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Kyoto University
LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

KOICHIRO YOKOYAMA
DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY

1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential \( \mathbf{a} = (a_j(x))_{j=1, \ldots, d} \) is expressed by the following form

\[
H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m \gamma_{d+1} + V,
\]

where \( P_j = \frac{i}{\hbar} \partial_{x_j} \) and \( V \) is a multiplication of an Hermitian matrix \( V(x) \). \( m \) is the mass of electron. The matrices \( \{\gamma_j\} \) satisfy the following relations

\[
\gamma_j \gamma_k + \gamma_k \gamma_j = 2 \delta_{jk} \mathbf{1} \quad (j, k = 1, \ldots, d+1).
\]

Here \( \delta_{jk} \) is Kronecker's delta and \( \mathbf{1} \) is an identity matrix. We assume that the speed of the light \( c = 1 \). When \( V \equiv 0 \), the square of \( H(\mathbf{a}) \) has the form

\[
H(\mathbf{a})^2 = \sum_{j=1}^{d} (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k,
\]

where

\[
b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x).
\]

It is called Pauli's Hamiltonian. The skew symmetric matrix \( (b_{jk}(x)) \) is the magnetic field associated with \( \mathbf{a} \). We say the magnetic field is asymptotically constant if it satisfies the following conditions as \( |x| \to \infty \):

\[
b_{jk}(x) \to \begin{cases} \Lambda_{jk} & (1 \leq j, k \leq d), \end{cases}
\]

where \( (\Lambda_{jk})_{j,k} \) is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for \( H(\mathbf{a}) \) with a constant magnetic field \( (b_{jk}(x)) \) and a long-range electric potential \( V(x) \) when \( d = 3 \). Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for \( d = 2, 3 \). As can be inferred from (1.3), the spectrum of \( H(\mathbf{a}) \) is closely related...
with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1(P_1 + \frac{\lambda}{2}x_2) + \sigma_2(P_2 - \frac{\lambda}{2}x_1) + m\sigma_3,$$  \hfill (1.6)$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2}x_2)^2 + (P_2 - \frac{\lambda}{2}x_1)^2 + m^2 - \lambda\sigma_3.$$  \hfill (1.7)$$

The right hand side is a decoupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2 \ldots \}.$$ 

In fact we have

$$\sigma(h(\lambda)) = \{ \sqrt{2\lambda n + m^2}, -\sqrt{2\lambda(n+1) + m^2} \mid n = 0, 1, 2 \ldots \}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along $x_3$-axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ : 

$$H_0(\lambda) = \alpha_1(P_1 + \frac{\lambda x_2}{2}) + \alpha_2(P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m\beta,$$  \hfill (1.8)$$

where $\{\alpha_j\}$ and $\beta$ are $4 \times 4$ Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hfill (1.9)$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$  \hfill (1.10)$$

with $D_0 = \sigma \cdot (P - a_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. 

By using Foldy-Wouthuysen transform, explained in detail in the following section, $H_0(\lambda)$ can be diagonalized by a unitary operator $U_{FW}$.

$$U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}.$$ \hspace{1cm} (1.11)

From the commutation relation (1.2) we have

$$D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda \beta.$$ \hspace{1cm} (1.12)

We can easily see that $\sigma(D_0^2) = [0, \infty)$. So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of $H_0(\lambda)$ is absolutely continuous.

Let us consider the perturbation of $H_0(\lambda)$: We put

$$H(\lambda) = H_0(\lambda) + V.$$ \hspace{1cm} (1.13)

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent $(z - H(\lambda))^{-1}$ on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

$$\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x).$$ \hspace{1cm} (1.14)

The existence of the boundary values

$$\langle x_3 \rangle^{-s}(\tilde{H} - \mu \mp i0)^{-1}\langle x_3 \rangle^{-s}$$

is proved for $s > 1/2$ and $\mu \in \mathbb{R} \setminus \{\lambda(2n+1)|n = 0, 1, 2, \ldots\} \cup \sigma_{pp}(\tilde{H})$.

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as $|x| \to \infty$. (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential $V(x)$. Roughly speaking, his assumption means that the absolute value of each components of $V$ is dominated from above by $C(x')^{-1-c}(x)^{-c}$ ($x' = (x_2, x_3)$) for sufficiently large $x$. We remark that $\epsilon > 0$ is used as a sufficiently small parameter throughout this paper. To be accurate, $\langle x' \rangle^{1+c}V(x)$ is required to be a $H_0(\lambda)$-compact operator.

In this paper we treat directly the following operator

$$H(\lambda) = \alpha_1(P_1 + \frac{\lambda}{2}x_2) + \alpha_2(P_2 - \frac{\lambda}{2}x_1) + \alpha_3P_3 + m\beta + V(x),$$ \hspace{1cm} (1.15)

where $V(x)$ is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for $V(x)$. In this case it
seems that an appropriate choice of the conjugate operator is
\[
\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle},
\]
which is inspired by \cite{9}, when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with \cite{9}. Namely we rewrite \(H_0(\lambda)\) by a direct integral and the conjugate operator \(A\) acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. CONJUGATE OPERATOR

Let us recall
\[
Q_0 = \left( \begin{array}{cc} 0 & D_0 \\ D_0 & 0 \end{array} \right), \quad D_0 = \sigma(P - a_0),
\]
(2.1)
with
\[
a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0).
\]
(2.2)
The Dirac Hamiltonian \(Q_0 + m\beta\) can be diagonalized by sandwiching it between a unitary operator \(U\) and \(U^* = U^{-1}\). In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator \(H_0(\lambda)\). Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians \(H_0(\lambda)\) and \(H(\lambda)\).

Let \(Q_0\) be the self-adjoint operator as in (2.1) and \(|Q_0| = \sqrt{Q_0^2}, |H_0(\lambda)| = \sqrt{H_0(\lambda)^2}\). We define a unitary operator \(U_{FW}\), which diagonalize \(H_0(\lambda)\), in the following way.

**Definition 2.1.** (i). At first we define a signature function associated with \(Q_0\) by
\[
sgn Q_0 = \begin{cases} Q_0/|Q_0|, & \text{on } (ker Q_0)^\perp \\ 0, & \text{on } (ker Q_0) \end{cases}
\]
(2.3)
We note that \(sgn Q_0\) is isometir on \((ker Q_0)^\perp\).

(ii). We can easily see that \(m/|H_0(\lambda)| \leq 1\). So we denote the square root of \(1 \pm m/|H_0(\lambda)|\) as \(a_\pm\). i.e.
\[
a_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm m/|H_0(\lambda)|}.
\]
(2.4)

(iii). Combining these operators we define the operator \(U_{FW}\) as
\[
U_{FW} = a_+ + \beta(sgn Q_0)a_-.
\]
(2.5)

**Lemma 2.2.** (i). \(U_{FW}\) is a unitary operator on \(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4\).

Further,
\[
U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(sgn Q_0)a_-.
\]
(2.6)
(ii). $H_0(\lambda)$ can be diagonalized by $U_{FW}$ as follows.

\[ U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)| \beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix} \cdot \beta. \quad (2.7) \]

**Proof.** See 5.6.1 in [8].

We denote the diagonalized Dirac Hamiltonian as $\tilde{H}_0(\lambda)$. i.e.

\[ \tilde{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}. \]

We rewrite (1.12) as follows.

\[ D_0^2 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}. \]

Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

\[ D_\pm = (P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2 + P_3^2 \pm \lambda. \]

It is well-known that $(P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2$ has eigenvalues

\[ \{ \lambda(2n+1) | n = 0, 1, 2, \ldots \}. \]

We denote the eigenprojection on each eigenspace as $\Pi_n$. With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

\[ \sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix}, \quad (2.8) \]

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

\[ f(\tilde{H}_0(\lambda)) = \sum_{n=0}^{\infty} \begin{pmatrix} f(d_n) \otimes \Pi_n & f(d_{n+1}) \otimes \Pi_n \\ f(-d_n) \otimes \Pi_n & f(-d_{n+1}) \otimes \Pi_n \end{pmatrix}, \]

for any Borel function $f$.

Now we define the conjugate operator. At first we define

\[ \hat{A} = \frac{1}{2} \{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \}. \quad (2.9) \]

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson’s commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A} \beta$ between $U_{FW}^{-1}$ and $U_{FW}$:

\[ A = U_{FW}^{-1}(\hat{A} \beta) U_{FW}. \quad (2.10) \]

Before we show Mourre’s inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.
Suppose that $f \in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0 \in \mathbb{R}$.

$$|f^{(k)}(t)| \leq C_k(1 + |t|)^{m_0-k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{2.11}$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R},$$

$$\text{supp} \tilde{f} \subset \{z; |Imz| \leq 1 + |Rez|\},$$

$$|\partial_z \tilde{f}(z)| \leq C_N |Imz|^N (z)^{m_0-1-N}, \quad \forall N \in \mathbb{N}. \tag{2.12}$$

Then for all $f$, satisfying (2.11) for $m_0 < 0$ and a self-adjoint operator $H$, we have

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H)^{-1}dz \wedge d\overline{z}. \tag{2.13}$$

3. Limiting Absorption Principle for Long-Range Potentials

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing $A$ defined in the previous section as the conjugate operator.

**Lemma 3.1.** Let $\mathbb{R}_N$ be the following discrete subset of $\mathbb{R}$

$$\mathbb{R}_N = \{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \ldots \} \subset \mathbb{R}.$$  

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_N$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C^\infty_0(I)$

$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), A]f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \tag{3.1}$$

**Proof.** By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \tag{3.2}$$

We rewrite the commutator as follow.

$$i[\hat{H}_0(\lambda), \hat{A}\beta] = \begin{pmatrix} i[\sqrt{D_0^2 + m^2}, \hat{A}] \\ i[\sqrt{D_0^2 + m^2}, \hat{A}] \end{pmatrix}. \tag{3.3}$$

We proceed the calculus more precisely to see that

$$i[\sqrt{D_0^2 + m^2}, \hat{A}] = \sum_{n=0}^{\infty} \begin{pmatrix} i[d_n, \hat{A}] \otimes \Pi_n \\ i[d_{n+1}, \hat{A}] \otimes \Pi_n \end{pmatrix} \tag{3.4}$$
by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

\[ f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix} \tag{3.5} \]

where

\[ I_1 = \sum_{n=0}^{\infty} f(d_n)i[d_n, \hat{A}]f(d_n) \otimes \Pi_n, \]
\[ I_2 = \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_n, \]
\[ I_3 = \sum_{n=0}^{\infty} f(-d_n)i[d_n, \hat{A}]f(-d_n) \otimes \Pi_n, \]
\[ I_4 = \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_n. \]

We note that all the sum in \( I_1, \ldots, I_4 \) are finite since \( f \) is a compactly supported function. By an elementary calculus, we have

\[ i[d, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2}} \langle P_3 \rangle \quad (l \in \mathbb{N} \cup \{0\}). \tag{3.6} \]

Since \( \text{supp} f \subset I \subset \mathbb{R} \setminus \mathbb{R}_3, P_3 \) is away from zero when \( P_3 \in \text{supp} f(d_i(P_3)) \) or \( P_3 \in \text{supp} f(-d_i(P_3)) \). So there exist \( C_l > 0 \) such that

\[ f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l \geq C_l f(d_l)^2 \otimes \Pi_l, \]
\[ f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l \geq C_l f(-d_l)^2 \otimes \Pi_l. \]

Since only a finite number of \( l = l_j \quad (j = 1, \ldots, N) \) is concerned, we have (3.2) with \( \alpha = \inf_{j=1, \ldots, N} C_{l_j}. \]

Now we give the assumption for the potential, which is necessary to Mourre’s inequality associated to \( H(\lambda) \). After that we give an example of \( V \) satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means \( V(x) = O(\langle x \rangle^{-\epsilon}(x_3)^{-1-\epsilon}) \) as \( |x| \to \infty \). And long-range part is a multiplication of a real valued function \( \varphi(x) \) such that \( \varphi(x) = O(\langle x \rangle^{-\epsilon}) \) as \( |x| \to \infty \). More precisely we assume that \( V \) satisfies the following.

**Assumption 3.2.** \( V = V(x) \) is a multiplicative operator of a \( 4 \times 4 \) Hermitian matrix satisfying the following properties.

(i). \( V \) is a \( H_0(\lambda) \)-compact operator.

(ii). The form \([V, A]\) can be extended to a \( H_0(\lambda) \)-compact operator.
For example a $4 \times 4$ matrix $V(x)$ satisfying the following inequality

$$|V(x)| \leq C(x)^{-\epsilon} \quad (x \in \mathbb{R}^3). \quad (3.7)$$

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for $H(\lambda)$.

**Lemma 3.3.** Suppose $V$ satisfies Assumption 3.2.

(i). We take $\mu \in \mathbb{R} \setminus \mathbb{R}_{\text{I}}$ and $\delta > 0$ so that the closed interval $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\text{I}}$. There exist $\alpha > 0$ and a compact operator $K$ such that the following inequality holds for all $f \in C_0^\infty(I)$.

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K. \quad (3.8)$$

(ii). There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R} \setminus \mathbb{R}_{\text{I}}$. For $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{I}} \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu - \delta_0, \mu + \delta_0])$.

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2. \quad (3.9)$$

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

**Theorem 3.4.** Suppose $V$ satisfies Assumption 3.2. Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{I}} \cup \sigma_{pp}(H(\lambda)))$, the following limits

$$R^\pm(\mu) = \lim_{\epsilon \downarrow 0}(x_3)^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}(x_3)^{-s} \quad (3.10)$$

exist and $R^\pm(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{I}} \cup \sigma_{pp}(H(\lambda)))$.

**Sketch of proof**

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value $\langle A \rangle^{-s}(H(\lambda) - \mu \mp i0)^{-1}\langle A \rangle^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{I}} \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient if we show the boundness of $\langle A \rangle^s(x_3)^{-s}$. Since $\langle \hat{A} \rangle^s(x_3)^{-s}$ is bounded, it is sufficient to show $\langle x_3 \rangle^sU_{FW}(x_3)^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let $\chi(t) \in C_0^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 
1/\sqrt{2} & (t > -m^2/3) \\
0 & (t < -2m^2/3).
\end{cases} \quad (3.11)$$

With this function we define $F_{\pm}(t)$ and $F_{\chi,\pm}$ as follows.

$$F_+(t) = \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}}$$

$$F_-(t) = \chi(t) \left( \sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}}$$

$$F_{\chi,+}(t) = F_+(t) - \chi(t)$$

$$F_{\chi,-}(t) = \sqrt{t + m^2}F_-(t) - \chi(t)$$
Then we can easily verify that
\[ a_+ = F_+(Q_0^2), \]
\[ a_- sgn Q_0 = F_-(Q_0^2)Q_0 = Q_0 F_-(Q_0^2). \]

As for the proof of \([Q_0, F_- (Q_0^2)] \equiv 0,\) see 5.2.4 in [8]. By the construction of these functions, we can also see that \(F_{\chi, \pm}(t)\) satisfy (2.11) with \(m_0 < 0.\) So we apply the functional calculus in section 2 to \(F_{\chi, \pm}(t)\) and see the following properties hold.

**Lemma 3.5.** Suppose \(0 \leq s \leq 2\) and \(z \in \mathbb{C} \setminus \mathbb{R}.\) Then

(i). For \(0 < s \leq 1,\) there exists \(C_s > 0\) such that
\[
\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathcal{H}} \leq C_s(|\text{Im}z|^{-1} + |\text{Im}z|^{-2}s(z)). \tag{3.12}
\]

(ii). For \(1 < s \leq 2,\) there exists \(C'_s > 0\) such that
\[
\|\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathcal{H}} \leq C'_s(|\text{Im}z|^{-1} + |\text{Im}z|^{-2}s(z) + |\text{Im}z|^{-3}s(z)^2). \tag{3.13}
\]

(iii). \(\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}\) and \(\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}\) are bounded operators.

**Proof.** For the proof of (i) and (ii), we use the resolvent equation. Suppose \(0 < s \leq 1.\) Then
\[
\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1}(Q_0^2 + 1)
\times (Q_0^2 + 1)^{-1}[(x)^s, Q_0^2](z - Q_0^2)^{-1} \langle x \rangle^{-s}. \tag{3.14}
\]
From the boundness of \((Q_0^2 + 1)^{-1}[(x)^s, Q_0^2]\) and the following estimate
\[
\|(z - Q_0^2)^{-1}(Q_0^2 + 1)\|_{\mathbb{H}} \leq C(|\text{Im}z|^{-1} s(z) + 1), \tag{3.16}
\]
we obtain (i). As for the case \(1 < s \leq 2,\) we rewrite the last term \((z - Q_0^2)^{-1}[(x)^s, Q_0^2](z - Q_0^2)^{-1} \langle x \rangle^{-s}\) as
\[
(z - Q_0^2)^{-1}(Q_0^2 + 1)(Q_0^2 + 1)^{-1}[(x)^s, Q_0^2] \langle x \rangle^{-s+1}
\times \langle x \rangle^{s-1}(z - Q_0^2)^{-1} \langle x \rangle^{-s+1} \langle x \rangle^{-1}. \tag{3.17}
\]
By using the result for \(0 < s \leq 1,\) we have the inequality for \(1 < s \leq 2.\)

With these estimates, we prove (iii). Since \(\chi(Q_0^2) \equiv 1,\) we can easily see that
\[
\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} = \langle x \rangle^s F_{\chi, +}(Q_0^2) \langle x \rangle^{-s} + I. \tag{3.19}
\]
Since \(F_{\chi, +}(t)\) satisfies (2.12) for \(m_0 = -1/2,\) \(F_{\chi, +}(Q_0^2)\) can be rewritten as follows.
\[
\frac{1}{2\pi i} \int_C \partial_s \tilde{F}_{\chi, +}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}. \tag{3.20}
\]
From this formula and (i) (ii) we have
\[
\|\partial_s \tilde{F}_{\chi, +}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}\|_{\mathcal{H}}
\leq C\|\partial_s \tilde{F}_{\chi, +}(z)\|(|\text{Im}z|^{-1} + |\text{Im}z|^{-2}s(z) + |\text{Im}z|^{-3}s(z)^2).
From (2.12) we have
\[
\|\partial_z \hat{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} \|_{\mathbb{H}} \leq C \langle z \rangle^{-5/2}.
\]
This implies the boundness of \( \langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} \).

In a similar way, we rewrite \( \langle x \rangle^s F_- (Q_0^2) Q_0 \langle x \rangle^{-s} \) as
\[
\langle x \rangle^s F_-(Q_0^2) (Q_0^2 + m^2)^{-1} \langle x \rangle^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}} (Q_0^2 + m^2)^{-1} \langle x \rangle^{-s}.
\]
(3.21)

It is sufficient to show the boundness of \( \langle x \rangle^s Q_0 \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s} \). To see this, we denote \( \chi(t)/\sqrt{t + m^2} \in C^\infty(\mathbb{R}^3) \) as \( S(t) \) and its almost analytic extension as \( \tilde{S}(z) \). We can easily see that \( S(Q_0^2 \langle x \rangle^s Q_0 \langle x \rangle^{-s} \) is bounded. So we obtain the boundness of \( \langle x \rangle^s F_- (Q_0^2) Q_0 \langle x \rangle^{-s} \) if we show that \( \| (\langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \) is bounded. We rewrite it as follows.
\[
\frac{1}{2 \pi i} \int_{\mathbb{C}} \partial_z \tilde{S}(z) \frac{Q_0^2 + 1}{z - Q_0^2} (Q_0^2 + 1)^{-1} \langle \langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \langle z - Q_0^2 \rangle^{-1} (Q_0^2 + m^2)^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}.
\]

By an elementary calculus, we have \( (Q_0^2 + 1)^{-1} \langle \langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \) bounded. Combining (i) and (ii), we have
\[
\| (\langle x \rangle^s, S(Q_0^2) \rangle Q_0 \langle x \rangle^{-s} \|_{\mathbb{H}} \leq C \int_{\mathbb{C}} \| \partial_z \tilde{S}(z) \| \{ 1 + |Imz|^{-1} \}
\times \{|Imz|^{-1} + |Imz|^{-2} |z| + |Imz|^{-3} |z|^2 \} dz \wedge d\bar{z} < \infty.
\]

This implies the boundness of \( \langle x \rangle^s F_- (Q_0^2) Q_0 \langle x \rangle^{-s} \). \hfill \Box

Next we give an example of \( V \). It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.6.** Let \( V \) be a \( 4 \times 4 \) Hermitian matrix of the form
\[
V(x) = (v_{ij}(x)) + \varphi(x) I_4 \equiv V_s(x) + V_l(x)
\]
(3.22)
where \( V_s(x) = (v_{ij}(x)) \) is an Hermitian matrix and \( I_4 \) is an identity matrix. Suppose the following conditions hold. Then \( V(x) \) satisfies Assumption 3.2.

There exist \( \delta > 0 \) such that the following inequalities hold for all multi-index \( \alpha \).
\[
|\partial_{x_i}^\alpha v_{ij}(x)| \leq C_\alpha \langle x \rangle^{-\delta-|\alpha|} \langle x_3 \rangle^{-1} \quad (1 \leq i, j \leq 4).
\]
(3.23)
\( \varphi(x) \in C^\infty(\mathbb{R}^3) \) is real valued and satisfies
\[
|\partial_{x_i}^\alpha \varphi(x)| \leq C'_{\alpha} \langle x \rangle^{-\delta-|\alpha|}.
\]
(3.24)

The relatively compactness of \( V(x) \) itself is clear since \( V \) satisfies (3.7). So we only have to show the relatively compactness of \([V, A]\). We prove the relatively compactness of \([V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}] \) at first. From the boundness of \( \langle x_3 \rangle^{-1} \hat{A} \beta \) and the relatively compactness of \( V_s(x_3) \), it is sufficient to show that \( \langle x_3 \rangle U_{FW} \langle x_3 \rangle^{-1} \) and \( \langle x_3 \rangle U_{FW}^{-1} \langle x_3 \rangle^{-1} \) are bounded operators in \( \mathbb{H} \). We have already proved it in Lemma 3.5.
Next we treat the long-range term. The conjugate operator $A$ can be decomposed into the sum of $J_1, \ldots, J_4$ where

\[
J_1 = F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2), \\
J_2 = F_+(Q_0^2)\hat{A}\beta^2 F_-(Q_0^2)Q_0, \\
J_3 = \beta F_-(Q_0^2)Q_0\hat{A}\beta F_+(Q_0^2), \\
J_4 = \beta F_-(Q_0^2)Q_0\hat{A}\beta^2 F_-(Q_0^2)Q_0.
\]

We prove that the $H_0(\lambda)\text{-}compactness$ holds for each of $[V_i, J_1], \ldots, [V_i, J_4]$. To see this we use the functional calculus again and rewrite $J_1$ as follows.

\[
F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2) = \hat{A}\beta F_+(Q_0^2)^2 + [F_{\chi,+}(Q_0^2), \hat{A}\beta]F_+(Q_0^2)
\]

\[
\equiv J'_1 + J''_1
\]

At first we prove the boundness of $J''_1$ and consequently the relatively compactness of $[V_i, J''_1]$. By using (2.13), we rewrite $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ as follows.

\[
\frac{1}{2\pi i} \int_C \partial_z \tilde{F}_{\chi,+}(z)(z - Q_0^2)^{-1} [Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1} dz \wedge d\bar{z}. \quad (3.25)
\]

From (3.16) we have $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$ is dominated from above by $C \{1 + |Imz|\}$. So we have

\[
\| [F_{\chi,+}(Q_0^2), \hat{A}\beta] \| \leq C \int_C |\partial_z \tilde{F}_{\chi,+}(z)| \{ |Imz|^{-1} + |Imz|^{-2} \} dz \wedge d\bar{z}.
\]

\[
\|(3.26)
\]

Since the almost analytic extension $\tilde{F}_{\chi,+}(z)$ satisfies

\[
|\partial_z \tilde{F}_{\chi,+}(z)| \leq C_N |Imz|^N (z)^{-3/2-N} \quad (4 N \in \mathbb{N}), \quad (3.27)
\]

we have $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ is bounded and inductively $[V_i, J''_1]$ is $H_0(\lambda)\text{-}compact$. So we only have to show the relatively compactness of $[V_i, J'_1]$. We have

\[
[V_i, J'_1] = [V_i, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta[V_i, F_+(Q_0^2)^2]. \quad (3.28)
\]

Clearly $[V_i, \hat{A}\beta]F_+(Q_0^2)$ is $H_0(\lambda)\text{-}compact$. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $(x)^{1+\delta}[V_i, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_i, J'_1]$. As for the commutator $[V_i, J_2], \ldots, [V_i, J_4]$ we also replace $F_\pm$ by $F_{\chi,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_i, J_2]$ and $[V_i, J_3]$ are almost the same. We only give the proof for $J_2$. We also estimate the 'principle' part before we compute the commutator with $V_i$.

\[
J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0 \quad (3.29)
\]
It is sufficient to show that $[V, \hat{A} F_+(Q_0^2) F_-(Q_0^2)]$ is a $H_0(\lambda)$-compact operator. We decompose it into the following sum.

$$
[V, \hat{A} F_+(Q_0^2) F_-(Q_0^2)] = \hat{A} [V, F_+(Q_0^2) F_-(Q_0^2)] + \hat{A} F_+(Q_0^2) F_-(Q_0^2) [V, Q_0].
$$

We can easily see that the first and the third term is relatively compact since $\langle x \rangle^{1+\delta} [V, Q_0]$ is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for $J_4$, the proof is similar. We rewrite it as

$$
\hat{A} F_-(Q_0^2)^2 Q_0^2 + [F_-(Q_0^2) Q_0, \hat{A}] F_-(Q_0^2) Q_0
$$

We can also obtain the relatively compactness by estimating the term $[V, \hat{A} F_-(Q_0^2)^2 Q_0^2]$.

**Corollary 3.7.** Let $V$ be a $4 \times 4$ Hermitian matrix. Suppose $V$ satisfies the condition in Lemma 3.6. Then the following limits

$$
R^\pm(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-\epsilon} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-\epsilon}
$$

exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_\mathbb{N} \cup \sigma_{pp}(H(\lambda)))$ and $R^\pm(\mu)$ are continuous with respect to $\mu$.

**References**


Department of Mathematics, Graduate School of Science, Osaka University Toyonaka, 560-0043, Japan

E-mail address: yokoyama@math.sci.osaka-u.ac.jp