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Application of Multitype Dawson-Watanabe Superprocesses to PDEs*

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Abstract. We consider the long term asymptotic behaviors for solutions to a system of nonlinear partial differential equations. We prove that under some suitable conditions the unique global nonnegative solution has L^1-norm non-degeracy as the time t approaches to infinity. Our method is totally probabilistic, where a limit theorem for measure -valued processes is used.

1. Introduction

We consider the following system of nonlinear reaction diffusion equations with interaction terms:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = & \Delta_{\alpha_i} u_i + \sum_{j\neq i} a_{ij}(u_j - u_i) + b_i u_i^{\beta_i} \\ u_i(0,x) = & f_i(x), \qquad i = 1, 2, \cdots, k \quad \text{in} \quad \mathbf{R}^d, \quad t > 0, \end{cases} \tag{1}$$

where $\Delta_{\alpha_i} = -(-\Delta)^{\alpha_i/2}$, $0 < \alpha_i \leq 2$, and $a_{ij} > 0$, $b_i < 0$, $0 < \beta_i \leq 2$ for all i and j. We assume the additional condition between these parameters, so that we may discuss the asymptotic behaviors of solutions for the Cauchy problem (1) in terms of the corresponding stochastic processes.

$$\text{(Assumption)} \qquad a_{ii} + b_i \beta_i \geq 0 \qquad (\forall i) \tag{2}$$

It is well known that the Cauchy problem (1) has the unique global mild solution $u \equiv u(t,x)$ for the initial function $\mathbf{f} = (f_1, \cdots, f_k) \in C_p^k$, and that (1) has the unique global strong sense solution $u \equiv u(t,x) = (u_1(t,x), \cdots, u_k(t,x))$ for the initial functions $f_i \in \mathrm{Dom}(\Delta_{\alpha_i}) \subset C_p$ for all i, with $\mathbf{f} = (f_1, \cdots, f_k) \in C_p^k$, and also that if the initial data $f_i \geq 0$ for all i, then the solution satisfies $u_i(t,x) > 0$ for any i, ($1 \leq i \leq k$). Nemaly, the last assertion implies that the positivity is preserved for the system (1). Here the space C_p is a subset of $C(\mathbf{R}^d)$. As a matter of fact, for $p > d$ the reference function φ_p is defined by $\varphi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbf{R}^d$. The norm $\|\cdot\|_p$ id given by $\|f\|_p := \sup \{|f(x)/\varphi_p(x)|;\, x \in \mathbf{R}^d\}$.

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Then the sapce of p-continuous functions is given by $C_p = C(\mathbf{R}^d) \cap \{\|f\|_p < \infty\}$. Let $\mathcal{M}_p$ denote the space of nonnegative Radon measures on $\mathbf{R}^d$ such that $\langle \mu, \varphi_p \rangle < \infty$ holds for $\mu \in \mathcal{M}$ with the p-vague topology. Here $\langle \mu, f \rangle$ indicates the integral of measurable function f with respect to measure $d\mu$. The p-vague topology means the topology induced by the elements of $pC_c \cup \{\varphi_p\}$. Note that $\mathcal{M}_p$ is a Polish space, and also that C_p and $\mathcal{M}_p$ are in duality. It is interesting to note that the Lebesgue measure $\lambda = \lambda(dx)$ on $\mathbf{R}^d$ is in $\mathcal{M}_p$ for $p > d$.

The aim of this paper is to show the L^1-norm non-degeneracy of nonnegative solutions as the long time asymptotic behaviors of solutions for the system (1) by taking advantage of probability theory. Our peculiar feature of this paper is as follows. The problem (1) is deterministic and the result (asymptotic behavior) is also deterministic, while the method we adopt for the proof is based on the theory of stochastic processes. Our probabilistic approach to this sort of asymptotic analysis is greatly due to the recent works on measure-valued stochastic processes, especially, on some limit theorems for Dawson-Watanabe superprocesses (cf. [9],[10],[11]).

2. Branching Particle System

For simplicity we put $V_i = \sum_{j=1}^{k} a_{ij} > 0$, $m_{ij} := a_{ij}/V_i > 0$ for all $i, j = 1, 2, \cdots, k$. Moreover, set $c_i := -b_i/V_i > 0$ and $\beta_i' := \beta_i - 1$ ($0 < \beta_i' \leq 1$) for all i. Notice that c_i ranges over interval $(0, m_{ii}/(1+\beta_i')]$. In this section let us consider the branching particle system associated with problem (1). The system in question is supposed to consist of an aggregate of k distinct types of particles in $\mathbf{R}^d$ ($1 \leq k < \infty$). We denote by P_i the particle of i-type ($1 \leq i \leq k$). The particle P_i of i-type migrates according to a symmetric stable law in $\mathbf{R}^d$ with exponent $\alpha_i \in (0, 2]$, namely, P_i behaves as a symmetric α_i-stable process in $\mathbf{R}^d$. Each particle P_i has its own lifetime denoted by ζ_i, and the lifetime ζ_i is distributed as random variable in exponential distribution with parameter V_i, i.e. $\zeta_i \sim Ex(V_i)$, for each i. At the end of lifetime, we suppose that either case of the following two cases should occur.

(i) P_i mutates into another particle P_j $(j \neq i)$ with probability $m_{ij} > 0$.
(ii) It produces n-particles of type i with probability $m_{ii} \cdot p_n^{(i)}$ ($p_n^{(i)} > 0$).

It is very convenient to give the above branching mechanism of each type i with the following generating function $F_i(s)$. In fact, $F_i(s)$ is given by

$$\begin{aligned} F_i(s) &= s - \frac{b_i}{a_{ii}}(1-s)^{\beta_i} = s + \frac{c_i}{m_{ii}}(1-s)^{1+\beta_i'} \\ &= s - \frac{b_i}{V_i m_{ii}}(1-s)^{1+\beta_i'}, \end{aligned}$$

where $0 \leq s \leq 1$. Alternatively, we can also give the probabilities $(p_n^{(i)})_n$ directly. That is to say, for each i

$$p_n^{(i)} = \begin{cases} \dfrac{c_i}{m_{ii}} = -\dfrac{b_i}{a_{ii}} = -\dfrac{b_i}{V_i m_{ii}} & (n=0) \\ \dfrac{c_i}{m_{ii}}(1-\beta_i') = -\dfrac{b_i}{a_{ii}}(2-\beta_i) & (n=1) \\ \dfrac{c_i}{m_{ii}}(-1)^k \dbinom{1+\beta_i'}{k} = \dfrac{b_i}{a_{ii}}(-1)^{k+1}\dbinom{\beta_i}{k} & (n=2,3,\cdots). \end{cases} \tag{3}$$

The matrix $M = (m_{ij}) \in M(k \times k)$ is irreducible, implying that the particle can mutate from any type into any other type with positive probability. Note that under (3), the system is critical, that is, an average of one particle is produced. In addition we assume that at the initial time $t = 0$ the configuration of the particle P_i is described by a Poisson random measure with intensity measure $\gamma_i\lambda$, where $\gamma_i > 0$ $(\forall i)$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_k = 1$. Therefore, $(\gamma_1, \gamma_2, \cdots, \gamma_k)$ indicates the initial proportion of particles of each type. Suppose that migration, lifetime and initial configuration of the particles are mutually independent. We write

$$\mathbf{N} = \{\mathbf{N}(t),\, t > 0\} = \{(N_1(t), N_2(t), \cdots, N_k(t)), t > 0\},$$

where $N_i(t)$ is the random point measure on $\mathbf{R}^d$, defined by the location of the particle of type i at time t. Here notice that $\mathbf{EN}(0) = \mathbf{E}(N_1(0), \cdots, N_k(0)) = (\gamma_1\lambda, \cdots, \gamma_k\lambda) \equiv \Lambda$.

3. Renormalization Procedure and Dawson-Watanabe Superprocess

We consider a limit procedure for the branching particle system, named the renormalization procedure, or the short time high density limit. It is sufficient to describe the n-th step of renormalizing procedure. Actually we let the mass of particle be $1/n$, and change the lifetime parameter of the i type particle into $n^{\beta_i'} V_i$. In accordance with the above-mentioned change, we let the mutation probability change into $n^{-\beta_i'} m_{ij}$. The intensity of the initial Poisson configuration is changed from Λ into $n\Lambda$. Then we have the mass process

$$\mathbf{N}^{(n)} = \{\mathbf{N}^{(n)}(t)\} = \{(N_1^{(n)}(t), N_2^{(n)}(t), \cdots, N_k^{(n)}(t)), t > 0\}$$

instead of $\mathbf{N} = \{\mathbf{N}(t)\}$. Under the aforementioned limiting procedure the process $\mathbf{N}^{(n)}$ converges in distribution as $n \to \infty$ in the Skorohod space

$$D([0,\infty), \underbrace{\mathcal{M}_p \times \cdots \times \mathcal{M}_p}_{k \text{ times}})$$

to the $\mathcal{M}_p^k$-valued Markov process $\mathbf{X} = \{\mathbf{X}(t)\} = \{(X_1(t), \cdots, X_k(t)), t \geq 0\}$ with $\mathbf{X}(0) = \Lambda$ (cf. [10]). For any $\mu = (\mu_1, \cdots, \mu_k) \in \mathcal{M}_p^k$ and $\mathbf{f} = (f_1, \cdots, f_k) \in b\mathcal{B}(\mathbf{R}^d)$ we define

$$\langle \mu, \mathbf{f} \rangle_k = \sum_{i=1}^k \int f_i(x) d\mu_i(x).$$

The Laplace transition functional for measure-valued processes $\mathbf{X}$ is given by

$$\mathbf{E}_\mu \exp\{-\langle \mathbf{X}(t), \mathbf{f}\rangle_k\} = \exp\{-\langle \mu, \mathbf{u}(t)\rangle_k\}, \quad t > 0, \tag{4}$$

where $\mathbf{u}(t) = (u_1(t,x), \cdots, u_k(t,x)) \in C_p^k$ is the unique solution of the Cauchy problem (1). Then we call this $\mathbf{X} = \{\mathbf{X}(t), t > 0\}$ a k multitype $\{\alpha_i, d, \beta_i - 1, V_i\}$- Dawson-Watanabe superprocess.

4. Long Term Asymptotic Behavior

First of all we define the critical dimension d_c. Indeed, for our models d_c is given by $d_c := \min\{\alpha_i\text{: } 1 \le i \le k\} \ / \ \min\{\beta_i - 1\text{: } 1 \le i \le k\}$. For $d < d_c$ (resp. $d = d_c$), namely, for the subcritical (resp. critical) case, it is well known (cf. [9],[10],[11]) that there exists local extinction in the branching particle system $\mathbf{N}$, hence the measure-valued process $\mathbf{X}$ as its limit has also local extinction. Therefore, if we take into account the probabilistic representation of the solution $\mathbf{u}(t)$ for problem (1), i.e.,

$$\mathbf{u}(t) = -\log \mathbf{E}_\delta \exp\{-\langle \mathbf{X}(t), \mathbf{f}\rangle_k\}, \tag{5}$$

then as asymptotic behavior, clearly the solution proves to be degenerate as time t approaches to infinity. On the other hand, for the case $d > d_c$, that is, for the supercritical case, both systems $\mathbf{N}$ and $\mathbf{X}$ may possibly possess non-trivial equilibrium state as long time asymptotic behaviors [10], [11].

For simplicity, we set $\mathcal{M} := M_F(\mathbf{R}^d)$, which is the totality of finite measures on $\mathbf{R}^d$. Define the set D_m by

$$D_m := \{(x_1, \cdots, x_d) \in \mathbf{R}^d : \ 0 \le x_i < m, \quad \text{for} \quad i = 1, 2, \cdots, d\}.$$

We set $\tilde{N}_0^\gamma := \sum_{i=1}^k \gamma_i N_i(0) \in \mathcal{M}$ for $\mathbf{N}(0) = (N_1(0), \cdots, N_k(0)) \in \mathcal{M}^k$. Note that here we do not assume Poisson random configration for the initial state $\mathbf{N}(0)$. We are now in a position to state the main result in this paper.

Theorem 1. (L^1-Norm Non-Degeneracy of Unique Global Nonnegative Solution) *Let $d > d_c$. If the condition*

$$\lim_{m \to \infty} \mathbf{E} \left| \tilde{N}_0^\gamma (x - D_m) - m^d \right| = 0 \tag{6}$$

holds uniformly in $x \in \mathbf{R}^d$, then for any $\mathbf{f}(\neq 0) \in (C_c^+)^k$

$$\lim_{t \to \infty} \left\| u_i^{[\mathbf{f}]}(t) \right\|_{L^1} > 0. \tag{7}$$

Proof. Set $N(k) := \{1, 2, \cdots, k\} \subset \mathbf{N}$ and $S_k := \mathbf{R}^d \times N(k)$. We consider the basic process Z_t, which is S_k-valued Markov process whose generator A is given by the following unbounded operator on C_p^k :

$$(A\mathbf{f})_i(x) = \Delta_{\alpha_i} f_i(x) + V_i \sum_{j \neq i} m_{ij}(f_j(x) - f_i(x)). \tag{8}$$

Let T_t denote the correponding semigroup on C_p^k. Actually, for the measure

$$\nu(d(x,k)) := \left(\sum_{i=1}^{k} \mu_i \otimes \delta_i\right)(d(x_1,\cdots,x_d,k)) \qquad \text{on} \quad S_k,$$

T_t has an integral expression with respect to the transition probability. Namely,

$$\begin{aligned} T_t\nu(\tilde{\mathbf{y}}) &\equiv T_t\nu(\{y_k\}_{k=1}^d, j) \\ &= \int_{S_k} \nu(d\{x_k\}, d\{i\}) \mathbf{P}\left(Z_t \in d\tilde{\mathbf{y}} = d(\{y_k\}, j)/Z_0 = \tilde{\mathbf{x}} = (\{x_k\}, i) = (x_1,\cdots,x_d,i)\right). \end{aligned}$$

Lemma 2. *When $d > d_c$, then under* (6) *we have the convergence $T_t\mathrm{N}(0) \to \Lambda$ in $\mathcal{M}_p^k$.*

This is due to Dobrushin's asymptotically Poisson argument [1] (see also [12]). As a consequence, it is obvious that $T_t\mathrm{N}^{(n)}(0) \to \Lambda$ in $\mathcal{M}_p^k$ as $t \to \infty$. Repeating the renormalization procedure in §3, we readily deduce that $T_t\mathbf{X}(0)$ also converges to Λ in $\mathcal{M}_p^k$ as $t \to \infty$. So that, this convergence is valid even in the sense of probability as well. Thus we attain

Proposition 3. *If $d > d_c$ and $T_t\mathbf{X}(0) \to \Lambda$ in probability (as $t \to \infty$), then $\mathbf{X}(t)$ converges in distribution to the non-trivial equilibrium state $\mathbf{X}(0)$ as $t \to \infty$ and $\mathbf{EX}(\infty) = \Lambda$ holds.*

While, we have

$$\mathbf{u}(t) = T_t\mathbf{f} - \int_0^t T_{t-s}\mathbf{h}(s)ds, \tag{9}$$

where

$$\mathbf{h}(t,x) = (h_1(t,x),\cdots,h_i(t,x),\cdots,h_k(t,x)) = (\cdots, V_i c_i u_i^{1+\beta_i}(t,x),\cdots).$$

Hence, from Proposition 3, an easy computation leads to the following expression.

Lemma 4. *We have*

$$\mathbf{E}_\mu \exp\{-\langle \mathbf{X}(\infty), \mathbf{f}\rangle_k\} = \exp\left\{-\langle \Lambda, \mathbf{f}\rangle_k + \int_0^\infty \langle \Lambda, \mathbf{h}(s)\rangle_k ds\right\}.$$

By choosing special cases

$$\Lambda_i = (0,\cdots,0,\overset{i}{\check{\lambda}},0,\cdots,0) \quad \text{and} \quad \mu = (0,\cdots,0,\overset{i}{\check{\delta}}_x,0,\cdots,0),$$

we can proceed calculation to obtain

$$\begin{aligned} \lim_{t\to\infty} \|u_i^{[\mathbf{f}]}(t)\|_{L^1} &= \lim_t \langle \Lambda_i, u_i^{[\mathbf{f}]}(t)\rangle \\ &= \lim_t (-\log \mathbf{E}_{\Lambda_i} \exp\{-\langle \mathbf{X}(t), \mathbf{f}\rangle_k\}) \\ &= -\log \mathbf{E}_{\Lambda_i} \exp\{-\langle \mathbf{X}(\infty), \mathbf{f}\rangle_k\} > 0 \end{aligned}$$

where we employed Lemma 4. This completes the proof.

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