) - f^{(2r-1)}(x)] \\
&\quad + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{e^{nhD}-1}{(2\pi ij)^{2m}(hD+2\pi ij)hD} h^{2m+1} f^{(2m+1)}(x),
\end{aligned} \tag{4.4}
$$

where we have used the formula (4.3). And further we can show that the last term on the most right-hand side of (4.4) is equal to the remainder term in the Euler-Maclaurin summation formula (4.1) as follows:

$$
\begin{aligned}
&\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{e^{nhD}-1}{(2\pi ij)^{2m}(hD+2\pi ij)hD} h^{2m+1} f^{(2m+1)}(x) \\
&= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi ij)^{2m+1}} \left\{ -\frac{e^{nhD}-1}{hD+2\pi ij} + \frac{e^{nhD}-1}{hD} \right\} h^{2m+1} f^{(2m+1)}(x) \\
&= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi ij)^{2m+1}} \left\{ -\frac{e^{nhD}-1}{D+2\pi ij/h} + \frac{e^{nhD}-1}{D} \right\} h^{2m} f^{(2m+1)}(x) \\
&= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi ij)^{2m+1}} \left\{ -\frac{e^{nh(D+2\pi ij/h)}-1}{D+2\pi ij/h} + \frac{e^{nhD}-1}{D} \right\} h^{2m} f^{(2m+1)}(x) \\
&= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi ij)^{2m+1}} \left\{ \int_0^{nh} (-e^{t(D+2\pi ij/h)} + e^{tD})dt \right\} h^{2m} f^{(2m+1)}(x) \\
&= \int_0^{nh} \left\{ -\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{e^{2\pi ijt/h}}{(2\pi ij)^{2m+1}} + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(2\pi ij)^{2m+1}} \right\} e^{tD} dt\ h^{2m} f^{(2m+1)}(x) \\
&= \int_0^{nh} \frac{B_{2m+1}(t/h-[t/h])}{(2m+1)!} e^{tD} dt\ h^{2m} f^{(2m+1)}(x) \\
&= \int_0^{nh} \frac{B_{2m+1}(t/h-[t/h])}{(2m+1)!} h^{2m} f^{(2m+1)}(x+t)\ dt,
\end{aligned}
$$

where we have used the formula (4.2). The desire is realized just now.

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