Ground state measure and its applications

Fumio Hiroshima* (北大理学 鳥住雄)

1 Introduction

In this paper we shall consider structures of ground states of a model describing an interaction between a particle and a quantized scalar boson field, which is called the "Nelson model"[15],[18]. Basic ideas in this paper is due to a fairly nice work of H.Spohn [22], in which he studies the spin-boson model. The Hamiltonian, $H$, of the Nelson model is defined as a self-adjoint operator acting on Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}$, where $\mathcal{F}$ denotes a Boson Fock space. The existence of the ground states, $\Psi_g$, of $H$ is established in e.g., [2],[4],[12],[23]. The main results presented here is to give the expectation-value of the number of bosons of $\Psi_g$ and its boson distribution by means of a ground state measure constructed in this paper. Especially the localization of bosons of $\Psi_g$ is proved. The ground state measure, $\mu$, on the set of paths, $\Omega$, gives an integral representation of the expectation-value of certain operator $A$ in $\mathcal{H}$, i.e.,

$$(\Psi_g, A\Psi_g) = \int_{\Omega} f_A(q) \mu(dq),$$

where $f_A$ is a density function corresponding to $A$. This integral representation leads us to the goal of this paper. Detailed arguments shall be published elsewhere [2], and refer to see [17],[21],[22]. This paper is organized as follows: section 2 gives a definition of models considered in this paper. In section 3 we review the second quantizations. Section 4 is devoted to investigating the ground states. In section 5 we give further problems on the Pauli-Fierz model in nonrelativistic quantum electrodynamics.

*I thank Japan Society for the Promotion of Science for the financial support.
2 Scalar quantum field models

Let $\mathcal{F} := \oplus_{n=0}^{\infty} \otimes_{s}^{n} L^2(\mathbb{R}^d) := \oplus_{n=0}^{\infty} \mathbb{C}$, where $\otimes_{s}^{n}$ denotes the $n$-fold symmetric tensor product with $\otimes_{s}^{0} L^2(\mathbb{R}^d) := \mathbb{C}$. The bare vacuum, $\Omega \in \mathcal{F}$, is defined by $\Omega := \{1,0,0,\ldots\}$. Let $a(f)$ and $a(g)$ be the creation operator and the annihilation operator smeared by $f, g \in L^2(\mathbb{R}^d)$, respectively, which are linear in $f$ and $g$. Let $\mathcal{F}_{\text{fin}}$ be the finite particle subspace of $\mathcal{F}$:

$\mathcal{F}_{\text{fin}} := \{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F} | \text{there exists } n_0 \text{ such that } \Psi^{(m)} = 0, m \geq n_0 \}$

They satisfy canonical commutation relations (CCR), i.e.,

$[a(f), a^\dagger(g)] = (f, g)_{L^2(\mathbb{R}^d)}$, \quad $[a^\dagger(f), a^\dagger(g)] = 0$,

on $\mathcal{F}_{\text{fin}}$, where $a^\dagger$ denotes $a$ or $a^\dagger$, and $(\cdot, \cdot)_{\mathcal{K}}$ the scalar product on Hilbert space $\mathcal{K}$. We denote by $\|\cdot\|_{\mathcal{K}}$ its associated norm. Unless confusion arises we omit $\mathcal{K}$ in $(\cdot, \cdot)_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$, respectively. $a^\dagger$ also satisfies that $(\Psi, a(f)\Phi) = (a^\dagger(f)\Psi, \Phi)$ for $\Psi, \Phi \in \mathcal{F}_{\text{fin}}$. For dense subset $\mathcal{K} \subset L^2(\mathbb{R}^d)$,

$\mathcal{F}(\mathcal{K}) := l.h.\{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega | f_j \in \mathcal{K}, j = 1, \ldots, n, n \in \mathbb{N}\}$

is dense in $\mathcal{F}$. We define the free Hamiltonian, $H_f$, in $\mathcal{F}$ by

$H_f \Omega := 0,$

$H_f a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega := \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(\omega f_j) \cdots a^\dagger(f_n)\Omega,$

$f_j \in D(\omega), \quad j = 1, \ldots, n, \quad n \in \mathbb{N},$

where $D(T)$ denotes the domain of $T$, $\omega := \omega(k) := \sqrt{|k|^2 + m^2}$, $m \geq 0$. Here $m$ denotes the mass of the quantized scalar boson field. Field operators $\phi(f)$ are defined by

$\phi(f) := \frac{1}{\sqrt{2}} (a^\dagger(f) + a(f)), \quad f \in L^2(\mathbb{R}^d).$

Note that $H_f[\mathcal{F}(D(\omega))]$ and $\phi(f)[\mathcal{F}_{\text{fin}}]$ are essentially self-adjoint, respectively.

It is known that $\sigma(H_f) = [0, \infty)$ and $\sigma_p(H_f) = \{0\}$. The Hamiltonian, $H$, considered in this paper is defined by

$H := H_p \otimes 1 + 1 \otimes H_f + \alpha H_I$
on $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}; \mathcal{F}d)$, where $\alpha \in \mathbb{R}$ is a coupling constant, and

$$H_1 := \phi(e^{ikz} \hat{\lambda}),$$

$$H_p := -\Delta/2 + V,$$

where $\hat{\lambda}$ is the Fourier transform of $\lambda$. A reasonable physical choice of $\hat{\lambda}$ is of the form

$$\hat{\lambda} = \frac{\hat{\rho}}{\sqrt{(2\pi)^d \omega}},$$

where $\rho$ describes a charge distribution, i.e.,

$$\sqrt{(2\pi)^d} \hat{\rho}(0) = \int_{\mathbb{R}^d} \rho(x) dx = \alpha.$$

For simplicity we assume that external potential $V = V_+ - V_-$ satisfies that $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and that $V_-$ is infinitesimally small with respect to $\Delta$ in the sense of form. Throughout this paper we assume that

$$\overline{\lambda(k)} = \hat{\lambda}(-k).$$

Let $\lambda, \hat{\lambda}/\sqrt{\omega}, \hat{\lambda}/\omega, \hat{\lambda} \in L^2(\mathbb{R}^d)$. Then it is known that, for arbitrary $\alpha$, $H$ is self-adjoint on $D(H_p \otimes 1) \cap D(1 \otimes H_f)$ and bounded from below. Moreover it is essentially self-adjoint on any core of $H_p \otimes 1 + 1 \otimes H_f$.

**Proposition 2.1 ([2],[12])** Let $\lambda/\omega, \hat{\lambda}/\sqrt{\omega}, \hat{\lambda} \in L^2(\mathbb{R}^d)$. Then there exists $\alpha_*$ such that for $|\alpha| \leq \alpha_*$ the ground states, $\Psi_g$, of $H$ exist. Moreover $(f \otimes \Omega, \Psi_g) > 0$ for arbitrary nonnegative $f \in L^2(\mathbb{R}^d)$ with $f \not\equiv 0$.

See Figure 2 for more explicit results on the existence of the ground states of $H$.

### 3 Second quantizations

For later use we review the second quantization of operator $T$ on $L^2(\mathbb{R}^d)$. Let $T$ be a contraction operator on $L^2(\mathbb{R}^d)$, i.e., $\|T\| \leq 1$. Then we define $\Gamma(T) : \mathcal{F}_\text{fin} \to \mathcal{F}_\text{fin}$ by

$$\Gamma(T)\Omega := \Omega,$$
\[ \Gamma(T)a^\uparrow(f_1)\cdots a^\uparrow(f_n)\Omega := a^\dagger(Tf_1)\cdots a^\dagger(Tf_n)\Omega, \]
\[ f_j \in L^2(\mathbb{R}^d), \quad j = 1, \ldots, n, \quad n \in \mathbb{N}. \]

For \( \Phi \in \mathcal{F}_{\text{fin}} \) we have \( \|\Gamma(T)\Phi\| \leq \|\Phi\| \). Thus \( \Gamma(T) \) extends to a contraction operator on \( \mathcal{F} \). We denote its extension by the same symbol. It is seen that \( \Gamma(\cdot) \) is linear in \( \cdot \) and that \( \Gamma(T)^* = \Gamma(T^*) \). Let \( h \) be a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^d) \). Then we see that \( \Gamma(e^{-th}) \) is a strongly continuous symmetric contraction one-parameter semigroup in \( t \geq 0 \). The second quantization of \( h \), \( d\Gamma(h) \), is defined by the generator of \( \Gamma(e^{-th}) \), i.e.,

\[ \Gamma(e^{-th}) = e^{-td\Gamma(h)}, \quad t \geq 0. \]

Actually \( H_t \) is the second quantization of multiplication operator \( \omega \). For nonnegative multiplication operator \( h \) in \( L^2(\mathbb{R}^d) \), formally, it is written as

\[ d\Gamma(h) = \int h(k)a^\dagger(k)a(k)dk. \tag{3.1} \]

The number operator, \( N \), in \( \mathcal{F} \) is defined by the second quantization of the identity operator in \( L^2(\mathbb{R}^d) \), i.e.,

\[ D(N) := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} \left| \sum_{n=0}^\infty n^2 \|\Psi^{(n)}\|_{\mathcal{F}_n}^2 < \infty \right. \right\}, \]

\[ (N\Psi)^{(n)} := n\Psi^{(n)}. \]

Let \( h \) be a multiplication operator in \( L^2(\mathbb{R}^d) \) such that \( s = s_{R+} - s_{R-} + i(s_{I+} - s_{I-}) \), where \( s_{R+} \) (resp. \( s_{R-}, s_{I+}, s_{I-} \)) denotes the real positive (resp. real nonpositive, imaginary positive, imaginary nonpositive) part of \( s \). Then we define

\[ d\Gamma(h) := d\Gamma(s_{R+}) - d\Gamma(s_{R-}) + i(d\Gamma(h_{I+}) - d\Gamma(h_{I-})), \]

\[ D(d\Gamma(h)) := D(d\Gamma(s_{R+})) \cap D(d\Gamma(s_{R-})) \cap D(d\Gamma(h_{I+})) \cap D(d\Gamma(h_{I-})). \]

### 4 Ground state measures

Let \( \Omega := (\mathbb{R}^d)^{(-\infty,\infty)} \) be the set of \( \mathbb{R}^d \)-valued paths and \( \mathcal{B}(\Omega) \) the \( \sigma \)-field constructed by cylinder sets. For \( T : \mathcal{H} \to \mathcal{H} \), we define

\[ \langle T \rangle := (\Psi_g, T\Psi_g)_{\mathcal{H}}. \]
For a convenience we denote by $\langle S \rangle$ for $\langle 1 \otimes S \rangle$, for $S : \mathcal{F} \to \mathcal{F}$. Our fundamental theorem is as follows:

**Theorem 4.1 ([2])** Let $s$ be such that $\sup_{k \in \mathbb{R}^d} |s(k)| < \infty$. Let $\hat{\lambda}/\omega$, $\hat{\lambda}/\sqrt{\omega}$, $\hat{\lambda} \in L^2(\mathbb{R}^d)$, and $|\alpha| \leq \alpha_*$. We assume that $A_1, \ldots, A_m$ are measurable sets in $\mathbb{R}^d$ and let $1_A$ denote the characteristic function of $A$. Then there exists a probability measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$ such that, for $t_1 \leq \cdots \leq t_m$,

$$
\langle 1_{A_1} e^{-(t_2-t_1)H} 1_{A_2} \cdots e^{-(t_m-t_{m-1})H} 1_{A_m} \rangle = \int_{\Omega} 1_{A_1}(q(t_1)) \cdots 1_{A_m}(q(t_m)) \mu(dq),
$$

where

$$
Z(\beta) := \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \int_{\mathbb{R}^d} |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} (e^{-\beta s(k)} - 1) e^{ik(q(t)-q(s))} dk.
$$

We give a remark on $Z(\beta)$. Since $\|\hat{\lambda}/\omega\| < \infty$, we see that

$$
|Z(\beta)| \leq 2 \|\hat{\lambda}/\omega\|^2 < \infty
$$

uniformly in paths $q \in \Omega$. Thus $Z(\beta)$ is well defined. It is proved in [2] that $\mu$ is a Gibbs measure. We call $\mu$ the "ground state measure for $H". It is easily seen that the right-hand side of (4.1) is analytically continued to $\beta \in \mathbb{C}$. Although it does not imply that $\langle e^{-\beta \mathcal{D}(s)} \rangle$ is well defined for all $\beta \in \mathbb{C}$, we have the following theorem:

**Theorem 4.2 ([2])** Let $s$, $\hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then we have $\Psi_g \in D(1 \otimes e^{-\beta \mathcal{D}(s)})$ for all $\beta \in \mathbb{C}$, and (4.1) holds true for all $\beta \in \mathbb{C}$.

We immediately have the following corollary.

**Corollary 4.3** Let $\hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then, for arbitrary $\epsilon \in \mathbb{R}$, we have $\Psi_g \in D(1 \otimes e^{\epsilon N})$. Moreover

$$
\langle N \rangle = \frac{\alpha^2}{2} \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \int_{\mathbb{R}^d} dk |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} \int_{\Omega} e^{ik(q(t)-q(s))} \mu(dq).
$$

(4.2)
Proof: Putting $s = 1$ in Theorem 4.2, we get $\Psi_g \in D(1 \otimes e^{\epsilon N})$ for all $\epsilon \in \mathbb{R}$. (4.2) follows from (4.1) and
\[
\langle N \rangle = -\frac{d(e^{-\beta N})}{d\beta} \Bigg|_{\beta=0}.
\]
The proof is complete. Q.E.D.

Corollary 4.3 implies that
\[
\sum_{n=0}^{\infty} e^{2\epsilon n} \| \Psi_g^{(n)} \|_{L^2(\mathbb{R}^d) \otimes F_n}^2 < \infty, \quad \text{for all } \epsilon > 0.
\]
Hence we conclude that $\| \Psi_g^{(n)} \|$ decays super-exponentially as $n \to \infty$; it decays faster than $e^{-\epsilon n}$ for arbitrary $\epsilon > 0$. Let $s \in C_0^\infty(\mathbb{R}^d)$. Then, by Theorem 4.2, we see that $\Psi_g \in D(d\Gamma(s))$ and
\[
|\langle d\Gamma(s) \rangle| \leq (\alpha^2/2) \| s \|_{\infty} \| \hat{\lambda}/\omega \|_{2}^2.
\]
Thus map
\[
D : C_0^\infty(\mathbb{R}^d) \ni s \to \langle d\Gamma(s) \rangle \in \mathbb{C}
\]
defines a distribution on $C_0^\infty(\mathbb{R}^d)$. Taking into account of the formal expression of $d\Gamma(s)$ (3.1), we denote by $\langle a^\dagger(k)a(k) \rangle$ the kernel of $D$. From Corollary 4.3 it immediately follows:

**Corollary 4.4** Let $\hat{\lambda}$ and $\alpha$ be in Theorem 4.1. Then for a.e. $k \in \mathbb{R}^d$,
\[
\langle a^\dagger(k)a(k) \rangle = \frac{\alpha^2}{2} |\hat{\lambda}(k)|^2 \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \ e^{-|t-s|\omega(k)} \int_{\Omega} e^{ik(q(t)-q(s))} \mu(dq).
\]
Note that
\[
\int_{\mathbb{R}^d} \langle a^\dagger(k)a(k) \rangle dk = \langle N \rangle.
\]
Moreover we see that
\[
|\langle a^\dagger(k)a(k) \rangle| \leq \frac{\alpha^2 |\hat{\lambda}(k)|^2}{2 \omega(k)^2}, \quad \text{a.e. } k \in \mathbb{R}^d.
\]
See Figure 1.
5 Nonrelativistic QED

5.1 The Pauli-Fierz model

The Pauli-Fierz model [1],[3],[5]-[11],[19],[20] in nonrelativistic QED describes an interaction of particles (electrons) and a quantized radiation field (photons). The quantized radiation field is quantized in a Coulomb gage. We assume that the number of the electrons is one and that the electron has spineless. Let

\[ \mathcal{F}_{PF} := \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} \frac{L^{2}(\mathbb{R}^{d}) \oplus \cdots \oplus L^{2}(\mathbb{R}^{d})}{d-1} \cong \frac{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}{d-1}. \]

Let \( \{b^{r}(f), b^\dagger r(g)\}_{r=1}^{d-1} \) be the annihilation operators and the creation operators, respectively, which satisfy CCR:

\[ [b^{r}(f), b^{\dagger s}(g)] = \delta_{rs}(\overline{f}, g)_{L^{2}(\mathbb{R}^{d})}, \quad [b^{\dagger r}(f), b^{s}(g)] = 0. \]

Let \( H_{f}^{PF} \) be the free Hamiltonian in \( \mathcal{F}_{PF} \), i.e.,

\[ H_{f}^{PF} := \sum_{r=1}^{d-1} \int \omega(k) b^{\dagger r}(k)b^{r}(k)dk. \]
The Hamiltonian of the Pauli-Fierz model is defined as an operator in 
\[ \mathcal{H}_{\text{PF}} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\text{PF}} \cong L^2(\mathbb{R}^d; \mathcal{F}_{\text{PF}}) \]
and reads
\[ H_{\text{PF}} := \frac{1}{2} (-i\nabla \otimes 1 - e\mathbf{A}(x))^2 + 1 \otimes H^\text{PF}_f + V \otimes 1, \]
where \( e \) is a coupling constant, \( \mathbf{A}(x) := (A_1(x), \cdots, A_d(x)) \),
\[ A_\mu(x) := \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} (b_r^\dagger e_\mu \lambda_r e^{-ikx} + br e_\mu^r \lambda_r e^{ikx}), \]
and \( e_r := (e_1^r, \cdots, e_d^r) \), polarization vectors; \( e^r(k) \cdot e^s(k) = \delta_{rs} \) and \( e^r(k) \cdot k = 0 \). Note that
\[ \text{div} \mathbf{A} = 0. \]

For the Nelson model, the self-adjointness of \( H \) for arbitrary \( \alpha \) is trivial, since \( H_1 \) is infinitesimally small with respect to \( H_\text{p} \otimes 1 + 1 \otimes H_f \). It is not so easy to show self-adjointness of \( H_{\text{PF}} \) for arbitrary \( e \in \mathbb{R} \). Let \( N_{\text{PF}} \) be the number operator in \( \mathcal{F}_{\text{PF}} \). We have the following proposition:

**Proposition 5.1** ([9]) \(^1\) Let \( \hat{\lambda}, \omega^2 \hat{\lambda} \in L^2(\mathbb{R}^d) \). We assume that \( V \) is relatively bounded with respect to \( \Delta \). Then, for arbitrary \( e \in \mathbb{R} \), \( H_{\text{PF}} \) is essentially self-adjoint on
\[ D(\Delta \otimes 1) \cap D(1 \otimes (H^\text{PF}_f)^2) \cap_{k=1}^\infty D(1 \otimes N_{\text{PF}}^k). \]

The existence of the ground states of \( H_{\text{PF}} \) are studied in [1],[6], and their multiplicities in [7],[11]. Moreover \( \inf \sigma(H_{\text{PF}}) \) is investigated in [3],[16].

### 5.2 Ground states of \( H \) and \( H_{\text{PF}} \)

Let
\[ \text{gap}(T) := \inf \sigma_{\text{ess}}(T) - \inf \sigma(T). \]

The existence of the ground states of \( H \) and \( H_{\text{PF}} \) are deeply related to conditions on \( m \), \( \text{gap} \), \( \hat{\lambda} \) and coupling constants. Let \( \hat{\lambda}/\omega \in L^2(\mathbb{R}^d) \). Then sufficient conditions for the existence of the ground states of \( H \) and \( H_{\text{PF}} \), as far as we know, are in Figures 2 and 3, respectively.
Note that see [4],[23] for a proof of the existence of ground states for case \( \text{gap}(H) = \infty \) and \( m \geq 0 \) in Figure 2, and [8],[9] for case \( \text{gap}(H_{\text{PF}}) = \infty \) and \( m > 0 \) in Figure 3. In [13],[14] the authors give examples such that the ground states of \( H \) and \( H_{\text{PF}} \) exist for the case where \( \text{gap}(H) = 0 \) and \( \text{gap}(H_{\text{PF}}) = 0 \), respectively. In [17] no existence of the ground states of \( H \) for arbitrary \( \alpha \neq 0 \) is proved if \( ||\hat{\lambda}/\omega|| = \infty \).

### 5.3 Distribution of bosons for \( \Psi_{\text{PF}} \)

Let \( \Psi_{\text{PF}} \) be the ground state of \( H_{\text{PF}} \) and

\[
\langle T \rangle_{\text{PF}} := (\Psi_{\text{PF}}, T \Psi_{\text{PF}})_{\mathcal{H}_{\text{PF}}}.
\]

\(^1\)In [9] essential self-adjointness of \( H_{\text{PF}} \) is proved only for the case where the number of the electrons is one. As far as we know it is not clear whether the statement in Proposition 5.1 with \( N \)-electrons holds true or not. In [19] self-adjointness of \( H_{\text{PF}} \) on \( D(\Delta \otimes 1) \cap D(1 \otimes H_{\text{PF}}^{\text{PF}}) \) is proved for sufficiently small \(|e|\).

\(^2\)It is not necessarily to assume \( \hat{\lambda}/\omega \in L^{2}(\mathbb{R}^{d}) \) for \( H_{\text{PF}} \). See [1].
Our next problem is to study the distribution of bosons of $\Psi_{PF}$, e.g., $\langle N_{PF}\rangle_{PF}$, $\langle e^{-\beta N_{PF}}\rangle_{PF}$, etc. In [10] a ground state measure, $\mu_{PF}$, on $(\Omega, B(\Omega))$ for $H_{PF}$ is constructed, which satisfies

$$\langle 1_{A_1} e^{-(t_2-t_1)H_{PF}} 1_{A_2} \cdots e^{-(t_m-t_{m-1})H_{PF}} 1_{A_m} \rangle_{PF} = \int_{\Omega} 1_{A_1}(q(t_1)) \cdots 1_{A_m}(q(t_m)) \mu_{PF}(dq).$$

Moreover a "formal" calculation gives a "formal" expression [5],[21]:

$$\langle e^{-\beta N_{PF}}\rangle_{PF} = \int_{\Omega} e^{(-e^2/2)Z_{PF}(\beta)} \mu_{PF}(dq),$$

where

$$Z_{PF}(\beta) := (e^{-\beta} - 1) \sum_{\mu,\nu=1}^{d} \int_{-\infty}^{0} dq_{\mu}(t) \int_{0}^{\infty} dq_{\nu}(s) \times \int_{\mathbb{R}^d} d\mu\nu(k) |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} e^{ik(q(t)-q(s))} dk.$$ 

Here $d\mu\nu(k) := \sum_{\tau=1}^{d} e_{\mu}^{\tau}(k) e_{\nu}^{\tau}(k)$ and $\int \cdots dq_{\mu}(t)$ denotes a stochastic integral. For the Nelson model $|Z(\beta)| \leq 2 \|\hat{\lambda}/\omega\|^2 < \infty$ guarantees that $\int_{\Omega} e^{(\alpha^2/2)Z(\beta)} \mu(dq)$ is well defined. We do not have such an estimate for $Z_{PF}(\beta)$, which is a crucial points to study $\langle N_{PF}\rangle_{PF}$ in terms of the ground state measure. Actually the definition of $Z_{PF}(\beta)$ is not clear, e.g., it is needed to give a rigorous definition of $\int_{-\infty}^{0} dq_{\mu}(t) \int_{0}^{\infty} dq_{\nu}(s)$.

### 5.4 Conjectures and problems

In view of subsections 5.1-5.3, we give the following conjectures. We assume some conditions on $\hat{\lambda}$ and $V$.

**Conjecture 5.2** For arbitrary $e \in \mathbb{R}$, $H_{PF}$ is self-adjoint and bounded from below on $D(\Delta \otimes 1) \cap D(1 \otimes H_{1}^{PF})$.

**Conjecture 5.3** Let $\text{gap}(H_{PF}) = \infty$ and $m \geq 0$. Then the ground states of $H_{PF}$ exist for arbitrary $e \in \mathbb{R}$.

**Conjecture 5.4** $\Psi_{PF} \in D(1 \otimes e^{\epsilon N_{PF}})$ for all $\epsilon \in \mathbb{R}$. 
References


