LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential \( \mathbf{a} = (a_j(x))_{j=1, \ldots, d} \) is expressed by the following form

\[
H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m\gamma_{d+1} + V, \tag{1.1}
\]

where \( P_j = \frac{i}{\hbar} \partial_{x_j}, \) \( V \) is a multiplication of an Hermitian matrix \( V(x) \). \( m \) is the mass of electron. The matrices \( \{\gamma_j\} \) satisfy the following relations

\[
\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} \mathbf{1} \quad (j, k = 1, \ldots, d+1). \tag{1.2}
\]

Here \( \delta_{jk} \) is Kronecker’s delta and \( \mathbf{1} \) is an identity matrix. We assume that the speed of the light \( c = 1 \). When \( V \equiv 0 \), the square of \( H(\mathbf{a}) \) has the form

\[
H(\mathbf{a})^2 = \sum_{j=1}^{d} (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k, \tag{1.3}
\]

where

\[
b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x). \tag{1.4}
\]

It is called Pauli’s Hamiltonian. The skew symmetric matrix \( (b_{jk}(x)) \) is the magnetic field associated with \( \mathbf{a} \). We say the magnetic field is asymptotically constant if it satisfies the following conditions as \( |x| \to \infty \):

\[
b_{jk}(x) \to 3\Lambda_{jk} \quad (1 \leq j, k \leq d), \tag{1.5}
\]

where \( (\Lambda_{jk})_{j,k} \) is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for \( H(\mathbf{a}) \) with a constant magnetic field \( (b_{jk}(x)) \) and a long-range electric potential \( V(x) \) when \( d = 3 \). Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for \( d = 2, 3 \). As can be inferred from (1.3), the spectrum of \( H(\mathbf{a}) \) is closely related
with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1 (P_1 + \frac{\lambda}{2}x_2) + \sigma_2 (P_2 - \frac{\lambda}{2}x_1) + m\sigma_3,$$  \hspace{1cm} (1.6)

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + m^2 - \lambda \sigma_3.$$  \hspace{1cm} (1.7)

The right hand side is a decoupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2 \ldots \}.$$ 

In fact we have

$$\sigma(h(\lambda)) = \{ \sqrt{2\lambda n + m^2}, \ -\sqrt{2\lambda (n+1) + m^2} \mid n = 0, 1, 2 \ldots \}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along $x_3$-axis:

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1 (P_1 + \frac{\lambda x_2}{2}) + \alpha_2 (P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m\beta,$$  \hspace{1cm} (1.8)

where $\{\alpha_j\}$ and $\beta$ are $4 \times 4$ Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (1.9)

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$  \hspace{1cm} (1.10)

with $D_0 = \sigma \cdot (P - a_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. 
By using Foldy-Wouthuysen transform, explained in detail in the following section, \( H_0(\lambda) \) can be diagonalized by a unitary operator \( U_{FW} \).

\[
U_{FW} H_0(\lambda) U_{FW}^{-1} = \begin{pmatrix}
\sqrt{D_0^2 + m^2} & 0 \\
0 & -\sqrt{D_0^2 + m^2}
\end{pmatrix}. \tag{1.11}
\]

From the commutation relation (1.2) we have

\[
D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda \beta. \tag{1.12}
\]

We can easily see that \( \sigma(D_0^2) = [0, \infty) \). So we have

\[
\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).
\]

Therefore in the 3 dimensional case, the spectrum of \( H_0(\lambda) \) is absolutely continuous.

Let us consider the perturbation of \( H_0(\lambda) \) : We put

\[
H(\lambda) = H_0(\lambda) + V. \tag{1.13}
\]

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent \((z - H(\lambda))^{-1}\) on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

\[
\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x). \tag{1.14}
\]

The existence of the boundary values

\[
\langle x_3 \rangle^{-s} (\tilde{H} - \mu \mp i0)^{-1} \langle x_3 \rangle^{-s}
\]

is proved for \( s > 1/2 \) and \( \mu \in \mathbb{R} \setminus \left( \{ \lambda (2n+1) | n = 0, 1, 2, \ldots \} \cup \sigma_{pp}(\tilde{H}) \right) \).

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as \( |x| \to \infty \). (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential \( V(x) \).

Roughly speaking, his assumption means that the absolute value of each components of \( V \) is dominated from above by \( C \langle x' \rangle^{-1-\epsilon} \langle x \rangle^{-\epsilon} \) \( (x' = (x_2, x_3)) \) for sufficiently large \( x \). We remark that \( \epsilon > 0 \) is used as a sufficiently small parameter throughout this paper. To be accurate, \( \langle x' \rangle^{1+\epsilon} V(x) \) is required to be a \( H_0(\lambda) \)-compact operator.

In this paper we treat directly the following operator

\[
H(\lambda) = \alpha_1 (P_1 + \frac{\lambda}{2} x_2) + \alpha_2 (P_2 - \frac{\lambda}{2} x_1) + \alpha_3 P_3 + m \beta + V(x), \tag{1.15}
\]

where \( V(x) \) is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for \( V(x) \). In this case it
seems that an appropriate choice of the conjugate operator is
\[ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle}, \]
which is inspired by [9], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [9]. Namely we rewrite \( H_0(\lambda) \) by a direct integral and the conjugate operator \( A \) acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. Conjugate Operator

Let us recall
\[ Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - a_0), \tag{2.1} \]
with
\[ a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0). \tag{2.2} \]
The Dirac Hamiltonian \( Q_0 + m\beta \) can be diagonalized by sandwiching it between a unitary operator \( U \) and \( U^* = U^{-1} \). In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator \( H_0(\lambda) \). Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians \( H_0(\lambda) \) and \( H(\lambda) \).

Let \( Q_0 \) be the self-adjoint operator as in (2.1) and \( |Q_0| = \sqrt{Q_0^2}, |H_0(\lambda)| = \sqrt{H_0(\lambda)^2} \). We define a unitary operator \( U_{FW} \), which diagonalize \( H_0(\lambda) \), in the following way.

**Definition 2.1.** (i). At first we define a signature function associated with \( Q_0 \) by
\[ \text{sgn} Q_0 = \begin{cases} \frac{Q_0}{|Q_0|}, & \text{on } (\ker Q_0)^\perp \\ 0, & \text{on } (\ker Q_0) \end{cases} \tag{2.3} \]
We note that \( \text{sgn} Q_0 \) is isometry on \( (\ker Q_0)^\perp \).

(ii). We can easily see that \( m/|H_0(\lambda)| \leq 1 \). So we denote the square root of \( \frac{1}{2}(1 \pm \frac{m}{|H_0(\lambda)|}) \) as \( a_\pm \). i.e.
\[ a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}. \tag{2.4} \]

(iii). Combining these operators we define the operator \( U_{FW} \) as
\[ U_{FW} = a_+ + \beta(\text{sgn} Q_0)a_- \tag{2.5} \]

**Lemma 2.2.** (i). \( U_{FW} \) is a unitary operator on \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \).

Further,
\[ U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\text{sgn} Q_0)a_- \tag{2.6} \]
(ii). $H_0(\lambda)$ can be diagonalized by $U_{FW}$ as follows.

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)| \beta = \left( \begin{array}{cc} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{array} \right). \quad (2.7)$$

Proof. See 5.6.1 in [8]. □

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$ 

We rewrite (1.12) as follows.

$$D_0^2 = \left( \begin{array}{cc} D_- & 0 \\ 0 & D_+ \end{array} \right).$$

Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_\pm = (P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2 + P_3^2 \pm \lambda.$$ 

It is well-known that $(P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2$ has eigenvalues

$$\{ \lambda(2n + 1) | n = 0, 1, 2, \ldots \}.$$ 

We denote the eigenprojection on each eigenspace as $\Pi_n$. With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \left( \begin{array}{cc} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{array} \right), \quad (2.8)$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^{\infty} \left( \begin{array}{cc} f(d_n) \otimes \Pi_n & 0 \\ 0 & f(-d_{n+1}) \otimes \Pi_n \end{array} \right),$$

for any Borel function $f$.

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2} \{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \}. \quad (2.9)$$

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson’s commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A}\beta$ between $U_{FW}^{-1}$ and $U_{FW}$:

$$A = U_{FW}^{-1}(\hat{A}\beta) U_{FW}. \quad (2.10)$$

Before we show Mourre’s inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.
Suppose that \( f \in C^\infty(\mathbb{R}) \) satisfies the following condition for some \( m_0 \in \mathbb{R} \).

\[
|f^{(k)}(t)| \leq C_k(1 + |t|)^{m_0-k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{2.11}
\]

Then we can construct an almost analytic extension \( \tilde{f}(z) \) of \( f(t) \) having the following properties

\[
\tilde{f}(t) = f(t), \quad t \in \mathbb{R},
\]

\[
supp \tilde{f} \subset \{ z ; |Imz| \leq 1 + |Rez| \},
\]

\[
|\partial_z \tilde{f}(z)| \leq C_N |Imz|^N (z)^{m_0-1-N}, \quad \forall N \in \mathbb{N}. \tag{2.12}
\]

Then for all \( f \), satisfying (2.11) for \( m_0 < 0 \) and a self-adjoint operator \( H \), we have

\[
f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H)^{-1}dz \wedge d\overline{z}. \tag{2.13}
\]

### 3. Limiting Absorption Principle for Long-Range Potentials

Now we show the Mourre’s inequality for the Dirac Hamiltonian by choosing \( A \) defined in the previous section as the conjugate operator.

**Lemma 3.1.** Let \( \mathbb{R}_N \) be the following discrete subset of \( \mathbb{R} \)

\[
\mathbb{R}_N = \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \ldots \} \subset \mathbb{R}.
\]

We take a compact interval \( I \subset \mathbb{R} \setminus \mathbb{R}_N \) arbitrarily. Then there exists \( \alpha > 0 \) such that the following inequality holds for any real valued \( f \in C_0^\infty(I) \)

\[
f(H_0(\lambda))i[H_0(\lambda), A]f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \tag{3.1}
\]

**Proof.** By the relations (2.7) and (2.10), it is sufficient to show the inequality

\[
f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}]f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \tag{3.2}
\]

We rewrite the commutator as follow.

\[
i[\hat{H}_0(\lambda), \hat{A}] = \begin{pmatrix} i[\sqrt{D_0^2 + m^2}, \hat{A}] \\ i[\sqrt{D_0^2 + m^2}, \hat{A}] \end{pmatrix}. \tag{3.3}
\]

We proceed the calculus more precisely to see that

\[
i[\sqrt{D_0^2 + m^2}, \hat{A}] = \sum_{n=0}^{\infty} \begin{pmatrix} i[d_n, \hat{A}] \otimes \Pi_n \\ i[d_{n+1}, \hat{A}] \otimes \Pi_n \end{pmatrix} \tag{3.4}
\]
by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as
\[
f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix}
\] (3.5)
where
\[
I_1 = \sum_{n=0}^{\infty} f(d_n) [d, \hat{A}] f(d_n) \otimes \Pi_n,
\]
\[
I_2 = \sum_{n=0}^{\infty} f(d_{n+1}) [d_{n+1}, \hat{A}] f(d_{n+1}) \otimes \Pi_n,
\]
\[
I_3 = \sum_{n=0}^{\infty} f(-d_n) [d_n, \hat{A}] f(-d_n) \otimes \Pi_n,
\]
\[
I_4 = \sum_{n=0}^{\infty} f(-d_{n+1}) [d_{n+1}, \hat{A}] f(-d_{n+1}) \otimes \Pi_n.
\]
We note that all the sum in $I_1, \cdots, I_4$ are finite since $f$ is a compactly supported function. By an elementary calculus, we have
\[
i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2l + P_3^2 + m^2}} \langle P_3 \rangle (l \in \mathbb{N} \cup \{0\}).
\] (3.6)
Since $supp f \subset I \subset \mathbb{R} \setminus \mathbb{N}$, $P_3$ is away from zero when $P_3 \in supp f(d_l(P_3))$ or $P_3 \in supp f(-d_l(P_3))$. So there exist $C_l > 0$ such that
\[
f(d_l) i[d_l, \hat{A}] f(d_l) \otimes \Pi_l \geq C_l f(d_l)^2 \otimes \Pi_l,
\]
\[
f(-d_l) i[d_l, \hat{A}] f(-d_l) \otimes \Pi_l \geq C_l f(-d_l)^2 \otimes \Pi_l.
\]
Since only a finite number of $l = l_j$ ($j = 1, \ldots, N$) is concerned, we have (3.2) with $\alpha = \inf_{j=1,\ldots,N} C_{l_j}$. \hfill\Box

Now we give the assumption for the potential, which is necessary to Mourre’s inequality associated to $H(\lambda)$. After that we give an example of $V$ satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means $V(x) = O((x)^{-\epsilon}(x_3)^{-1-\epsilon})$ as $|x| \to \infty$. And long-range part is a multiplication of a real valued function $\varphi(x)$ such that $\varphi(x) = O((x)^{-\epsilon})$ as $|x| \to \infty$. More precisely we assume that $V$ satisfies the following.

**Assumption 3.2.** $V = V(x)$ is a multiplicative operator of a $4 \times 4$ Hermitian matrix satisfying the following properties.

(i). $V$ is a $H_0(\lambda)$-compact operator.

(ii). The form $[V, A]$ can be extended to a $H_0(\lambda)$-compact operator.
For example a $4 \times 4$ matrix $V(x)$ satisfying the following inequality is $H_0(\lambda)$-compact.

$$|V(x)| \leq C \langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3). \quad (3.7)$$

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre’s inequality for $H(\lambda)$.

**Lemma 3.3.** Suppose $V$ satisfies Assumption 3.2.

(i). We take $\mu \in \mathbb{R} \setminus \mathbb{R}_I$ and $\delta > 0$ so that the closed interval $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_I$. There exist $\alpha > 0$ and a compact operator $K$ such that the following inequality holds for all $f \in C_0^\infty(I)$.

$$f(H(\lambda)) [H(\lambda), A] f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K. \quad (3.8)$$

(ii). There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R} \setminus \mathbb{R}_I$. For $\mu \in \mathbb{R} \setminus (\mathbb{R}_I \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu - \delta, \mu + \delta])$.

$$f(H(\lambda)) [H(\lambda), A] f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2. \quad (3.9)$$

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

**Theorem 3.4.** Suppose $V$ satisfies Assumption 3.2. Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_I \cup \sigma_{pp}(H(\lambda)))$, the following limits

$$R^\pm(\mu) = \lim_{\iota \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i0)^{-1} \langle x_3 \rangle^{-s} \quad (3.10)$$

exist and $R^\pm(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_I \cup \sigma_{pp}(H(\lambda)))$.

**Sketch of proof**

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value $\langle \hat{A} \rangle^{-s} (H(\lambda) - \mu \mp i0)^{-1} \langle \hat{A} \rangle^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_I \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient if we show the boundness of $\langle \hat{A} \rangle^{-s} \langle x_3 \rangle^{-s}$. Since $\langle \hat{A} \rangle^{-s} \langle x_3 \rangle^{-s}$ is bounded, it is sufficient to show $\langle x_3 \rangle^{-s} U_{FW} \langle x_3 \rangle^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let $\chi(t) \in C_0^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1/\sqrt{2} & (t > -m^2/3) \\ 0 & (t < -2m^2/3). \end{cases} \quad (3.11)$$

With this function we define $F_\pm(t)$ and $F_{x,\pm}$ as follows.

$$F_+(t) = \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}}$$

$$F_-(t) = \chi(t) \left( \sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}}$$

$$F_{x,+}(t) = F_+(t) - \chi(t)$$

$$F_{x,-}(t) = \sqrt{t + m^2} F_-(t) - \chi(t)$$
Then we can easily verify that
\[ a_+ = F_+(Q_0^2), \]
\[ a_- sgn Q_0 = F_-(Q_0^2)Q_0 = Q_0F_-(Q_0^2). \]

As for the proof of \([Q_0, F_-(Q_0^2)] \equiv 0\), see 5.2.4 in [8]. By the construction of these functions, we can also see that \(F_{\chi, \pm}(t)\) satisfy (2.11) with \(m_0 < 0\). So we apply the functional calculus in section 2 to \(F_{\chi, \pm}(t)\) and see the following properties hold.

**Lemma 3.5.** Suppose \(0 \leq s \leq 2\) and \(z \in \mathbb{C} \setminus \mathbb{R}\). Then

(i). For \(0 < s \leq 1\), there exists \(C_s > 0\) such that
\[
\|\langle x \rangle^s (z - Q_0^2)^{-1}(x)^{-s}\|_\mathbb{H} \leq C_s(|Imz|^{-1} + |Imz|^{-2}(z)). \tag{3.12}
\]

(ii). For \(1 < s \leq 2\), there exists \(C_s' > 0\) such that
\[
\|\langle x \rangle^s (z - Q_0^2)^{-1}(x)^{-s}\|_\mathbb{H} \leq C_s'(|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)^2). \tag{3.13}
\]

(iii). \(\langle x \rangle^s F_+(Q_0^2)\langle x \rangle^{-s}\) and \(\langle x \rangle^s F_-(Q_0^2)Q_0\langle x \rangle^{-s}\) are bounded operators.

**Proof.** For the proof of (i) and (ii), we use the resolvent equation. Suppose \(0 < s \leq 1\). Then
\[
\langle x \rangle^s(z - Q_0^2)^{-1}(x)^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1}(Q_0^2 + 1) \tag{3.14}
\]
\[
\times (Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0]\langle x \rangle^{-s}. \tag{3.15}
\]

From the boundness of \((Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0]\) and the following estimate
\[
\|(z - Q_0^2)^{-1}(Q_0^2 + 1)\|_\mathbb{H} \leq C(|Imz|^{-1}(z) + 1), \tag{3.16}
\]
we obtain (i). As for the case \(1 < s \leq 2\), we rewrite the last term \((z - Q_0^2)^{-1}[\langle x \rangle^s, Q_0]\langle x \rangle^{-s}\) as
\[
(z - Q_0^2)^{-1}(Q_0^2 + 1)(Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0]\langle x \rangle^{-s+1} \tag{3.17}
\]
\[
\times \langle x \rangle^{-1}(z - Q_0^2)^{-1}\langle x \rangle^{-s+1}\langle x \rangle^{-1}. \tag{3.18}
\]

By using the result for \(0 < s \leq 1\), we have the inequality for \(1 < s \leq 2\). With these estimates, we prove (iii). Since \(\chi(Q_0^2) \equiv 1\), we can easily see that
\[
\langle x \rangle^s F_+(Q_0^2)\langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2)\langle x \rangle^{-s} + I. \tag{3.19}
\]

Since \(F_{\chi,+}(t)\) satisfies (2.12) for \(m_0 = -1/2\), \(F_{\chi,+}(Q_0^2)\) can be rewritten as follows.
\[
\frac{1}{2\pi i} \int_C \partial_{\bar{z}} \tilde{F}_{\chi, +}(z) \langle x \rangle^s(z - Q_0^2)^{-1}\langle x \rangle^{-s}dz \wedge d\bar{z}. \tag{3.20}
\]

From this formula and (i) (ii) we have
\[
\|\partial_{\bar{z}} \tilde{F}_{\chi, +}(z) \langle x \rangle^s(z - Q_0^2)^{-1}\langle x \rangle^{-s}\|_\mathbb{H}
\leq C|\partial_{\bar{z}} \tilde{F}_{\chi, +}(z)|(|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)^2).
From (2.12) we have
\[ \| \partial_x \hat{F}_{\chi^+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} \|_H \leq C \langle z \rangle^{-5/2}. \]
This implies the boundness of \( \langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} \).

In a similar way, we rewrite \( \langle x \rangle^s F_-(Q_0^2) \langle x \rangle^{-s} \) as
\[ \langle x \rangle^s F_-(Q_0^2) \langle x \rangle^{-s} \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s}. \]
(3.21)

It is sufficient to show the boundness of \( \langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s} \). To see this, we denote \( \chi(t)/\sqrt{t + m^2} \in C^\infty(\mathbb{R}^3) \) as \( S(t) \) and its almost analytic extension as \( \tilde{S}(z) \). We can easily see that \( S(Q_0^2) \langle x \rangle^s Q_0 \langle x \rangle^{-s} \) is bounded. So we obtain the boundness of \( \langle x \rangle^s F_-(Q_0^2) \langle x \rangle^{-s} \) if we show that \( \langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \) is bounded. We rewrite it as follows.
\[ \frac{1}{2 m i} \int_\mathbb{C} \partial_z \tilde{S}(z) \frac{Q_0^2 + 1}{z - Q_0^2} (Q_0^2 + 1)^{-1} \langle x \rangle^s Q_0 \langle x \rangle^{-s} \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}. \]

By an elementary calculus, we have \( (Q_0^2 + 1)^{-1} \langle x \rangle^s Q_0 \langle x \rangle^{-s} \) bounded. Combining (i) and (ii), we have
\[ \| \langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \|_H \leq C \int_\mathbb{C} |\partial_z \tilde{S}(z)| \{1 + |Imz|^{-1}\} \times \{ |Imz|^{-1} + |Imz|^{-2} (z) + |Imz|^{-3} (z)^2 \} dz \wedge d\bar{z} < \infty. \]
This implies the boundness of \( \langle x \rangle^s F_-(Q_0^2) \langle x \rangle^{-s} \).

Next we give an example of \( V \). It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.6.** Let \( V \) be a \( 4 \times 4 \) Hermitian matrix of the form
\[ V(x) = (v_{ij}(x)) + \varphi(x) I_4 \equiv V_s(x) + V_l(x) \]
(3.22)
where \( V_s(x) = (v_{ij}(x)) \) is an Hermitian matrix and \( I_4 \) is an identity matrix. Suppose the following conditions hold. Then \( V(x) \) satisfies Assumption 3.2.

There exist \( \delta > 0 \) such that the following inequalities hold for all multi-index \( \alpha \).
\[ |\partial^\alpha v_{ij}(x)| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|} \langle x_3 \rangle^{-1} \quad (1 \leq i, j \leq 4). \]
(3.23)
\[ \varphi(x) \in C^\infty(\mathbb{R}^3) \text{ is real valued and satisfies} \]
\[ |\partial^\alpha \varphi(x)| \leq C'_\alpha \langle x \rangle^{-\delta - |\alpha|}. \]
(3.24)

The relatively compactness of \( V(x) \) itself is clear since \( V \) satisfies (3.7). So we only have to show the relatively compactness of \([V, A] \). We prove the relatively compactness of \([V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}] \) at first. From the boundness of \( \langle x_3 \rangle^{-1} \hat{A} \beta \) and the relatively compactness of \( V_s(x_3) \), it is sufficient to show that \( \langle x_3 \rangle U_{FW} \langle x_3 \rangle^{-1} \) and \( \langle x_3 \rangle U_{FW}^{-1} \langle x_3 \rangle^{-1} \) are bounded operators in \( \mathbb{H} \). We have already proved it in Lemma 3.5.
Next we treat the long-range term. The conjugate operator $A$ can be decomposed into the sum of $J_1, \cdots, J_4$ where
\[
J_1 = F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2), \\
J_2 = F_+(Q_0^2)\hat{A}\beta^2 F_- (Q_0^2)Q_0, \\
J_3 = \beta F_- (Q_0^2)Q_0\hat{A}\beta F_+(Q_0^2), \\
J_4 = \beta F_- (Q_0^2)Q_0\hat{A}\beta^2 F_- (Q_0^2)Q_0.
\]
We prove that the $H_0(\lambda)$-compactness holds for each of $[V_i, J_1], \cdots, [V_i, J_4]$. To see this we use the functional calculus again and rewrite $J_1$ as follows.
\[
F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2) = \hat{A}\beta F_+(Q_0^2)^2 + [F_+(Q_0^2), \hat{A}\beta]F_+(Q_0^2)
\equiv J_1' + J_1''
\]
At first we prove the boundness of $J_1''$ and consequently the relatively compactness of $[V_i, J_1'']$. By using (2.13), we rewrite $[F_+(Q_0^2), \hat{A}\beta]$ as follows.
\[
\frac{1}{2\pi i} \int_{C} \partial_{\bar{z}} \tilde{F}_{x,+}(z)(z - Q_0^2)^{-1}[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1} dz \wedge d\bar{z}.
\] (3.25)
From (3.16) we have $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |Imz|\}$. So we have
\[
\| [F_+(Q_0^2), \hat{A}\beta] \| \leq C \int_{C} |\partial_{\bar{z}} \tilde{F}_{x,+}(z)| \{ |Imz|^{-1} + |Imz|^{-2} \} dz \wedge d\bar{z}.
\] (3.26)
Since the almost analytic extension $\tilde{F}_{x,+}(z)$ satisfies
\[
|\partial_{\bar{z}} \tilde{F}_{x,+}(z)| \leq C_N |Imz|^N |z|^{-3/2-N} \quad (\forall N \in \mathbb{N}),
\] (3.27)
we have $[F_+(Q_0^2), \hat{A}\beta]$ is bounded and inductively $[V_i, J_1'']$ is $H_0(\lambda)$-compact. So we only have to show the relatively compactness of $[V_i, J_1']$.
\[
[V_i, J_1'] = [V_i, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta [V_i, F_+(Q_0^2)^2].
\] (3.28)
Clearly $[V_i, \hat{A}\beta]F_+(Q_0^2)$ is $H_0(\lambda)$-compact. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $(x)^{1+\delta}[V_i, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_i, J_1]$.

As for the commutator $[V_i, J_2], \cdots, [V_i, J_4]$ we also replace $F_{\pm}$ by $F_{x,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_i, J_2]$ and $[V_i, J_3]$ are almost the same. We only give the proof for $J_2$. We also estimate the 'principle' part before we compute the commutator with $V_i$.
\[
J_2 = \hat{A}F_+(Q_0^2)F_- (Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_- (Q_0^2)Q_0
\] (3.29)
It is sufficient to show that \([V, \hat{A}F_{+}(Q_{0}^{2})F_{-}(Q_{0}^{2})Q_{0}]\) is a \(H_{0}(\lambda)\)-compact operator. We decompose it into the following sum.

\[
[V, \hat{A}]F_{+}(Q_{0}^{2})F_{-}(Q_{0}^{2})Q_{0}
+ \hat{A}[V, F_{+}(Q_{0}^{2})F_{-}(Q_{0}^{2})]Q_{0}
+ \hat{A}F_{+}(Q_{0}^{2})F_{-}(Q_{0}^{2})[V, Q_{0}].
\]

We can easily see that the first and the third term is relatively compact since \((x)^{1+\delta}[V, Q_{0}]\) is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for \(J_{4}\), the proof is similar. We rewrite it as

\[
\hat{A}F_{-}(Q_{0}^{2})^{2}Q_{0}^{2} + [F_{-}(Q_{0}^{2})Q_{0}, \hat{A}]F_{-}(Q_{0}^{2})Q_{0}
\]

(3.30)

We can also obtain the relatively compactness by estimating the term \([V, \hat{A}F_{-}(Q_{0}^{2})^{2}Q_{0}^{2}]\).

**Corollary 3.7.** Let \(V\) be a \(4 \times 4\) Hermitian matrix. Suppose \(V\) satisfies the condition in Lemma 3.6. Then the following limits

\[
R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0}(x_{3})^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}(x_{3})^{-s}
\]

exist for \(\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))\) and \(R^{\pm}(\mu)\) are continuous with respect to \(\mu\).

**REFERENCES**


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