LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. INTRODUCTION

The Dirac Hamiltonian with magnetic vector potential \( \mathbf{a} = (a_j(x))_{j=1,...,d} \) is expressed by the following form

\[
H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m \gamma_{d+1} + V,
\]

where \( P_j = \frac{i}{2} \partial_{x_j} \), \( V \) is a multiplication of an Hermitian matrix \( V(x) \). \( m \) is the mass of electron. The matrices \( \{\gamma_j\} \) satisfy the following relations

\[
\gamma_j \gamma_k + \gamma_k \gamma_j = 2 \delta_{jk} 1 \quad (j,k = 1, \ldots, d+1).
\]

Here \( \delta_{jk} \) is Kronecker's delta and \( 1 \) is an identity matrix. We assume that the speed of the light \( c = 1 \). When \( V \equiv 0 \), the square of \( H(\mathbf{a}) \) has the form

\[
H(\mathbf{a})^2 = \sum_{j=1}^{d} (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \leq j < k \leq d} b_{jk}(x) \gamma_j \gamma_k,
\]

where

\[
b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x).
\]

It is called Pauli's Hamiltonian. The skew symmetric matrix \( (b_{jk}(x)) \) is the magnetic field associated with \( \mathbf{a} \). We say the magnetic field is asymptotically constant if it satisfies the following conditions as \( |x| \to \infty \):

\[
b_{jk}(x) \to 3 \Lambda_{jk} \quad (1 \leq j,k \leq d),
\]

where \((\Lambda_{jk})_{j,k}\) is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for \( H(\mathbf{a}) \) with a constant magnetic field \( (b_{jk}(x)) \) and a long-range electric potential \( V(x) \) when \( d = 3 \). Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for \( d = 2,3 \). As can be inferred from (1.3), the spectrum of \( H(\mathbf{a}) \) is closely related
with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2} a_1(x) - \partial_{x_1} a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1(P_1 + \frac{\lambda}{2} x_2) + \sigma_2(P_2 - \frac{\lambda}{2} x_1) + m \sigma_3,$$  \hspace{1cm} (1.6)

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = (P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2 + m^2 - \lambda \sigma_3.$$ \hspace{1cm} (1.7)

The right hand side is a de-coupled 2 dimensional magnetic Schrödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \{ \pm \sqrt{2\lambda n + m^2} | n = 0, 1, 2 \ldots \}.$$ 

In fact we have

$$\sigma(h(\lambda)) = \{ \sqrt{2\lambda n + m^2}, -\sqrt{2\lambda(n+1) + m^2} | n = 0, 1, 2 \ldots \}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [8].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$a_0(x) = (-\lambda x_2 / 2, \lambda x_1 / 2, 0) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along $x_3$-axis:

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1(P_1 + \frac{\lambda x_2}{2}) + \alpha_2(P_2 - \frac{\lambda x_1}{2}) + \alpha_3 P_3 + m \beta,$$ \hspace{1cm} (1.8)

where $\{\alpha_j\}$ and $\beta$ are $4 \times 4$ Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ \hspace{1cm} (1.9)

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [8].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m \beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$ \hspace{1cm} (1.10)

with $D_0 = \sigma \cdot (P - a_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. 

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By using Foldy-Wouthuysen transform, explained in detail in the following section, \( H_0(\lambda) \) can be diagonalized by a unitary operator \( U_{FW} \).

\[
U_{FW} H_0(\lambda) U_{FW}^{-1} = \begin{pmatrix}
\sqrt{D_0^2 + m^2} & 0 \\
0 & -\sqrt{D_0^2 + m^2}
\end{pmatrix}.
\] (1.11)

From the commutation relation (1.2) we have

\[
D_0^2 = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 - \lambda \beta.
\] (1.12)

We can easily see that \( \sigma(D_0^2) = [0, \infty) \). So we have

\[
\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).
\]

Therefore in the 3 dimensional case, the spectrum of \( H_0(\lambda) \) is absolutely continuous.

Let us consider the perturbation of \( H_0(\lambda) \) : We put

\[
H(\lambda) = H_0(\lambda) + V.
\] (1.13)

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent \((z - H(\lambda))^{-1}\) on the real axis. As for the Schrödinger operator with constant magnetic field, Iwashita [4] shows the limiting absorption principle for long-range potential by using commutator method. In [4] the following self-adjoint operator is considered.

\[
\tilde{H} = (P_1 + \frac{\lambda x_2}{2})^2 + (P_2 - \frac{\lambda x_1}{2})^2 + P_3^2 + V(x).
\] (1.14)

The existence of the boundary values

\[
\langle x_3 \rangle^{-s} (\tilde{H} - \mu \mp i0)^{-1} \langle x_3 \rangle^{-s}
\]
is proved for \( s > 1/2 \) and \( \mu \in \mathbb{R} \setminus \{\lambda(2n+1) | n = 0, 1, 2, \ldots\} \cup \sigma_{pp}(\tilde{H}) \).

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as \( |x| \to \infty \). (See [2].) Hachem [3] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential \( V(x) \).

Roughly speaking, his assumption means that the absolute value of each components of \( V \) is dominated from above by \( C\langle x' \rangle^{-1-\epsilon} \langle x \rangle^{-\epsilon} (x' = (x_2, x_3)) \) for sufficiently large \( x \). We remark that \( \epsilon > 0 \) is used as a sufficiently small parameter throughout this paper. To be accurate, \( \langle x' \rangle^{1+\epsilon} V(x) \) is required to be a \( H_0(\lambda) \)-compact operator.

In this paper we treat directly the following operator

\[
H(\lambda) = \alpha_1(P_1 + \frac{\lambda}{2} x_2) + \alpha_2(P_2 - \frac{\lambda}{2} x_1) + \alpha_3 P_3 + m\beta + V(x),
\] (1.15)

where \( V(x) \) is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for \( V(x) \). In this case it
seems that an appropriate choice of the conjugate operator is
\[
\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle},
\]
which is inspired by [9], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [9]. Namely we rewrite $H_0(\lambda)$ by a direct integral and the conjugate operator $A$ acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. Conjugate Operator

Let us recall
\[
Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - a_0),
\]
with
\[
a_0(x) = (-\lambda x_2/2, \lambda x_1/2, 0).
\]
The Dirac Hamiltonian $Q_0 + m\beta$ can be diagonalized by sandwiching it between a unitary operator $U$ and $U^* = U^{-1}$. In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator $H_0(\lambda)$. Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians $H_0(\lambda)$ and $H(\lambda)$.

Let $Q_0$ be the self-adjoint operator as in (2.1) and $|Q_0| = \sqrt{Q_0^2}$, $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$. We define a unitary operator $U_{FW}$, which diagonalize $H_0(\lambda)$, in the following way.

**Definition 2.1.**

(i). At first we define a signature function associated with $Q_0$ by
\[
\text{sgn}Q_0 = \begin{cases} \frac{Q_0}{|Q_0|}, & \text{on } (\ker Q_0)^{\perp} \\ 0, & \text{on } \ker Q_0 \end{cases}
\]
We note that $\text{sgn}Q_0$ is isometry on $(\ker Q_0)^{\perp}$.

(ii). We can easily see that $m/|H_0(\lambda)| \leq 1$. So we denote the square root of $\frac{1}{2}(1 \pm \frac{m}{|H_0(\lambda)|})$ as $a_{\pm}$. i.e.
\[
a_{\pm} = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}.
\]

(iii). Combining these operators we define the operator $U_{FW}$ as
\[
U_{FW} = a_+ + \beta(\text{sgn}Q_0)a_-.
\]

**Lemma 2.2.**

(i). $U_{FW}$ is a unitary operator on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. Further,
\[
U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\text{sgn}Q_0)a_-.
\]
(ii). $H_0(\lambda)$ can be diagonalized by $U_{FW}$ as follows.

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)| \beta = \left( \begin{array}{cc} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{array} \right). \quad (2.7)$$

Proof. See 5.6.1 in [8].

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$ 

We rewrite (1.12) as follows.

$$D_0^2 = \left( \begin{array}{cc} D_- & 0 \\ 0 & D_+ \end{array} \right).$$

Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_\pm = (P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2 + P_3^2 \pm \lambda.$$

It is well-known that $(P_1 + \frac{\lambda}{2} x_2)^2 + (P_2 - \frac{\lambda}{2} x_1)^2$ has eigenvalues

$$\{ \lambda (2n + 1) | n = 0, 1, 2, \ldots \}.$$ 

We denote the eigenprojection on each eigenspace as $\Pi_n$. With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \left( \begin{array}{cc} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{array} \right), \quad (2.8)$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^{\infty} \left( \begin{array}{ccc} f(d_n) \otimes \Pi_n & & \\ & f(d_{n+1}) \otimes \Pi_n & \\ & & f(-d_n) \otimes \Pi_n \end{array} \right),$$

for any Borel function $f$.

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2} \left\{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \right\}. \quad (2.9)$$

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson’s commutator theorem [7].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A} \beta$ between $U_{FW}^{-1}$ and $U_{FW}$:

$$A = U_{FW}^{-1}(\hat{A} \beta) U_{FW}. \quad (2.10)$$

Before we show Mourre’s inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.
Suppose that $f \in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0 \in \mathbb{R}$.

$$|f^{(k)}(t)| \leq C_k (1 + |t|)^{m_0-k}, \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (2.11)$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R},$$

$$supp \tilde{f} \subset \{z; |Imz| \leq 1 + |Rez|\},$$

$$|\partial_z \tilde{f}(z)| \leq C_N |Imz|^N (z)^{m_0-1-N}, \quad \forall N \in \mathbb{N}. \quad (2.12)$$

Then for all $f$, satisfying (2.11) for $m_0 < 0$ and a self-adjoint operator $H$, we have

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z) (z - H)^{-1} dz \wedge d\overline{z}. \quad (2.13)$$

3. LIMITING ABSORPTION PRINCIPLE FOR LONG-RANGE POTENTIALS

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing $A$ defined in the previous section as the conjugate operator.

**Lemma 3.1.** Let $\mathbb{R}_N$ be the following discrete subset of $\mathbb{R}$

$$\mathbb{R}_N = \{\pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \ldots \} \subset \mathbb{R}. \quad (3.1)$$

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_N$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C_0^\infty(I)$

$$f(H_0(\lambda)) i[H_0(\lambda), A] f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \quad (3.2)$$

**Proof.** By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda)) i[\hat{H}_0(\lambda), \hat{A}] f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \quad (3.3)$$

We rewrite the commutator as follow.

$$i[\hat{H}_0(\lambda), \hat{A}] = \left( \begin{array}{c}
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \\
 i[\sqrt{D_0^2 + m^2}, \hat{A}] \end{array} \right). \quad (3.4)$$

We proceed the calculus more precisely to see that

$$i[\sqrt{D_0^2 + m^2}, \hat{A}] = \sum_{n=0}^{\infty} \left( i[d_n, \hat{A}] \otimes \Pi_n \right). \quad (3.5)$$

where $d_n = \sum_{\ell=-\infty}^{\infty} \{ z \in \mathbb{R} : |z| \leq n \}$.
by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

$$f(\tilde{H}_0(\lambda))i[\tilde{H}_0(\lambda), \hat{A} \beta]f(\tilde{H}_0(\lambda)) = \begin{pmatrix}
I_1 & I_2 \\
I_3 & I_4
\end{pmatrix}$$

(3.5)

where

$$I_1 = \sum_{n=0}^{\infty} f(d_n) i[d_n, \hat{A}] f(d_n) \otimes \Pi_n,$$

$$I_2 = \sum_{n=0}^{\infty} f(d_{n+1}) i[d_{n+1}, \hat{A}] f(d_{n+1}) \otimes \Pi_n,$$

$$I_3 = \sum_{n=0}^{\infty} f(-d_n) i[d_n, \hat{A}] f(-d_n) \otimes \Pi_n,$$

$$I_4 = \sum_{n=0}^{\infty} f(-d_{n+1}) i[d_{n+1}, \hat{A}] f(-d_{n+1}) \otimes \Pi_n.$$

We note that all the sum in $I_1, \cdots, I_4$ are finite since $f$ is a compactly supported function. By an elementary calculus, we have

$$i[d, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2}(P_3)} \quad (l \in \mathbb{N} \cup \{0\}).$$

(3.6)

Since $supp f \subset I \subset \mathbb{R} \setminus \mathbb{R}_N$, $P_3$ is away from zero when $P_3 \in supp f(d_l(P_3))$ or $P_3 \in supp f(-d_l(P_3))$. So there exist $C_l > 0$ such that

$$f(d_l) i[d_l, \hat{A}] f(d_l) \otimes \Pi_l \geq C_l f(d_l)^2 \otimes \Pi_l,$$

$$f(-d_l) i[d_l, \hat{A}] f(-d_l) \otimes \Pi_l \geq C_l f(-d_l)^2 \otimes \Pi_l.$$

Since only a finite number of $l = l_j$ ($j = 1, \ldots, N$) is concerned, we have (3.2) with $\alpha = \inf_{j=1, \ldots, N} C_{l_j}$. \hfill \Box

Now we give the assumption for the potential, which is necessary to Mourre's inequality associated to $H(\lambda)$. After that we give an example of $V$ satisfying this assumption. It consists of a sum of long-range part and short-range part. In our case short-range potential means $V(x) = O(\langle x \rangle^{-\epsilon} \langle x_3 \rangle^{-1-\epsilon})$ as $|x| \to \infty$. And long-range part is a multiplication of a real valued function $\varphi(x)$ such that $\varphi(x) = O(\langle x \rangle^{-\epsilon})$ as $|x| \to \infty$. More precisely we assume that $V$ satisfies the following.

**Assumption 3.2.** $V = V(x)$ is a multiplicative operator of a $4 \times 4$ Hermitian matrix satisfying the following properties.

(i). $V$ is a $H_0(\lambda)$-compact operator.

(ii). The form $[V, A]$ can be extended to a $H_0(\lambda)$-compact operator.
For example a $4 \times 4$ matrix $V(x)$ satisfying the following inequality is $H_0(\lambda)$-compact.

$$|V(x)| \leq C\langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3).$$

(3.7)

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre’s inequality for $H(\lambda)$.

**Lemma 3.3.** Suppose $V$ satisfies Assumption 3.2.

(i). We take $\mu \in \mathbb{R} \setminus \mathbb{R}_{\text{II}}$ and $\delta > 0$ so that the closed interval $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\text{II}}$. There exist $\alpha > 0$ and a compact operator $K$ such that the following inequality holds for all $f \in C_0^\infty(I)$.

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha f(H(\lambda))^2 + K. \quad (3.8)$$

(ii). There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R} \setminus \mathbb{R}_{\text{II}}$. For $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{II}} \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu - \delta_0, \mu + \delta_0])$.

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \geq \alpha_0 f(H(\lambda))^2. \quad (3.9)$$

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

**Theorem 3.4.** Suppose $V$ satisfies Assumption 3.2. Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{II}} \cup \sigma_{pp}(H(\lambda)))$, the following limits

$$R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s} \quad (3.10)$$

exist and $R^{\pm}(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{II}} \cup \sigma_{pp}(H(\lambda)))$.

**Sketch of proof**

From (3.9) and Theorem 2.2 in [5], we can see that the boundary value $(\hat{A})^{-s}(H(\lambda) - \mu \mp i0)^{-1} (\hat{A})^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\text{II}} \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient if we show the boundness of $(\hat{A})^{-s}(x_3)\langle x_3 \rangle^{-s}$. Since $(\hat{A})^{-s}(x_3)\langle x_3 \rangle^{-s}$ is bounded, it is sufficient to show $(x_3)^s U_{\text{FW}} \langle x_3 \rangle^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions.

Let $\chi(t) \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1/\sqrt{2} & (t > -m^2/3) \\ 0 & (t < -2m^2/3). \end{cases} \quad (3.11)$$

With this function we define $F_{\pm}(t)$ and $F_{\chi,\pm}$ as follows.

$$F_+(t) = \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}}$$

$$F_-(t) = \chi(t) \left( \sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}}$$

$$F_{\chi,+}(t) = F_+(t) - \chi(t)$$

$$F_{\chi,-}(t) = \sqrt{t + m^2} F_-(t) - \chi(t)$$
Then we can easily verify that
\[ a_+ = F_+(Q_0^2), \]
\[ a_- sgn Q_0 = F_-(Q_0^2)Q_0 = Q_0 F_-(Q_0^2). \]

As for the proof of \([Q_0, F_-(Q_0^2)] \equiv 0\), see 5.2.4 in [8]. By the construction of these functions, we can also see that \(F_{\chi, \pm}(t)\) satisfy (2.11) with \(m_0 < 0\). So we apply the functional calculus in section 2 to \(F_{\chi, \pm}(t)\) and see the following properties hold.

**Lemma 3.5.** Suppose \(0 \leq s \leq 2\) and \(z \in \mathbb{C} \setminus \mathbb{R}\). Then

(i). For \(0 < s \leq 1\), there exists \(C_s > 0\) such that
\[
||\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}||_H \leq C_s (|Imz|^{-1} + |Imz|^{-2} (z)).
\]
(ii). For \(1 < s \leq 2\), there exists \(C'_s > 0\) such that
\[
||\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s}||_H \leq C'_s (|Imz|^{-1} + |Imz|^{-2} (z) + |Imz|^{-3} (z)^2).
\]
(iii). \(\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s}\) and \(\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}\) are bounded operators.

**Proof.** For the proof of (i) and (ii), we use the resolvent equation. Suppose \(0 < s \leq 1\). Then
\[
\langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1} (Q_0^2 + 1)
\times (Q_0^2 + 1)^{-1} [(\langle x \rangle^s, Q_0^2)] (z - Q_0^2)^{-1} \langle x \rangle^{-s}.
\]
From the boundness of \((Q_0^2 + 1)^{-1} [(\langle x \rangle^s, Q_0^2)]\) and the following estimate
\[
|| (z - Q_0^2)^{-1} (Q_0^2 + 1) ||_H \leq C (|Imz|^{-1} (z) + 1),
\]
we obtain (i). As for the case \(1 < s \leq 2\), we rewrite the last term \((z - Q_0^2)^{-1} [(\langle x \rangle^s, Q_0^2)] (z - Q_0^2)^{-1} \langle x \rangle^{-s}\) as
\[
(z - Q_0^2)^{-1} (Q_0^2 + 1) (Q_0^2 + 1)^{-1} [(\langle x \rangle^s, Q_0^2)] \langle x \rangle^{-s+1}
\times \langle x \rangle^{-1} (z - Q_0^2)^{-1} \langle x \rangle^{-s+1} \langle x \rangle^{-1}.
\]
By using the result for \(0 < s \leq 1\), we have the inequality for \(1 < s \leq 2\). With these estimates, we prove (iii). Since \(\chi(Q_0^2) \equiv 1\), we can easily see that
\[
\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2) \langle x \rangle^{-s} + I.
\]
Since \(F_{\chi,+}(t)\) satisfies (2.12) for \(m_0 = -1/2\), \(F_{\chi,+}(Q_0^2)\) can be rewritten as follows,
\[
\frac{1}{2\pi i} \int_C \partial_z \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} dz \wedge d\overline{z}.
\]
From this formula and (i) (ii) we have
\[
|| \partial_z \tilde{F}_{\chi,+}(z) \langle x \rangle^s (z - Q_0^2)^{-1} \langle x \rangle^{-s} ||_H
\leq C |\partial_z \tilde{F}_{\chi,+}(z)| (|Imz|^{-1} + |Imz|^{-2} (z) + |Imz|^{-3} (z)^2).
From (2.12) we have
\[ \| \partial_{\xi} \tilde{F}_\chi + (z) \langle x \rangle^s (z - Q_0^2)^{-1}(x)^{-s} \|_H \leq C \langle \gamma \rangle^{-5/2}. \]
This implies the boundness of \( \langle x \rangle^s F_+ (Q_0^2)(x)^{-s} \).

In a similar way, we rewrite \( \langle x \rangle^s F_- (Q_0^2)Q_0(x)^{-s} \) as
\[
\langle x \rangle^s F_-(Q_0^2)(x)^{-s} \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}}(x)^{-s} + \langle x \rangle^s \frac{Q_0}{\sqrt{Q_0^2 + m^2}}(x)^{-s}.
\]
(3.21)

It is sufficient to show the boundness of \( \langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2}(x)^{-s} \). To see this, we denote \( \chi(t)/\sqrt{t + m^2} \in C^\infty(\mathbb{R}^3) \) as \( S(t) \) and its almost analytic extension as \( \tilde{S}(z) \). We can easily see that \( S(Q_0^2)(x)^s Q_0(x)^{-s} \) is bounded. So we obtain the boundness of \( \langle x \rangle^s F_- (Q_0^2)Q_0(x)^{-s} \) if we show that \( \langle x \rangle^s, S(Q_0^2)Q_0(x)^{-s} \) is bounded. We rewrite it as follows.
\[
\frac{1}{2ni} \int_C \partial_z \tilde{S}(z) \frac{Q_0^2 + 1}{z - Q_0^2}(Q_0^2 + 1)^{-1}(\langle x \rangle^s, Q_0^2)Q_0(x)^{-s}(z - Q_0^2)^{-1}\langle x \rangle^{-s}dz + d\bar{z}.
\]

By an elementary calculus, we have \( (Q_0^2 + 1)^{-1}(\langle x \rangle^s, Q_0^2)Q_0(x)^{-s} \) bounded. Combining (i) and (ii), we have
\[
\| \langle x \rangle^s, S(Q_0^2)Q_0(x)^{-s} \|_H \leq C \int_C |\partial_z \tilde{S}(z)| \{1 + |Imz|^{-1}\} \times \{|Imz|^{-1} + |Imz|^{-2}(z) + |Imz|^{-3}(z)^2\}dz \wedge d\bar{z} < \infty.
\]

This implies the boundness of \( \langle x \rangle^s F_- (Q_0^2)Q_0(x)^{-s} \).

Next we give an example of \( V \). It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.6.** Let \( V \) be a \( 4 \times 4 \) Hermitian matrix of the form
\[
V(x) = (v_{ij}(x)) + \varphi(x)I_4 \equiv V_s(x) + V_i(x)
\]
(3.22)
where \( V_s(x) = (v_{ij}(x)) \) is an Hermitian matrix and \( I_4 \) is an identity matrix. Suppose the following conditions hold. Then \( V(x) \) satisfies Assumption 3.2.

There exist \( \delta > 0 \) such that the following inequalities hold for all multi-index \( \alpha \).
\[
|\partial^\alpha v_{ij}(x)| \leq C_\alpha(x)^{-\delta - |\alpha|} (x_3)^{-1} \quad (1 \leq i, j \leq 4). \]
(3.23)
\( \varphi(x) \in C^\infty(\mathbb{R}^3) \) is real valued and satisfies
\[
|\partial^\alpha \varphi(x)| \leq C'_\alpha(x)^{-\delta - |\alpha|}.
\]
(3.24)

The relatively compactness of \( V(x) \) itself is clear since \( V \) satisfies (3.7). So we only have to show the relatively compactness of \( [V, A] \). We prove the relatively compactness of \( [V_s, A] = [V_s, U_{FW}^{-1}\hat{A}\beta U_{FW}] \) at first. From the boundness of \( \langle x_3 \rangle^{-1} \hat{A}\beta \) and the relatively compactness of \( V_s(x_3) \), it is sufficient to show that \( \langle x_3 \rangle U_{FW}^{-1} \hat{A}\beta U_{FW} \langle x_3 \rangle^{-1} \) and \( \langle x_3 \rangle U_{FW}^{-1} \hat{A}\beta U_{FW} \langle x_3 \rangle^{-1} \) are bounded operators in \( \mathbb{H} \). We have already proved it in Lemma 3.5.
Next we treat the long-range term. The conjugate operator $A$ can be decomposed into the sum of $J_1, \ldots, J_4$ where

$$J_1 = F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2),$$
$$J_2 = F_+(Q_0^2) \hat{A} \beta^2 F_-(-Q_0^2) Q_0,$$
$$J_3 = \beta F_-(-Q_0^2) Q_0 \hat{A} \beta F_+(Q_0^2),$$
$$J_4 = \beta F_-(-Q_0^2) Q_0 \hat{A} \beta^2 F_-(-Q_0^2) Q_0.$$

We prove that the $H_0(\lambda)$-compactness holds for each of $[V_i, J_1], \ldots, [V_i, J_4]$. To see this we use the functional calculus again and rewrite $J_1$ as follows.

$$F_+(Q_0^2) \hat{A} \beta F_+(Q_0^2) = \hat{A} \beta F_+(Q_0^2)^2 + [F_{x,+}(Q_0^2), \hat{A} \beta] F_+(Q_0^2) \equiv J'_1 + J''_1$$

At first we prove the boundness of $J''_1$ and consequently the relatively compactness of $[V_i, J''_1]$. By using (2.13), we rewrite $[F_{x,+}(Q_0^2), \hat{A} \beta]$ as follows.

$$\frac{1}{2 \pi i} \int_{C} \partial_{\bar{z}} \tilde{F}_{x,+}(z)(z - Q_0^2)^{-1} [Q_0^2, \hat{A} \beta](z - Q_0^2)^{-1} dz \land d \bar{z}. \quad (3.25)$$

From (3.16) we have $[Q_0^2, \hat{A} \beta](z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |Imz|\}$. So we have

$$\| [F_{x,+}(Q_0^2), \hat{A} \beta] \| \leq C \int_{C} |\partial_{\bar{z}} \tilde{F}_{x,+}(z)| \{ |Imz|^{-1} + |Imz|^{-2} \} dz \land d \bar{z}. \quad (3.26)$$

Since the almost analytic extension $\tilde{F}_{x,+}(z)$ satisfies

$$|\partial_{\bar{z}} \tilde{F}_{x,+}(z)| \leq C_N |Imz|^N \langle z \rangle^{-3/2 - N} \quad (\forall N \in \mathbb{N}), \quad (3.27)$$

we have $[F_{x,+}(Q_0^2), \hat{A} \beta]$ is bounded and inductively $[V_i, J''_1]$ is $H_0(\lambda)$-compact. So we only have to show the relatively compactness of $[V_i, J'_1]$.

$$[V_i, J'_1] = [V_i, \hat{A} \beta] F_+(Q_0^2)^2 + \hat{A} \beta [V_i, F_+(Q_0^2)^2]. \quad (3.28)$$

Clearly $[V_i, \hat{A} \beta] F_+(Q_0^2)$ is $H_0(\lambda)$-compact. Again we rewrite the commutator in the second term, by use of (2.13). Then we have $\langle x \rangle^{1+\delta} [V_i, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_i, J_1]$.

As for the commutator $[V_i, J_2], \ldots, [V_i, J_4]$ we also replace $F_{x, \pm}$ by $F_{x, \pm}$ and use the functional calculus. The proof of relatively compactness of $[V_i, J_2]$ and $[V_i, J_3]$ are almost the same. We only give the proof for $J_2$. We also estimate the 'principle' part before we compute the commutator with $V_i$.

$$J_2 = \hat{A} F_+(Q_0^2) F_-(-Q_0^2) Q_0 + [F_+(Q_0^2), \hat{A}] F_-(-Q_0^2) Q_0 \quad (3.29)$$
It is sufficient to show that $[V, \hat{A}F_+(Q_0^2)\hat{A}F_- (Q_0^2)Q_0]$ is a $H_0(\lambda)$-compact operator. We decompose it into the following sum.

$$
[V, \hat{A}]F_+(Q_0^2)\hat{A}F_- (Q_0^2)Q_0
+ \hat{A}[V, F_+(Q_0^2)\hat{A}F_- (Q_0^2)]Q_0
+ \hat{A}F_+(Q_0^2)\hat{A}F_- (Q_0^2)[V, Q_0].
$$

We can easily see that the first and the third term is relatively compact since $\langle x \rangle^{1+\delta}[V, Q_0]$ is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for $J_4$, the proof is similar. We rewrite it as

$$
\hat{A}F_- (Q_0^2)^2Q_0^2 + [F_- (Q_0^2)Q_0, \hat{A}]F_- (Q_0^2)Q_0
$$

(3.30)

We can also obtain the relatively compactness by estimating the term $[V, \hat{A}F_- (Q_0^2)^2Q_0^2]$.

**Corollary 3.7.** Let $V$ be a $4 \times 4$ Hermitian matrix. Suppose $V$ satisfies the condition in Lemma 3.6. Then the following limits

$$
R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-\epsilon} (H(\lambda) - \mu \mp i\epsilon)^{-1}\langle x_3 \rangle^{-\epsilon}
$$

(3.31)

eexist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_\mathbb{N} \cup \sigma_{pp}(H(\lambda)))$ and $R^{\pm}(\mu)$ are continuous with respect to $\mu$.

**REFERENCES**


