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Operator Monotone Functions Which Are Defined Implicitly And Operator Inequalities

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Abstract

A function t^{α} $(0<\alpha<1)$ is operator monotone on $0\leq t<\infty$. This is well-known as Löwner- Heinz inequality. We will seek operator monotone functions which are defined implicitly. This investigation seems to be new, and we will actually find a family of operator monotone functions which includes t^{α} $(0<\alpha<1)$. Moreover, by constructing one-parameter families of operator monotone functions, we will get many operator inequalities; especially, we will extend the Furuta inequality and the exponential inequality by Ando.

1. Introduction

Throughout this paper, A and B stand for bounded selfadjoint operators on a Hilbert space, and $sp(X)$ for the spectrum of an operator X. A real valued function $f(t)$ is called an *operator monotone function* on $(0,\infty)$ if, for A, B with $sp(A), sp(B)\subset(0,\infty)$

$$A\geq B \quad \text{implies} \quad f(A)\geq f(B).$$

Clearly a composite function of operator monotone functions is operator monotone too, provided it is well defined. A holomorphic function which maps the open upper half plane Π_+ into itself is called a *Pick function.* By **Löwner** theorem [13], $f(t)$ is an operator monotone function on $[0,\infty)$ if and only if $f(t)$ has an analytic continuation $f(z)$ to $\Pi_+ \cup (0,\infty)$ so that $f(z)$ is a Pick function; therefore $f(t)$ is analytically extended to $\mathbf{C}\backslash(-\infty,0]$ by reflection. Thus if $f(t) \geq 0$ and $g(t) \geq 0$ are operator monotone, then so is $f(t)^{\mu}g(t)^{\lambda}$ for $0 \leq \mu, \lambda \leq 1, \quad \mu + \lambda \leq 1$. Since an operator monotone function $f(t)$ on $(0,\infty)$ is increasing, if $f(t)$ is bounded from below, $f(t)$ can be continuously extended to the closed interval $[0,\infty)$. In this case, for A, B with their spectra in $[0,\infty)$ $A \geq B$ implies $f(A) \geq f(B)$. Such a function $f(t)$ is said to be *operator monotone* on $[0,\infty)$; that is, a function $f(t)$ is called an *operator monotone function* on $[0,\infty)$ if $f(t)$ is continuous at $t = 0$ and operator monotone on $(0,\infty)$. It is well-known that $t^{\alpha}(0 < \alpha \leq 1)$, $\log(1+t)$ and $\frac{t}{t+\lambda}$ $(\lambda > 0)$ are operator monotone on $[0,\infty)$, though operator monotone functions which have been known so far are not so many (see [4]). Thus,

$$A \geq B \geq 0 \quad \text{implies} \quad A^{\alpha} \geq B^{\alpha} \quad \text{for} \quad 0 < \alpha < 1, \tag{1}$$

which is called a *Löwner- Heinz inequality* [12,13]. But $A \geq B \geq 0$ does not generally imply $A^2 \geq B^2$; actually we have shown that if $A, B \geq 0$ and $(A + tB^n)^2 \geq A^2$ for every $t > 0$ and $n = 1, 2, \cdots$, then $AB = BA$ [16]. Refer [1,3,5,9,11,14] for the details about operator monotone functions.

Chan-Kwong [4] had posed a conjecture:

$$\text{Does} \quad A \geq B \geq 0 \quad \text{imply} \quad (BA^2B)^{1/2} \geq B^2\ ?$$

Furuta [7,8] affirmatively solved it as follows:

$$A \geq B \geq 0 \quad \text{implies} \quad \begin{cases} (B^{r/2}A^pB^{r/2})^{1/q} & \geq (B^{r/2}B^pB^{r/2})^{1/q}, \\ (A^{r/2}A^pA^{r/2})^{1/q} & \geq (A^{r/2}B^pA^{r/2})^{1/q}, \end{cases} \tag{2}$$

where $r, p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$. This is called a *Furuta inequality.* In this inequality, the case of $p \leq 1$ is the deformation of Löwner-Heinz inequality; further, the case of $(1+r)q > p+r$ follows from the case of $(1+r)q = p+r$ by Löwner-Heinz inequality again: so the essentially important part of Furuta inequality is the case of $p > 1$ and $(1+r)q = p+r$. The second inequality follows from the first one by taking the inverse. **Tanahashi** [15] showed that the exponential condition $(1+r)q \geq p+r$ is the best condition for (2). Related to this inequality, **Ando** [2] showed that for $t > 0$

$$A \geq B \quad \text{implies} \quad \begin{cases} (e^{t/2B}e^{tA}e^{t/2B})^{1/2} \geq e^{tB} \\ e^{tA} \geq (e^{t/2A}e^{tB}e^{t/2A})^{1/2}, \end{cases}$$

which was improved, by making use of this inequality itself and (2), by **Fujii, Kamei** [6] as follows:

for $p \geq 0,\ r \geq s \geq 0$

$$A \geq B \quad \text{implies} \quad \begin{cases} (e^{\frac{r}{2}B}e^{pA}e^{\frac{r}{2}B})^{\frac{s}{r+p}} \geq e^{sB} \\ e^{sA} \geq (e^{\frac{r}{2}A}e^{pB}e^{\frac{r}{2}A})^{\frac{s}{r+p}}. \end{cases} \tag{3}$$

It is evident that the essentially important part of this inequality is the case of $s = r$. Recently, by making use of only (2), we [18] got a simple proof of (3).

Now we give a simple example that motivated us for investigating operator monotone functions which are defined implicitly:

$$A, B \geq 0 \text{ and } A^2 \geq B^2 \quad \text{implies} \quad (A+1)^2 \geq (B+1)^2,$$

because $A \geq B$ follows from $A^2 \geq B^2$. But we can easily construct 2×2 matrices A, B such that $(A+1)^2 \geq (B+1)^2$, but $A^2 \not\geq B^2$; for example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1.94 \end{pmatrix}.$$

The above results mean that $\phi(t) = (t^{1/2}+1)^2$ is operator monotone on $[0,\infty)$, but $\psi(t) = (t^{1/2}-1)^2$ is not on $[1,\infty)$. We may say that ϕ and ψ are implicitly defined by $\phi(t^2) = (t+1)^2$ $(t \geq 0)$ and $\psi((t+1)^2) = t^2$ $(t \geq 0)$.
One of the aims of this paper is to seek operator monotone functions which are defined implicitly; this investigation seems to be new, and we will actually find a family of operator monotone functions which includes t^α $(0 < \alpha < 1)$: this means that we can get not merely an extension of (1) but also another proof of (1). The other is to extend simultaneously (2) and (3), by making use of a one-parameter family of operator monotone functions.

2. The construction of new operator monotone functions

Let us define a non-negative increasing function $u(t)$ on $[-a_1, \infty)$ by

$$u(t) = \prod_{i=1}^{k} (t + a_i)^{\gamma_i} \quad (a_1 < a_2 < \cdots < a_k, \quad 1 \leq \gamma_1, 0 < \gamma_i). \tag{4}$$

Theorem 2.1. *Let us consider a function $s = u(t)$, where $u(t)$ is defined by* (4). *Then the inverse function $u^{-1}(s)$ is operator monotone on* $[0, \infty)$.

Proof. Since $u^{-1}(s)$ is continuous on $[0, \infty)$, we have to show that $u^{-1}(s)$ is operator monotone on $(0, \infty)$. We may assume that $a_1 = 0$; for, setting $v(t) = u(t - a_1)$ we have $u^{-1}(s) = v^{-1}(s) - a_1$; hence the operator monotonicity of $u^{-1}(s)$ follows from that of $v^{-1}(s)$. Set $D = \mathbf{C} \backslash (-\infty, 0]$, and restrict the argument as $-\pi < \arg z < \pi$ for $z \in D$. For $\gamma > 0$ define a single valued holomorphic function z^γ on D by

$$z^\gamma = \exp \gamma (\log |z| + i \arg z),$$

which is the principal branch of analytic function $\exp(\gamma \log z)$. We also define a holomorphic function $u(z)$ on D by

$$u(z) = \prod_{i=1}^{k} (z + a_i)^{\gamma_i}, \quad 0 = a_1 < a_2 < \cdots < a_k$$

which is an extension of $u(t)$. Since

$$u'(z) = \{\prod_{i=1}^{k} (z + a_i)^{\gamma_i}\}(\sum_{j=1}^{k} \frac{\gamma_j}{z + a_j}),$$

it is necessary and sufficient for $u'(z) = 0$ in D that $\sum_{j=1}^{k} \frac{\gamma_j}{z+a_j} = 0$. Since $\gamma_j > 0$ and $a_j \geq 0$, the roots of $\sum_{j=1}^{k} \frac{\gamma_j}{z+a_j} = 0$ are all in $(-\infty, 0)$. Therefore, $u'(z)$ does not vanish in D. Let us consider the function $w = u(z)$ as a mapping from the z-plane to the w-plane. We denote D in the z-plane by D_z and D in the w-plane by D_w. Take a $t_0 > 0$ and set $s_0 = u(t_0)$. Since $u'(t_0) \neq 0$, by the inverse mapping theorem, there is a univalent holomorphic function $g_0(w)$ from a disk $\Delta(s_0)$ with the center s_0 onto an open set including t_0 such that $u(g_0(w)) = w$ for $w \in \Delta(s_0)$. We show that for an arbitrary point w_0 in

D_w and for an arbitrary path C in D_w from s_0 to w_0, the function element $(g_0, \Delta(s_0))$ admits an analytic continuation $(g_i, \Delta(\zeta_i))_{0\le i\le n}$ along C satisfying the following:

$$\star \quad \begin{cases} g_i(w) \quad \text{is univalent from} \quad \Delta(\zeta_i) \quad \text{into} \quad D_z, \\ u(g_i(w)) = w \quad \text{for} \quad w \in \Delta(\zeta_i). \end{cases}$$

For $\zeta \in C$ let us denote the subpath of C from s_0 to ζ by C_ζ, and let E be a set of point ζ in C such that $(g_0, \Delta(s_0))$ admits an analytic continuation satisfying $\star$ along C_ζ. Since E includes s_0 and is a relatively open subset of C, if E is closed in C, then $w_0 \in E$. Thus we need to show the closedness of E; actually we show that if $C_\zeta \backslash \{\zeta\}$ is included in E, so is ζ. Take a sequence $\{\zeta_n\}$ in $C_\zeta \backslash \{\zeta\}$ which converges to ζ, and construct a family $\{(g_n, \Delta(\zeta_n))\}$ so that $\{(g_i, \Delta(\zeta_i))\}_{1\le i\le n}$ is the analytic continuation of $(g_0, \Delta(s_0))$ along C_{ζ_n}, satisfying $\star$; $C_\zeta \backslash \{\zeta\}$ may be covered by finite numbers of $\Delta(\zeta_i)$, but even in this case we can construct infinite numbers of $\Delta(\zeta_i)$ given above. If an infinite numbers of the radii of disks $\Delta(\zeta_n)$ are larger than a positive constant, then ζ is in some $\Delta(\zeta_n)$ and hence in E. Therefore, we assume that the sequence of radii of $\Delta(\zeta_n)$ converges to 0. The sequence of $z_n := g_n(\zeta_n)$ is bounded in D_z, because the sequence of $\zeta_n = u(g_n(\zeta_n))$ is bounded. Hence it contains a convergent subsequence $\{z_{n_i}\}$, whose limit we denote by z_0. We prove that z_0 is in D_z by the reduction to absurdity.

Assume that $z_0 = 0$, then from the definition of $u(z)$, $\zeta_{n_i} = u(z_{n_i}) \to 0$; this implies $\zeta = 0$, which contradicts $C_\zeta \subset D_w$: assume that $\arg z_{n_i} \uparrow \pi$, then, because of $\gamma_1 \ge 1$ and $a_1 = 0$, $\lim \arg \zeta_{n_i} = \lim \arg u(z_{n_i}) \ge \pi$; this implies that C_ζ intersect $(-\infty, 0)$, which contradicts $C_\zeta \subset D_w$: similarly assume that $\arg z_{n_i} \downarrow -\pi$, then C_ζ intersect $(-\infty, 0)$, which contradicts $C_\zeta \subset D_w$.

Therefore, z_0 is in D_z. Thus $u(z)$ is continuous at z_0. Hence $u(z_0) = \lim u(z_{n_i}) = \lim \zeta_{n_i} = \zeta$. Since $u'(z_0) \neq 0$, by the inverse mapping theorem, there is a disk $\Delta(\zeta)$ and a holomorphic function g_ζ from $\Delta(\zeta)$ into D_z such that $w = u(g(w))$ for $w \in \Delta(\zeta)$. Since $\zeta_n \to \zeta$ and since the radii of disks $\Delta(\zeta_n)$ diminish to 0, $\Delta(\zeta) \supseteq \Delta(\zeta_n)$ for $n > N$. Therefore $g_\zeta(w) = g_n(w)$ for $n > N$ and for $w \in \Delta(\zeta_n)$. This implies $z_n \to z_0$; in fact, for $n > N$ $z_n = g_n(\zeta_n) = g_\zeta(\zeta_n)$ which converges to $g_\zeta(\zeta) = z_0$.

Let us join $(g_\zeta, \Delta(\zeta))$ to $\{(g_i, \Delta(\zeta_i))\}_{1\le i\le N}$. Then this new family is an analytic continuation of (g_0, s_0) satisfying $\star$. Hence $\zeta \in E$. Thus we have shown that an analytic element (g_0, s_0) has an analytic continuation satisfying $\star$ along every path in D_w. By the monodromy theorem, this analytic continuation is a single valued holomorphic function. We denote it by $g(w)$. Then $g(w)$ is a holomorphic function from D_w into D_z such that

$$u(g(w)) = w \quad (w \in D_w) \quad \text{and} \quad g(s) = u^{-1}(s) \quad (0 < s < \infty).$$

We finally show that $g(w)$ is a Pick function. We denote the open lower half plane by Π_-. Set $\Gamma = \sum_{i=1}^{n} \gamma_i$. Since $g(w)$ is continuous, there is a neighbourhood W of s_0 so that

$$g(W) \subseteq V := \{z : -\pi/\Gamma < \arg z < \pi/\Gamma\},$$

because V is an open set including $t_0 = g(s_0)$. Here we note that

$$u(V \cap \Pi_+) \subset \Pi_+, \quad u(V \cap \Pi_-) \subset \Pi_-, \quad \text{and} \quad u((0,\infty)) = (0,\infty).$$

In fact, take $z \in (V \cap \Pi_+)$; since $0 = a_1 < a_i$ for $i > 1$, $(z + a_i) \in V \cap \Pi_+$, and hence $0 < \arg(\prod_{i=1}^{k}(z + a_i)^{\gamma_i}) < \pi$, which means that $u(V \cap \Pi_+) \subset \Pi_+$; similarly we can see the rest. From these inclusions of sets, it follows that

$$g(W \cap \Pi_+) \subseteq \Pi_+ :$$

in fact, take an arbitrary $w \in W \cap \Pi_+$, then $g(w) \in V$; assume $g(w) \not\in \Pi_+$, then by the above argument, we have $w = u(g(w)) \not\in \Pi_+$; this is a contradiction. From $u((0,\infty)) = (0,\infty)$ and $u(g(w)) = w$ for $w \in D_w$ it follows that $g(\Pi_+) \cap (0,\infty) = \emptyset$. This and the connectedness of $g(\Pi_+)$ in D_z, by taking account of $\emptyset \neq g(W \cap \Pi_+) \subset \Pi_+$, show that $g(\Pi_+) \subseteq \Pi_+$. Hence g is a Pick function. □

For $0 < \alpha < 1$, a function $u(t) = t^{1/\alpha}$ satisfies (4). Hence the above theorem says $u^{-1}(s) = s^{\alpha}$ is operator monotone on $[0,\infty)$: this means (1).
In the above proof we used the condition $\gamma_1 \geq 1$. To see that we can not make this condition weak as $\sum_i r_i \geq 1$, we give

Counter example. Set $u(t) = t^{1/2}(t+1)$. Then $u'(t) = \frac{1}{2}t^{-1/2}(3t+1)$ and $u''(t) = \frac{1}{4}t^{-3/2}(3t-1)$. Therefore

$u''(t) < 0 \quad (0 < t < 1/3)$ hence $(u^{-1})''(s) > 0 \quad (0 < s < 4/27)$.
Since an operator monotone function is concave, this implies that $u^{-1}(s)$ is not operator monotone on $[0, \infty)$.

From now on we describe only result and we omit the detail for the length limit.

Theorem 2.2. *Define a function $v(t)$ by*

$$v(t) = \prod_{j=1}^{l} (t + b_j)^{\lambda_j} \quad (t \geq -b_1), \quad b_1 < b_2 < \cdots < b_l, \quad 0 < \lambda_j. \tag{5}$$

Then, for $u(t)$ represented as (4), if the following conditions

$$\begin{cases} a_1 \leq b_1, \\ \sum_{b_j < t} \lambda_j \leq \sum_{a_i < t} \gamma_i \quad \text{for every } t \in \mathbf{R} \end{cases} \tag{6}$$

are satisfied, a function ϕ defined on $[0, \infty)$ by

$$\phi(u(t)) = v(t) \quad (-a_1 \leq t), \quad \text{that is,} \quad \phi(s) = v(u^{-1}(s)) \quad (0 \leq s)$$

is an operator monotone function on $[0, \infty)$.

3. The further construction of operator monotone functions

This section is continued from the preceding section. We start with a simple lemma.

Lemma 3.1. *Let f_n $(n = 1, 2, \dots)$ be strictly increasing continuous functions on $[a, \infty)$ $(a \in \mathbf{R})$ with $f_n(a) = 0, f_n(\infty) = \infty$, and let $f_n(t) \leq f_{n+1}(t)$ for $t \in [a, \infty)$. If $f_n(t)$ converges pointwise to a strictly increasing continuous function $f(t)$, then $f_n^{-1}(s)$ converges uniformly to $f^{-1}(s)$ on every bounded closed interval $[0, b]$ $(0 < b < \infty)$. Furthermore, if a sequence $\{h_n\}$ of continuous functions on $[0, \infty)$ satisfies $h_n(t) \leq h_{n+1}(t)$ and converges to a continuous function $h(t)$, then $h_n(f_n^{-1}(s))$ converges uniformly to $h(f^{-1}(s))$ on $[0, b]$ as well.*

Theorem 3.2. *Let $u(t), v(t)$ be functions defined by (4), (5). Suppose that condition (6) is satisfied. Then, if $0 \leq \beta \leq \alpha$, a function ϕ on $[0, \infty)$ defined* *by*

$$\phi(u(t)e^{\alpha t}) = v(t)e^{\beta t} \quad (-a_1 \leq t < \infty)$$

is operator monotone on $[0,\infty)$.

By the above theorem we can easily construct a one-parameter family of operator monotone functions.

Corollary 3.3. *Let* $u(t)$, $v(t)$ *be functions given by* (4),(5). *Suppose that condition* (6) *is satisfied and that* $0 \leq \beta \leq \alpha$, $0 \leq c \leq 1$. *Then, for each* $r > 0$ *a function* $\phi_r(s)$ *on* $[0,\infty)$ *defined by*

$$\phi_r(u(t)v(t)^r e^{(\alpha+\beta r)t}) = (v(t)e^{\beta t})^{c+r} \quad (-a_1 \leq t < \infty)$$

is operator monotone.

It is not difficult to derive the next corollary from Lemma 3.1 and Theorem 3.2.

Corollary 3.4. *Suppose that two infinite products*

$$\tilde{u}(t) := \prod_{i=1}^{\infty} (t + a_i)^{\gamma_i} \quad (a_i < a_{i+1}, 1 \leq \gamma_1,\ 0 < \gamma_i)$$

and

$$\tilde{v}(t) := \prod_{j=1}^{\infty} (t + b_j)^{\lambda_j}, \quad (b_j < b_{j+1}, \quad 0 < \lambda_j)$$

are both convergent on $-a_1 \leq t < \infty$. *If condition* (6) *is satisfied and if* $0 \leq \beta \leq \alpha$, *then a function* ϕ *defined by*

$$\phi(\tilde{u}(t)e^{\alpha t}) = \tilde{v}(t)e^{\beta t} \quad (-a_1 \leq t < \infty)$$

is operator monotone on $[0,\infty)$. *Moreover, if* $0 \leq c \leq 1$ *and* $r > 0$, *then a function* $\phi_r(s)$ *on* $[0,\infty)$ *defined by*

$$\phi_r(\tilde{u}(t)\tilde{v}(t)^r e^{(\alpha+\beta r)t}) = (\tilde{v}(t)e^{\beta t})^{c+r} \quad (-a_1 \leq t < \infty)$$

is operator monotone.

4. An essential inequality and an extension of Furuta inequality

The aim of this section is to give an essential inequality which lead us to extensions of (2) and (3), and to extend (2). To do it we need some tools on operator inequality. Now we adopt the notion of the *connection* (or *mean*) that was introduced by **Kubo-Ando** [10]: a connection σ corresponding to an operator monotone function $\phi(t) \geq 0$ on $[0, \infty)$ is defined by

$$A\sigma B = A^{1/2}\phi(A^{-1/2}BA^{-\frac{1}{2}})A^{1/2}$$

if A is invertible, and $A\sigma B = \lim_{t\to+0}(A+t)\sigma B$ if A is not invertible. In this paper we need the following property:

$$A \geq C \text{ and } B \geq D \text{ imply } A\sigma B \geq C\sigma D.$$

From now on, we assume that a function means a *continuous* function, I, J represent intervals (may be unbounded) in the real line, and J^i the interior of J. To make proofs simply in future, we give a remark.

Remark. Suppose that $sp(A) \subseteq [a, b] \subseteq J$, and that f is a function on an interval J. Then for an arbitrary $\epsilon > 0$ there is an affine function $p_\epsilon(t) = ct + d$ such that $c > 0$, $p_\epsilon(a) = a + \epsilon$, $p_\epsilon(b) = b - \epsilon$ and $p_\epsilon(t)$ converges uniformly t on $[a, b]$ as $\epsilon \to 0$. Then we have

$$\|f(p_\epsilon(A)) - f(A)\| \to 0 \quad (\epsilon \to 0), \quad \text{and} \quad sp(p_\epsilon(A)) \subseteq [a+\epsilon, b-\epsilon].$$

Therefore, to show something about $f(A)$ under a condition $sp(A) \subseteq J$ we will often assume that $sp(A)$ is in the interior of J.

Lemma 4.1. *Let $\phi(t) \geq 0$ be an operator monotone function on $[0, \infty)$. Let $k(t)$ be a non-negative and strictly increasing function on an interval $I \subseteq [0, \infty)$. Suppose*

$$\phi(k(t)t) = t^2 \quad (t \in I).$$

Then

$$sp(A), sp(B) \subseteq I, \quad A \geq B \implies \begin{cases} \phi(B^{1/2}k(A)B^{1/2}) \geq B^2, \\ A^2 \geq \phi(A^{1/2}k(B)A^{1/2}). \end{cases}$$

Lemma 4.2. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative functions on $[0,\infty)$, and J an arbitrary interval. Let $f(t), h(t)$ be non-negative strictly increasing functions on J. If, for a fixed real number $c : 0 \leq c \leq 1$, the condition*

$$\phi_r(h(t)f(t)^r) = f(t)^{c+r} \quad (t \in J, r > 0) \tag{7}$$

is satisfied, then

$$\phi_{c+2r}(s\phi_r^{-1}(s)) = s^2 \qquad (s = f(t)^{c+r}).$$

Theorem 4.3. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0,\infty)$, and J an arbitrary interval. Let $f(t), h(t)$ be non-negative strictly increasing functions on J. If condition (7) is satisfied for a fixed $c : 0 \leq c \leq 1$, then*

$$\left.\begin{array}{r} sp(A), sp(B) \subseteq J^i, \\ f(A) \geq f(B) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \phi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^{c+r}, \\ f(A)^{c+r} \geq \phi_r(f(A)^{r/2}h(B)f(A)^{r/2}). \end{array}\right. \tag{8}$$

Proof. We will only show the first inequality of (8). Since $sp(A), sp(B)$ are in the interior of J, $f(A)$ and $f(B)$ are invertible, because $f(t)$ is strictly increasing. We first show (8) in the case of $0 < r \leq 1$. By making use of the connection σ corresponding to ϕ_r, we have

$$\begin{aligned} &f(B)^{-\frac{r}{2}}\phi_r(f(B)^{\frac{r}{2}}h(A)f(B)^{\frac{r}{2}})f(B)^{-\frac{r}{2}} = f(B)^{-r}\sigma h(A) \\ &\geq f(A)^{-r}\sigma h(A) = f(A)^{-r}f(A)^{c+r} = f(A)^c \geq f(B)^c. \end{aligned}$$

Thus (8) follows. We next assume (8) holds for all $r : 0 \leq r \leq n$. Take any $r : n < r \leq n+1$ and fix it. Because of $\frac{r-c}{2} \leq n$, we have

$$\phi_{\frac{r-c}{2}}(f(B)^{\frac{r-c}{4}}h(A)f(B)^{\frac{r-c}{4}}) \geq f(B)^{\frac{r+c}{2}}.$$

Here we simply denote the left hand side by H and the right hand side by K; clearly $H \geq K$. Set $I := \{f(t)^{\frac{r+c}{2}} : t \in J\}$. Then $I \subseteq [0,\infty)$ and $sp(K) \subseteq I$. To see $sp(H) \subseteq I$, take a, b in J such that $a \leq A, B \leq b$. Since $h(a) \leq h(A) \leq h(b)$,

$$h(a)f(a)^{\frac{r-c}{2}} \leq f(B)^{\frac{r-c}{4}}h(A)f(B)^{\frac{r-c}{4}} \leq h(b)f(b)^{\frac{r-c}{2}}.$$

In conjunction with (7), this shows $sp(H) \subseteq I$. It follows from Lemma 4.2 that

$$\phi_r(s\phi^{-1}_{\frac{r-c}{2}}(s)) = s^2 \quad \text{for } s \in I.$$

Thus we can apply Lemma 4.1 to get

$$\phi_r(K^{1/2}\phi^{-1}_{\frac{r-c}{2}}(H)K^{1/2}) \geq K^2,$$

which means

$$\phi_r(f(B)^{\frac{r}{2}}h(A)f(B)^{\frac{r}{2}}) \geq f(B)^{c+r}. \qquad \square$$

Theorem 4.4. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$, and J an arbitrary interval. Let $f(t), h(t)$ be non-negative strictly increasing functions on J. If $f(t)$ is operator monotone, and if condition (7) is satisfied for a fixed $c : 0 \leq c \leq 1$, then*

$$\left.\begin{array}{r} sp(A), sp(B) \subseteq J, \\ A \geq B \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \phi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^{c+r}, \\ f(A)^{c+r} \geq \phi_r(f(A)^{r/2}h(B)f(A)^{r/2}). \end{array}\right. \tag{9}$$

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We explain that the above theorem includes Furuta Inequality.
Let $p \geq 1$, and put

$$f(t) = t, \quad h(t) = t^p \quad (0 \leq t < \infty).$$

Define a one-parameter family of operator monotone functions $\{\phi_r : r > 0\}$ by

$$\phi_r(t) = t^{\frac{1+r}{p+r}} \quad (0 \leq t < \infty).$$

Then

$$\phi_r(h(t)f(t)^r) = t^{1+r} = f(t)^{1+r}.$$

Thus (7) with $c = 1$ and other required conditions in Theorem 4.4 is satisfied. Therefore, from Theorem 4.4 it follows that

$$A \geq B \geq 0 \Longrightarrow (B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq B^{1+r}.$$

If $q(1+r) \geq p+r$, take λ such that

$$\frac{1}{q} = \lambda \frac{1+r}{p+r}.$$

Then $0 < \lambda \leq 1$, hence by Löwner-Heinz inequality (1) we have

$$(B^{r/2} A^p B^{r/2})^{1/q} \geq B^{\frac{p+r}{q}}.$$

This is just the Furuta inequality.

Remark. In the above theorems, we assumed that condition (7) is satisfied for all $r > 0$. However, it is evident that if we assume that (7) is satisfied for r in an interval $(0, \alpha)$, then (8) and (9) hold for $r \in (0, \alpha)$.

(8) and (9) are abstract inequalities, however we can get concrete inequalities by using one-parameter families of non-negative operator monotone functions on $[0, \infty)$ in Corollary 3.3.

Corollary 4.5. *Under the condition of Corollary* 3.3, *suppose* $A, B \geq -a_1$. *Then*

$$v(A)e^{\beta A} \geq v(B)e^{\beta B} \Rightarrow \phi_r((v(B)e^{\beta B})^{r/2} u(A)e^{aA}(v(B)e^{\beta B})^{r/2}) \geq (v(B)e^{\beta B})^{c+r}.$$

Corollary 4.6. *Let* $u(t)$, $v(t)$ *be functions given by* (4),(5). *Let us assume that* $a_1 \leq b_1$ *and* $\sum \lambda_j < 1$. *For fixed* α, c: $0 \leq \alpha$, $0 \leq c \leq 1$, *define a function* $\phi_r(s)$ *on* $[0, \infty)$ *by*

$$\phi_r(u(t)v(t)^r e^{\alpha t}) = v(t)^{c+r} \quad (r > 0).$$

Then

$$A \geq B \geq -a_1 \implies \phi_r(v(B)^{\frac{r}{2}} u(A)e^{aA} v(B)^{\frac{r}{2}}) \geq v(B)^{c+r}.$$

5. Extensions of exponential type operator inequality by Ando

Let us remember the inequality (3): for $p \geq 0$, $r \geq s > 0$

$$A \geq B \implies (e^{\frac{r}{2}B} e^{pA} e^{\frac{r}{2}B})^{\frac{s}{r+p}} \geq e^{sB}.$$

In this section we will extend this. We consider (7) under the condition of $c = 0$, and denote the function by φ_r instead of ϕ_r. In addition to the conditions of Theorem 4.3 we assume that $\log f(t)$ is operator monotone. Then we have

Theorem 5.1. *Let $f(t)$ and $h(t)$ be non-negative strictly increasing functions on an interval J, and let $\{\varphi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$ satisfying*

$$\varphi_r(h(t)f(t)^r) = f(t)^r \quad (t \in J; r > 0). \tag{10}$$

If $\log f(t)$ is a non-constant operator monotone function in the interior of J, then

$$\left.\begin{array}{r} sp(A), sp(B) \subseteq J, \\ A \geq B \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \varphi_r(f(B)^{r/2}h(A)f(B)^{r/2}) \geq f(B)^r \\ f(A)^r \geq \varphi_r(f(A)^{r/2}h(B)f(A)^{r/2}). \end{array}\right. \tag{11}$$

Now we explain that this theorem is an extension of (3). For $p, r > 0$, put $\varphi_r(s) = s^{r/(p+r)}$ for $s \geq 0$, $f(t) = e^t$ and $h(t) = e^{pt}$ for $t \in J := (-\infty, \infty)$. Then (10) and all other conditions of Theorem 5.1 are satisfied. Thus $A \geq B$ implies

$$(e^{\frac{r}{2}B}e^{pA}e^{\frac{r}{2}B})^{\frac{r}{r+p}} \geq e^{rB}.$$

By Löwner-Heinz theorem, we get (3).

Since $\varphi_r(s) = s^{r/(p+r)}$ $(p, r > 0)$ is operator monotone on $[0, \infty)$ and satisfies $\varphi_r(f(t)^p f(t)^r) = f(t)^r$ for every function $f(t)$, we can obtain

Corollary 5.2. *Let $0 \leq f(t)$ be a strictly increasing function on an interval J, and let $sp(A), sp(B) \subseteq J$. If $\log f(t)$ is an operator monotone function in the interior of J, then for $r > 0$, $p > 0$*

$$A \geq B \Longrightarrow \left\{\begin{array}{l} (f(B)^{\frac{r}{2}}f(A)^p f(B)^{\frac{r}{2}})^{\frac{r}{p+r}} \geq f(B)^r \\ f(A)^r \geq (f(A)^{\frac{r}{2}}f(B)^p f(A)^{\frac{r}{2}})^{\frac{r}{p+r}}. \end{array}\right.$$

Corollary 5.3. *If $\alpha, p, r > 0$, then*

$$A \geq B \geq -a_1 \Rightarrow \left\{\begin{array}{l} [(u(B)e^{\alpha B})^{\frac{r}{2}}\ (u(A)e^{\alpha A})^p\ (u(B)e^{\alpha B})^{\frac{r}{2}}]^{\frac{r}{p+r}} \geq (u(B)e^{\alpha B})^r, \\ (u(A)e^{\alpha A})^r \geq [(u(A)e^{\alpha A})^{\frac{r}{2}}\ (u(B)e^{\alpha B})^p\ (u(A)e^{\alpha A})^{\frac{r}{2}}]^{\frac{r}{p+r}}. \end{array}\right.$$

By applying this inequality to $u(t) = 1$, we can get (3) again. We end this paper with a slightly complicated inequality:

Corollary 5.4. *Let $u(t), v(t)$ be functions defined by* (4), (5), *and let $a_1 \leq b_1$. For fixed $\alpha, \beta \geq 0$, define $\varphi_r(s)$ $(r > 0)$ on $[0, \infty)$ by*

$$\varphi_r(u(t)v(t)^r e^{(\alpha+\beta r)t}) = v(t)^r e^{\beta r t} \quad (t \geq -a_1).$$

Then, for each $r > 0$ $\varphi_r(s)$ is operator monotone and

$$A \geq B \geq -a_1 \Rightarrow \begin{cases} \varphi_r((v(B)e^{\beta B})^{\frac{r}{2}} \, (u(A)e^{\alpha A}) \, (v(B)e^{\beta B})^{\frac{r}{2}}) \geq (v(B)e^{\beta B})^r, \\ (v(A)e^{\beta A})^r \geq \varphi_r((v(A)e^{\beta A})^{\frac{r}{2}} \, (u(B)e^{\alpha B}) \, (v(A)e^{\beta A})^{\frac{r}{2}}). \end{cases}$$

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References

1. T. Ando, *Topics on operator inequalities*, Lecture note, Sapporo, 1978.

2. T. Ando, On some operator inequalities, Math. Ann. 279 (1987), 157–159.

3. R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York (1996).

4. N. Chan, M. Kwong, Hermitian matrix inequalities and a conjecture, Amer. Math. Monthly 92(1985), 533–541.

5. W. Donoghue, *Monotone matrix functions and analytic continuation*, Springer, 1974.

6. M. Fujii, E. Kamei, Furuta's inequality and a generalization of Ando's theorem, Proc. A. M. S. 115(1992), 409– 413.

7. T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. A.M.S. 101(1987), 85–88.

8. T. Furuta, An elementary proof of an order preserving inequality, Proc. Japan Acad. 65 ser. A (1989), 126.

9. R. Horn, C. Johnson, *Topics in matrix analysis*, Cambridge Univ. Press, 1991.

10. F. Kubo, T. Ando, Means of positive linear operators, Math. Ann. 246(1980),205–224.

11. F. Hansen, G. K. Pedersen, Jensen's inequality for operators and Löwner's theorem. Math. Ann. 258(1982), 229 - 241

12. E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann. 123(1951), 415-438.

13. K. Löwner, Über monotone Matrixfunktionen, Math. Z. 38(1934), 177-216.

14. M. Rosenblum, J. Rovnyak, *Hardy classes and operator theory,* Oxford University Press (1985).

15. K. Tanahashi, Best possibility of the Furuta inequality, Proc. A. M. S. 124(1996),141–146.

16. M. Uchiyama, Commutativity of selfadjoint opeators, Pacific J. Math. 161(1993),385–392.

17. M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality, Proc. A. M. S. to appear.

18. M. Uchiyama, Some exponential operator inequalities, Math. Inequal. Appl. 2(1999), 469–471.