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Kyoto University
LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential \( \mathbf{a} = (a_j(x))_{j=1,\ldots,d} \) is expressed by the following form

\[
H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m \gamma_{d+1} + V, \tag{1.1}
\]

where \( P_j = \frac{i}{\hbar} \partial_{x_j}, V \) is a multiplication of an Hermitian matrix \( V(x) \). \( m \) is the mass of electron. The matrices \( \{ \gamma_j \} \) satisfy the following relations

\[
\gamma_j \gamma_k + \gamma_k \gamma_j = 2 \delta_{jk} \mathbf{1} \quad (j, k = 1, \ldots, d+1). \tag{1.2}
\]

Here \( \delta_{jk} \) is Kronecker's delta and \( \mathbf{1} \) is an identity matrix. We assume that the speed of the light \( c = 1 \). When \( V \equiv 0 \), the square of \( H(\mathbf{a}) \) has the form

\[
H(\mathbf{a})^2 = \sum_{j=1}^{d} (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k, \tag{1.3}
\]

where

\[
b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x). \tag{1.4}
\]

It is called Pauli's Hamiltonian. The skew symmetric matrix \( (b_{jk}(x)) \) is the magnetic field associated with \( \mathbf{a} \). We say the magnetic field is asymptotically constant if it satisfies the following conditions as \( |x| \to \infty \):

\[
b_{jk}(x) \to 3 \lambda_{jk} \quad (1 \leq j, k \leq d), \tag{1.5}
\]

where \( (\lambda_{jk})_{j,k} \) is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for \( H(\mathbf{a}) \) with a constant magnetic field \( (b_{jk}(x)) \) and a long-range electric potential \( V(x) \) when \( d = 3 \). Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for \( d = 2,3 \). As can be inferred from (1.3), the spectrum of \( H(\mathbf{a}) \) is closely related
with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1(P_1 + \frac{\lambda}{2}x_2) + \sigma_2(P_2 - \frac{\lambda}{2}x_1) + m\sigma_3,$$  \hfill (1.6)

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + m^2 - \lambda\sigma_3.$$  \hfill (1.7)

The right hand side is a de-coupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{ \pm \sqrt{2\lambda n + m^2} | n = 0, 1, 2 \ldots \}.$$  

In fact we have

$$\sigma(h(\lambda)) = \{ \sqrt{2\lambda n + m^2}, -\sqrt{2\lambda(n + 1) + m^2} | n = 0, 1, 2 \ldots \}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along $x_3$-axis:

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1(P_1 + \frac{\lambda x_2}{2}) + \alpha_2(P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m\beta,$$  \hfill (1.8)

where $\{\alpha_j\}$ and $\beta$ are $4 \times 4$ Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (1.9)

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$  \hfill (1.10)

with $D_0 = \sigma \cdot (P - a_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. 

By using Foldy-Wouthuysen transform, explained in detail in the following section, $H_0(\lambda)$ can be diagonalized by a unitary operator $U_{FW}$.

$$U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \quad (1.11)$$

From the commutation relation (1.2) we have

$$D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda \beta. \quad (1.12)$$

We can easily see that $\sigma(D_0^2) = [0, \infty)$. So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of $H_0(\lambda)$ is absolutely continuous.

Let us consider the perturbation of $H_0(\lambda)$: We put

$$H(\lambda) = H_0(\lambda) + V. \quad (1.13)$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent $(z - H(\lambda))^{-1}$ on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

$$\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x). \quad (1.14)$$

The existence of the boundary values

$$\langle x_3 \rangle^{-s}(\tilde{H} - \mu \mp i0)^{-1}\langle x_3 \rangle^{-s}$$

is proved for $s > 1/2$ and $\mu \in \mathbb{R} \setminus \{\lambda(2n+1)|n = 0, 1, 2, \ldots\} \cup \sigma_{pp}(\tilde{H})$.

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as $|x| \to \infty$. (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential $V(x)$. Roughly speaking, his assumption means that the absolute value of each components of $V$ is dominated from above by $C\langle x' \rangle^{-1-\epsilon}\langle x \rangle^{-\epsilon}$ ($x' = (x_2, x_3)$) for sufficiently large $x$. We remark that $\epsilon > 0$ is used as a sufficiently small parameter throughout this paper. To be accurate, $\langle x' \rangle^{1+\epsilon}V(x)$ is required to be a $H_0(\lambda)$-compact operator.

In this paper we treat directly the following operator

$$H(\lambda) = \alpha_1(P_1 + \frac{\lambda}{2} x_2) + \alpha_2(P_2 - \frac{\lambda}{2} x_1) + \alpha_3 P_3 + m\beta + V(x), \quad (1.15)$$

where $V(x)$ is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for $V(x)$. In this case it
seems that an appropriate choice of the conjugate operator is
\[ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle}, \]
which is inspired by [9], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [9]. Namely we rewrite \( H_0(\lambda) \) by a direct integral and the conjugate operator \( A \) acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. Conjugate Operator

Let us recall
\[ Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - a_0), \quad (2.1) \]
with
\[ a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0). \quad (2.2) \]
The Dirac Hamiltonian \( Q_0 + m\beta \) can be diagonalized by sandwiching it between a unitary operator \( U \) and \( U^* = U^{-1} \). In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator \( H_0(\lambda) \). Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre’s inequality for original Hamiltonians \( H_0(\lambda) \) and \( H(\lambda) \).

Let \( Q_0 \) be the self-adjoint operator as in (2.1) and \(|Q_0| = \sqrt{Q_0^2}, |H_0(\lambda)| = \sqrt{H_0(\lambda)^2} \). We define a unitary operator \( U_{FW} \), which diagonalize \( H_0(\lambda) \), in the following way.

**Definition 2.1.** (i). At first we define a signature function associated with \( Q_0 \) by
\[ \text{sgn}Q_0 = \begin{cases} \frac{Q_0}{|Q_0|}, & \text{on } (\ker Q_0)^\perp \\ 0, & \text{on } (\ker Q_0) \end{cases} \quad (2.3) \]
We note that \( \text{sgn}Q_0 \) is isometry on \((\ker Q_0)^\perp\).

(ii). We can easily see that \( m/|H_0(\lambda)| \leq 1 \). So we denote the square root of \( \frac{1}{2}(1 \pm \frac{m}{|H_0(\lambda)|}) \) as \( a_\pm \). i.e.
\[ a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}. \quad (2.4) \]

(iii). Combining these operators we define the operator \( U_{FW} \) as
\[ U_{FW} = a_+ + \beta(\text{sgn}Q_0)a_- \quad (2.5) \]

**Lemma 2.2.** (i). \( U_{FW} \) is a unitary operator on \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \).

Further,
\[ U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\text{sgn}Q_0)a_- \quad (2.6) \]
(ii). $H_0(\lambda)$ can be diagonalized by $U_{FW}$ as follows.

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \quad (2.7)$$

Proof. See 5.6.1 in [8].

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$

We rewrite (1.12) as follows.

$$D_0 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}.$$ Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_\pm = (P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2 + P_3^2 \pm \lambda.$$

It is well-known that $(P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2$ has eigenvalues

$$\{\lambda(2n + 1) | n = 0, 1, 2, \ldots \}.$$

We denote the eigenprojection on each eigenspace as $\Pi_n$. With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^\infty \begin{pmatrix} d_n \otimes \Pi_n \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_{n+1} \otimes \Pi_n \end{pmatrix}, \quad (2.8)$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^\infty \begin{pmatrix} f(d_n) \otimes \Pi_n \\ f(d_{n+1}) \otimes \Pi_n \\ f(-d_n) \otimes \Pi_n \\ f(-d_{n+1}) \otimes \Pi_n \end{pmatrix},$$

for any Borel function $f$.

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2} \{ \langle P_3 \rangle \cdot x_3 + x_3 \cdot \langle P_3 \rangle \}.$$

(2.9)

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson’s commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A}\beta$ between $U_{FW}^{-1}$ and $U_{FW}$:

$$A = U_{FW}^{-1}(\hat{A}\beta)U_{FW}. \quad (2.10)$$

Before we show Mourre’s inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.
Suppose that $f \in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0 \in \mathbb{R}$.

$$|f^{(k)}(t)| \leq C_k (1 + |t|)^{m_0 - k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (2.11)$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R},$$

$$\text{supp} \tilde{f} \subset \{ z; |Imz| \leq 1 + |Rez| \},$$

$$|\partial_z \tilde{f}(z)| \leq C_N |Imz|^N (z)^{m_0 - 1 - N}, \quad \forall N \in \mathbb{N}. \quad (2.12)$$

Then for all $f$, satisfying (2.11) for $m_0 < 0$ and a self-adjoint operator $H$, we have

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z) (z - H)^{-1} dz \wedge d\overline{z}. \quad (2.13)$$

### 3. Limiting Absorption Principle for Long-Range Potentials

Now we show the Mourre’s inequality for the Dirac Hamiltonian by choosing $A$ defined in the previous section as the conjugate operator.

**Lemma 3.1.** Let $\mathbb{R}_N$ be the following discrete subset of $\mathbb{R}$

$$\mathbb{R}_N = \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \ldots \} \subset \mathbb{R}.$$

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_N$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C_0^\infty(I)$

$$f(H_0(\lambda)) i[H_0(\lambda), A] f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \quad (3.1)$$

**Proof.** By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda)) i[\hat{H}_0(\lambda), \hat{A}] f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \quad (3.2)$$

We rewrite the commutator as follow.

$$i[\hat{H}_0(\lambda), \hat{A}] = \left( i[\sqrt{D_0^2 + \lambda^2}, \hat{A}] \quad i[\sqrt{D_0^2 + \lambda^2}, \hat{A}] \right). \quad (3.3)$$

We proceed the calculus more precisely to see that

$$i[\sqrt{D_0^2 + \lambda^2}, \hat{A}] = \sum_{n=0}^{\infty} \left( i[d_n, \hat{A}] \otimes \Pi_n \quad i[d_{n+1}, \hat{A}] \otimes \Pi_n \right) \quad (3.4)$$
by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

\[
f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix}
I_1 & I_2 \\
I_3 & I_4
\end{pmatrix}
\] (3.5)

where

\[
I_1 = \sum_{n=0}^{\infty} f(d_n)i[d_n, \hat{A}]f(d_n) \otimes \Pi_n,
\]

\[
I_2 = \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_n,
\]

\[
I_3 = \sum_{n=0}^{\infty} f(-d_n)i[d_n, \hat{A}]f(-d_n) \otimes \Pi_n,
\]

\[
I_4 = \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_n.
\]

We note that all the sum in \(I_1, \cdots, I_4\) are finite since \(f\) is a compactly supported function. By an elementary calculus, we have

\[
i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2}(P_3)} \quad (l \in \mathbb{N} \cup \{0\}).
\] (3.6)

Since \(supp f \subset I \subset \mathbb{R} \setminus \mathbb{R}_\lambda\), \(P_3\) is away from zero when \(P_3 \in supp f(d_l(P_3))\) or \(P_3 \in supp f(-d_l(P_3))\). So there exist \(C_l > 0\) such that

\[
f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l \geq C_l f(d_l)^2 \otimes \Pi_l,
\]

\[
f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l \geq C_l f(-d_l)^2 \otimes \Pi_l.
\]

Since only a finite number of \(l = l_j, (j = 1, \ldots, N)\) is concerned, we have (3.2) with \(\alpha = \inf_{j=1,\ldots,N} C_{l_j}\).

Now we give the assumption for the potential, which is necessary to Mourre's inequality associated to \(H(\lambda)\). After that we give an example of \(V\) satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means \(V(x) = O((x)^{-\epsilon}(x_3)^{-1-\epsilon})\) as \(|x| \to \infty\). And long-range part is a multiplication of a real valued function \(\varphi(x)\) such that \(\varphi(x) = O((x)^{-\epsilon})\) as \(|x| \to \infty\). More precisely we assume that \(V\) satisfies the following.

**Assumption 3.2.** \(V = V(x)\) is a multiplicative operator of a 4 \(\times\) 4 Hermitian matrix satisfying the following properties.

(i). \(V\) is a \(H_0(\lambda)\)-compact operator.

(ii). The form \([V,A]\) can be extended to a \(H_0(\lambda)\)-compact operator.
For example a 4 × 4 matrix \( V(x) \) satisfying the following inequality is \( H_0(\lambda) \)-compact.

\[
|V(x)| \leq C \langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3).
\]

(3.7)

It is owing to the fact that \( V(x)(-\Delta_x + 1)^{-1} \) is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for \( H(\lambda) \).

**Lemma 3.3.** Suppose \( V \) satisfies Assumption 3.2.

(i). We take \( \mu \in \mathbb{R} \setminus \mathbb{R}_{\text{ill}} \) and \( \delta > 0 \) so that the closed interval \( I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\text{ill}} \). There exist \( \alpha > 0 \) and a compact operator \( K \) such that the following inequality holds for all \( f \in C^\infty_0(I) \).

\[
f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K.
\]

(3.8)

(ii). There is no accumulation point of \( \sigma_{pp}(H(\lambda)) \) in \( \mathbb{R} \setminus \mathbb{R}_{\text{ill}} \). For \( \mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{ill}} \cup \sigma_{pp}(H(\lambda))) \), there exist \( \delta_0 > 0 \) and \( \alpha_0 > 0 \) such that the following inequality holds for all \( f \in C^\infty_0([\mu - \delta_0, \mu + \delta_0]) \).

\[
f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2.
\]

(3.9)

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

**Theorem 3.4.** Suppose \( V \) satisfies Assumption 3.2. Then for \( \mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{ill}} \cup \sigma_{pp}(H(\lambda))) \), the following limits

\[
R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}\langle x_3 \rangle^{-s}
\]

(3.10)

exist and \( R^\pm(\mu) \) are continuous with respect to \( \mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{ill}} \cup \sigma_{pp}(H(\lambda))) \).

**Sketch of proof**

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value \( \langle A \rangle^{-s}(H(\lambda) - \mu \mp i0)^{-1}\langle A \rangle^{-s} \) exist for \( \mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{ill}} \cup \sigma_{pp}(H(\lambda))) \). To see the existence of (3.10), it is sufficient if we show the boundness of \( \langle A \rangle^s\langle x_3 \rangle^{-s} \). Since \( \langle \hat{A} \rangle^s\langle x_3 \rangle^{-s} \) is bounded, it is sufficient to show \( \langle x_3 \rangle^s U_{\text{FW}}\langle x_3 \rangle^{-s} \) is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let \( \chi(t) \in C^\infty(\mathbb{R}) \) such that

\[
\chi(t) = \begin{cases} 
1/\sqrt{2} & (t > -m^2/3) \\
0 & (t < -2m^2/3).
\end{cases}
\]

(3.11)

With this function we define \( F_\pm(t) \) and \( F_{\chi,\pm} \) as follows.

\[
F_+(t) = \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}}
\]

\[
F_-(t) = \chi(t) \left( \sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}}
\]

\[
F_{\chi,+}(t) = F_+(t) - \chi(t)
\]

\[
F_{\chi,-}(t) = \sqrt{t + m^2} F_-(t) - \chi(t)
\]
Then we can easily verify that
\[ a_{+} = F_{+}(Q_{0}^{2}), \]
\[ a_{-} sgn Q_{0} = F_{-}(Q_{0}^{2})Q_{0} = Q_{0}F_{-}(Q_{0}^{2}). \]

As for the proof of \([Q_{0}, F_{-}(Q_{0}^{2})] \equiv 0\), see 5.2.4 in [8]. By the construction of these functions, we can also see that \(F_{X;\pm}(t)\) satisfy (2.11) with \(m_{0} < 0\). So we apply the functional calculus in section 2 to \(F_{X;\pm}(t)\) and see the following properties hold.

**Lemma 3.5.** Suppose \(0 \leq s \leq 2\) and \(z \in \mathbb{C} \setminus \mathbb{R}\). Then

(i). For \(0 < s \leq 1\), there exists \(C_{s} > 0\) such that
\[
\|(x)^{s}((z - Q_{0}^{2})^{-1} - (x)^{-s})\|_{\mathcal{H}} \leq C_{s}(|Imz|^{-1} + |Imz|^{-2}(z)).
\] (3.12)

(ii). For \(1 < s \leq 2\), there exists \(C'_{s} > 0\) such that
\[
\|(x)^{s}((z - Q_{0}^{2})^{-1} - (x)^{-s})\|_{\mathcal{H}} \leq C'_{s}(|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)^{2}).
\] (3.13)

(iii). \((x)^{s}F_{+}(Q_{0}^{2})\langle x \rangle^{-s}\) and \((x)^{s}F_{-}(Q_{0}^{2})Q_{0}\langle x \rangle^{-s}\) are bounded operators.

**Proof.** For the proof of (i) and (ii), we use the resolvent equation. Suppose \(0 < s \leq 1\). Then
\[
(x)^{s}((z - Q_{0}^{2})^{-1} - (x)^{-s}) = (z - Q_{0}^{2})^{-1} + (z - Q_{0}^{2})^{-1}(Q_{0}^{2} + 1)
\]
\[
\times (Q_{0}^{2} + 1)^{-1}[(x)^{s}, Q_{0}^{2}](z - Q_{0}^{2})^{-1}(x)^{-s}.
\] (3.14)
\[
(x)^{s}((z - Q_{0}^{2})^{-1} - (x)^{-s}) \leq C_{s}(|Imz|^{-1} + |Imz|^{-2}(z)).
\] (3.15)

From the boundness of \((Q_{0}^{2} + 1)^{-1}[(x)^{s}, Q_{0}^{2}]\) and the following estimate
\[
\|(z - Q_{0}^{2})^{-1}(Q_{0}^{2} + 1)\|_{\mathcal{H}} \leq C(|Imz|^{-1} + 1),
\] (3.16)
we obtain (i). As for the case \(1 < s \leq 2\), we rewrite the last term \((z - Q_{0}^{2})^{-1}[(x)^{s}, Q_{0}^{2}](z - Q_{0}^{2})^{-1}(x)^{-s}\) as
\[
(z - Q_{0}^{2})^{-1}(Q_{0}^{2} + 1)(Q_{0}^{2} + 1)^{-1}[(x)^{s}, Q_{0}^{2}](x)^{-s+1}
\]
\[
\times (x)^{s+1}(z - Q_{0}^{2} - 1)(x)^{-s+1}.
\] (3.17)
\[
\times (x)^{s-1}(z - Q_{0}^{2} - 1)(x)^{-s+1}(x)^{-1}.
\] (3.18)

By using the result for \(0 < s \leq 1\), we have the inequality for \(1 < s \leq 2\).

With these estimates, we prove (iii). Since \(\chi(Q_{0}^{2}) \equiv 1\), we can easily see that
\[
(x)^{s}F_{+}(Q_{0}^{2})\langle x \rangle^{-s} = (x)^{s}F_{+,+}(Q_{0}^{2})\langle x \rangle^{-s} + I.
\] (3.19)

Since \(F_{+,+}(t)\) satisfies (2.12) for \(m_{0} = -1/2\), \(F_{+,+}(Q_{0}^{2})\) can be rewritten as follows.
\[
\frac{1}{2\pi i} \int_{C} \partial_{z} \tilde{F}_{+,+}(z)(x)^{s}(z - Q_{0}^{2})^{-1}(x)^{-s}dz \wedge d\overline{z}.
\] (3.20)

From this formula and (i) (ii) we have
\[
\|\partial_{z} \tilde{F}_{+,+}(z)(x)^{s}(z - Q_{0}^{2})^{-1}(x)^{-s}\|_{\mathcal{H}}
\]
\[
\leq C|\partial_{z} \tilde{F}_{+,+}(z)||(|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)^{2}).
\]
From (2.12) we have
\[ \| \partial^s_z \tilde{F}_{\chi,+} (z) \|_{\mathcal{H}} \leq C \langle z \rangle^{-5/2}. \]
This implies the boundness of \( \langle x \rangle^s F_+ \langle Q_0^2 \rangle \langle x \rangle^{-s} \).

In a similar way, we rewrite \( \langle x \rangle^s F_- \langle Q_0^2 \rangle \langle x \rangle^{-s} \) as
\[
\langle x \rangle^s F_+ \langle Q_0^2 \rangle \langle x \rangle^{-s} \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} \langle x \rangle^{-s} .
\]
(3.21)

It is sufficient to show the boundness of \( \langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s} \). To see this, we denote \( \chi(t) / \sqrt{t + m^2} \in C^\infty(\mathbb{R}^3) \) as \( S(t) \) and its almost analytic extension as \( \tilde{S}(z) \). We can easily see that \( S(Q_0^2) \langle x \rangle^s Q_0 \langle x \rangle^{-s} \) is bounded. So we obtain the boundness of \( \langle x \rangle^s F_- \langle Q_0^2 \rangle \langle x \rangle^{-s} \) if we show that \( \langle x \rangle^s, S(Q_0^2) \langle x \rangle^{-s} \) is bounded. We rewrite it as follows.
\[
\frac{1}{2ni} \int_\mathbb{C} \partial_z \tilde{S}(z) \langle x \rangle^s Q_0^2 + 1 \langle x \rangle^s (z - Q_0^2)^{-s} dz \wedge d\bar{z}.
\]
By an elementary calculus, we have \( (Q_0^2 + 1)^{-1} |(x)^s, Q_0^2 \rangle \langle x \rangle^{-s} \) bounded. Combining (i) and (ii), we have
\[
\| \langle x \rangle^s, S(Q_0^2) \langle x \rangle^{-s} \|_{\mathcal{H}} \leq C \int_\mathbb{C} |\partial_z \tilde{S}(z)| \{1 + |Imz|^{-1}\} \times \{|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)\} dz \wedge d\bar{z} < \infty.
\]
This implies the boundness of \( \langle x \rangle^s F_- \langle Q_0^2 \rangle \langle x \rangle^{-s} \).

Next we give an example of \( V \). It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.5.** Let \( V \) be a \( 4 \times 4 \) Hermitian matrix of the form
\[
V(x) = (v_{ij}(x)) + \varphi(x) I_4 \equiv V_s(x) + V_l(x)
\]
where \( V_s(x) = (v_{ij}(x)) \) is an Hermitian matrix and \( I_4 \) is an identity matrix. Suppose the following conditions hold. Then \( V(x) \) satisfies Assumption 3.2.

There exist \( \delta > 0 \) such that the following inequalities hold for all multi-index \( \alpha \).
\[
|\partial_\alpha^s v_{ij}(x)| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|} (x_3)^{-1} \quad (1 \leq i, j \leq 4).
\]
\[
\varphi(x) \in C^\infty(\mathbb{R}^3) \text{ is real valued and satisfies}
\]
\[
|\partial_\alpha^s \varphi(x)| \leq C_\alpha \langle x \rangle^{-\delta - |\alpha|}.
\]

The relatively compactness of \( V(x) \) itself is clear since \( V \) satisfies (3.7). So we only have to show the relatively compactness of \( [V, A] \). We prove the relatively compactness of \( [V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}] \) at first. From the boundness of \( (x_3)^{-1} \hat{A} \beta \) and the relatively compactness of \( V_s(x_3) \), it is sufficient to show that \( (x_3) U_{FW}(x_3)^{-1} \) and \( (x_3) U_{FW}^{-1}(x_3)^{-1} \) are bounded operators in \( \mathbb{H} \). We have already proved it in Lemma 3.5.
Next we treat the long-range term. The conjugate operator $A$ can be decomposed into the sum of $J_1, \cdots, J_4$ where

$$
J_1 = F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2), \\
J_2 = F_+(Q_0^2)\hat{A}\beta^2 F_-(Q_0^2)Q_0, \\
J_3 = \beta F_-(Q_0^2)Q_0\hat{A}\beta F_+(Q_0^2), \\
J_4 = \beta F_-(Q_0^2)Q_0\hat{A}\beta^2 F_-(Q_0^2)Q_0.
$$

We prove that the $H_0(\lambda)$-compactness holds for each of $[V_i, J_3], \cdots, [V_i, J_4]$. To see this we use the functional calculus again and rewrite $J_1$ as follows.

$$
F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2) = \hat{A}\beta F_+(Q_0^2)^2 + [F_{x,+}(Q_0^2), \hat{A}\beta]F_+(Q_0^2) \\
\equiv J'_1 + J''_1
$$

At first we prove the boundness of $J'_1$ and consequently the relatively compactness of $[V_i, J'_1]$. By using (2.13), we rewrite $[F_{x,+}(Q_0^2), \hat{A}\beta]$ as follows.

$$
\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{F}_{x,+}(z)(z - Q_0^2)^{-1}[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1} \, dz \wedge d\overline{z}.
$$

(3.25)

From (3.16) we have $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |Im z|\}$. So we have

$$
\|[F_{x,+}(Q_0^2), \hat{A}\beta]\| \leq C \int_{\mathbb{C}} |\partial_{\overline{z}} \tilde{F}_{x,+}(z)|||Im z|^{-1} + |Im z|^{-2}(z)\} \, dz \wedge d\overline{z}.
$$

(3.26)

Since the almost analytic extension $\tilde{F}_{x,+}(z)$ satisfies

$$
|\partial_{\overline{z}} \tilde{F}_{x,+}(z)| \leq C_N |Im z|^N |(z)^{-3/2-N} \quad (N \in \mathbb{N}),
$$

(3.27)

we have $[F_{x,+}(Q_0^2), \hat{A}\beta]$ is bounded and inductively $[V_i, J'_1]$ is $H_0(\lambda)$-compact. So we only have to show the relatively compactness of $[V_i, J'_1]$.

$$
[V_i, J'_1] = [V_i, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta [V_i, F_+(Q_0^2)^2].
$$

(3.28)

Clearly $[V_i, \hat{A}\beta]F_+(Q_0^2)$ is $H_0(\lambda)$-compact. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $(x)^{1+\delta}[V_i, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_i, J'_1]$.

As for the commutator $[V_i, J_2], \cdots, [V_i, J_4]$ we also replace $F_{\pm}$ by $F_{x,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_i, J_2]$ and $[V_i, J_3]$ are almost the same. We only give the proof for $J_2$. We also estimate the 'principle' part before we compute the commutator with $V_i$.

$$
J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0
$$

(3.29)
It is sufficient to show that \([V_t, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]\) is a \(H_0(\lambda)\)-compact operator. We decompose it into the following sum.

\[
[V_t, \hat{A}] F_+(Q_0^2)F_-(Q_0^2)Q_0
+ \hat{A}[V_t, F_+(Q_0^2)F_-(Q_0^2)]Q_0
+ \hat{A}F_+(Q_0^2)F_-(Q_0^2)[V_t, Q_0].
\]

We can easily see that the first and the third term is relatively compact since \((x)^{1+\delta}[V_t, Q_0]\) is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for \(J_4\), the proof is similar. We rewrite it as

\[
\hat{A}F_-(Q_0^2)^2Q_0 + [F_-(Q_0^2)Q_0, \hat{A}]F_-(Q_0^2)Q_0
\]

(3.30)

We can also obtain the relatively compactness by estimating the term \([V, \hat{A}F_-(Q_0^2)^2Q_0]\).

**Corollary 3.7.** Let \(V\) be a \(4 \times 4\) Hermitian matrix. Suppose \(V\) satisfies the condition in Lemma 3.6. Then the following limits

\[
R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-\delta}(H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-\delta}
\]

(3.31)

exist for \(\mu \in \mathbb{R} \setminus (\mathbb{R}_N \cup \sigma_{pp}(H(\lambda)))\) and \(R^\pm(\mu)\) are continuous with respect to \(\mu\).

**References**


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