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MEASURE-VALUED IMMIGRATION PROCESSES AND KUZNETSOV PROCESSES

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Abstract. An immigration process associated with the Borel right Dawson-Watanabe superprocess does not always have a right continuous realization. We construct the immigration process by adding up measure-valued paths in the Kuznetsov process determined by an entrance rule for the superprocess. The path behavior of the Kuznetsov process is studied, which gives insights into trajectory structures of the immigration process. Some known results on excessive measures are interpreted probabilistically in terms of stationary immigration processes.

Key words: superprocess; immigration structure; skew convolution semigroup; entrance rule; excessive measure; Kuznetsov measure

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1. Introduction

Let E be a Lusin topological space, i.e., a homeomorphism of a Borel subset of a compact metric space, with the Borel σ-algebra $\mathcal{B}(E)$. Let $B(E)$ denote the set of bounded $\mathcal{B}(E)$-measurable functions on E, and $B(E)^+$ the subspace of $B(E)$ of non-negative functions. Denote by $M(E)$ the space of finite measures on $(E, \mathcal{B}(E))$ endowed with the topology of weak convergence. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f d\mu$. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a Borel right process in E with semigroup $(P_t)_{t\geq 0}$ and $\phi(\cdot, \cdot)$ is a branching mechanism given by

$$(1.1) \qquad \phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)m(x, \mathrm{d}u), \quad x \in E, z \geq 0,$$

where $b \in B(E)$, $c \in B(E)^+$ and $[u \wedge u^2]m(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. Then for each $f \in B(E)^+$ the evolution equation

$$(1.2) \qquad V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x), \quad t \geq 0, x \in E,$$

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has a unique solution $V_t f \in B(E)^+$, and there is a Markov semigroup $(Q_t)_{t\geq 0}$ on $M(E)$ such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(V_t f)\} \tag{1.3}$$

for all $t \geq 0$, $\mu \in M(E)$ and $f \in B(E)^+$. A Markov process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ having semigroup $(Q_t)_{t\geq 0}$ is called a Dawson-Watanabe *superprocess* with parameters (ξ, ϕ). Under our hypotheses, X has a Borel right realization; see Fitzsimmons (1988, 1992). The (ξ, ϕ)-superprocess is a special form of the measure-valued branching process (MB-process), which is the mathematical model for the evolution of a population in some region; see e.g. Dawson (1992, 1993).

If we consider a situation where there are some additional sources of population from which immigration into the region occurs during the evolution, we need to introduce a measure-valued branching process with immigration (MBI-process). This type of modification is familiar from the branching process literature; see e.g. Arthreya and Ney (1972), Dawson and Ivanoff (1978), Kawazu and Watanabe (1971) and Shiga (1990). A class of measure-valued immigration processes were formulated in Li (1996ab) as follows. Let $(N_t)_{t\geq 0}$ be a family of probability measures on $M(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ if

$$N_{r+t} = (N_r Q_t) * N_t, \quad r, t \geq 0, \tag{1.4}$$

where "$*$" denotes the convolution operation. The relation (1.4) holds if and only if

$$Q_t^N(\mu, \cdot) := Q_t(\mu, \cdot) * N_t, \quad t \geq 0, \ \mu \in M(E), \tag{1.5}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. If Y is a Markov process in $M(E)$ having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *MBI-process*, or simply an *immigration process*, associated with X. The intuitive meaning of the immigration process is clear from (1.5), that is, $Q_t(\mu, \cdot)$ is the distribution of descendants of the people distributed as $\mu \in M(E)$ at time zero and N_t is the distribution of descendants of the people immigrating to E during the time interval $(0, t]$. Clearly, (1.5) gives the general formulation for the immigration independent of the inner population.

Needless to say, most of the theory of Dawson-Watanabe superprocesses carries over to their associated immigration processes and could be developed by techniques very close to those in Dawson (1992, 1993). It is interesting, however, that the immigration processes have many additional structures, as might be expected from (1.4) and (1.5). The formula (1.5) is similar to the construction of Lévy's transition semigroup from the *usual* convolution semigroup. It is well-known that a convolution semigroup on the Euclidean space is uniquely determined by an infinitely divisible probability measure. It was proved in Li (1996a) that the skew convolution semigroup may be characterized in terms of an infinitely divisible probability entrance law. In this sense, the immigration process is a generalized form of the celebrated Lévy process. Other examples of the immigration process are squares of Bessel diffusions and radial parts of Ornstein-Uhlenbeck diffusions; see Kawazu and Watanabe (1971) and Shiga and

Watanabe (1973). The above formulation also includes new kinds of immigration processes which have not been studied before. The MBI-process involves more complicated trajectory structures — an immigration process associated with the Borel right (ξ, ϕ)-superprocess does not always have a right continuous realization. This wild behavior of the process is caused by the immigrants coming in from some boundary points of the underlying space E. For instance, if ξ is a minimal (absorbing barrier) Brownian motion in $(0, \infty)$, a non-right-continuous immigration process may be generated by cliques of immigrants with infinite mass entering from the origin; see section 4.

This work arose from some curiosity about the trajectory structures of the immigration processes. A natural and realistic problem one would raise is "For a given immigration process, what is the largest possible space where all the immigrants enter from?" The problem is answered rigorously in this paper using the theory of Kuznetsov processes developed in Dellacherie et al (1992), Fitzsimmons and Maisonneuve (1986) and Getoor (1990). We first construct a general immigration process by adding up measure-valued paths $\{w_t : \alpha < t < \beta\}$ in the Kuznetsov process determined by an entrance rule for the (ξ, ϕ)-superprocess, and then we study the behavior of $\{w_t : \alpha < t < \beta\}$ near the birth time $\alpha = \alpha(w)$. We show that almost all these paths start propagation in an extension E_D^T of the underlying space, but some of them may grow up at points in this space from the null measure. In some special cases, the infinitely divisible entrance law for the (ξ, ϕ)-superprocess corresponds to a σ-finite entrance law and the associated immigration process can be constructed easily using a path-valued Poisson random process whose characteristic measure is the Markov measure determined by the σ-finite entrance law. The construction has been proved useful in studying those special immigration processes; see e.g. Li (1996b), Li and Shiga (1995) and Shiga (1990). Our general construction is based on the observation that every skew convolution semigroup is determined by a continuous increasing path $(\gamma_t)_{t \geq 0}$ from $[0, \infty)$ to $M(E)$ and an entrance rule $(G_t)_{t \geq 0}$. This fact yields a natural decomposition of the immigration into two parts — the deterministic immigration part given by $(\gamma_t)_{t \geq 0}$ and the random immigration part determined by $(G_t)_{t \geq 0}$. Our construction gives some insights into the structures of the immigration process.

As an application of the construction, we give interpretations of some well-known results on excessive measures in terms of stationary immigration processes. The stationary distributions of immigration processes may be represented by excessive measures of the original superprocess. Corresponding to the Riesz type decomposition of an excessive measure into purely excessive and invariant parts the stationary immigration process is decomposed naturally into "purely immigrative" and "native" parts. A measure potential determines a right continuous immigration process and the process determined by a general excessive measure may be obtained as the increasing limit of a sequence of right continuous ones.

For simplicity we shall only work with measure-valued processes having state space $M(E)$. There is no much change when $M(E)$ is replaced by the more general space $M_\rho(E) := \{\text{Borel measures } \mu \text{ on } E \text{ satisfying } \mu(\rho) < \infty\}$, where ρ is some bounded, strictly positive, continuous function on E. Indeed, most of our results can be translated into the $M_\rho(E)$ space case using the mapping $\mu(\mathrm{d}x) \mapsto \rho(x)^{-1}\mu(\mathrm{d}x)$.

The paper is organized as follows. Section 2 contains some preliminaries. The construction of immigration processes using Kuznetsov processes is given in section 3. Almost sure behavior of the Kuznetsov processes are studied in section 4. In sections 5 we discuss stationary immigration processes determined by excessive measures. Fluctuation limits of immigration processes are studied in section 6.

2. Preliminaries

Recall that $M(E)$ is the space of finite Borel measures on the Lusin topological space E. It is well-known that $M(E)$ endowed with the weak convergence topology is also a Lusin space. Let $M(E)^\circ = M(E)\setminus\{0\}$, where 0 denotes the null measure. A probability measure F on $M(E)$ is infinitely divisible if and only if its Laplace functional has the following representation:

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} F(\mathrm{d}\nu) = \exp\left\{ -\eta(f) - \int_{M(E)^\circ} \left(1-\mathrm{e}^{-\nu(f)}\right) H(\mathrm{d}\nu)\right\}, \tag{2.1}$$

where $\eta \in M(E)$ and $[1\wedge\nu(E)]H(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. See e.g. Kallenberg (1975). We write $F = I(\eta, H)$ if F is determined by (2.1).

Suppose that $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ is a Borel right process in $M(E)$ with transition semigroup $(Q_t)_{t\geq 0}$. For $f \in B(E)^+$, set

$$V_t f(x) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \quad t \geq 0, x \in E, \tag{2.2}$$

where δ_x denote the unit mass concentrated at $x \in E$. Throughout this paper we assume that, for every $l \geq 0$ and $f \in B(E)^+$, the function $V_t f(x)$ of (t,x) restricted to $[0,l] \times E$ is bounded. We call X an (regular) *MB-process* if $(Q_t)_{t\geq 0}$ is determined by (1.3) with $(V_t)_{t\geq 0}$ being defined by (2.2). If this is satisfied, $Q_t(\mu, \cdot)$ is infinitely divisible for all $t \geq 0$ and $\mu \in M(E)$, and the operators $(V_t)_{t\geq 0}$ form a semigroup which is called the *cumulant semigroup* of X. See e.g. Silverstein (1969) and Watanabe (1968). The (ξ, ϕ)-superprocess defined in the introduction is a special form of the MB-process. Clearly, the associated skew convolution semigroups and immigration processes can also be introduced for a general MB-process.

THEOREM 2.1. (Li, 1996a) *The family of probability measures $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with the MB-process if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ such that*

$$\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \int_0^t \left[\log \int_{M(E)} \mathrm{e}^{-\nu(f)} K_s(\mathrm{d}\nu)\right] \mathrm{d}s \tag{2.3}$$

for all $t \geq 0$ and $f \in B(E)^+$.

Let $\mathcal{K}^1(Q)$ denote the set of probability entrance laws $K = (K_t)_{t>0}$ for the semigroup $(Q_t)_{t\geq 0}$ such that

$$\int_0^1 \mathrm{d}s \int_{M(E)^\circ} \nu(E) K_s(\mathrm{d}\nu) < \infty. \tag{2.4}$$

Let $\mathcal{K}(P)$ be the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for the semigroup $(P_t)_{t\geq 0}$ that satisfy $\int_0^1 \kappa_s(E)\mathrm{d}s < \infty$. For $\kappa \in \mathcal{K}(P)$, set

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y))\kappa_{t-s}(\mathrm{d}y), \quad t > 0. \tag{2.5}$$

Clearly, if $\kappa_t = \gamma P_t$ for some $\gamma \in M(E)$, then $S_t(\kappa, f) = \gamma(V_t f)$. Now we have the following

THEOREM 2.2. (Li, 1996b) *Suppose that $(Q_t)_{t\geq 0}$ is the semigroup of the (ξ,ϕ)-superprocess. Then $K \in \mathcal{K}^1(Q)$ is infinitely divisible if and only if its Laplace functional has the representation*

$$\begin{aligned}\int_{M(E)} &\mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) \\ &= \exp\Big\{ - S_t(\kappa, f) - \int_{\mathcal{K}(P)} (1 - \exp\{-S_t(\eta, f)\})\, F(\mathrm{d}\eta)\Big\},\end{aligned}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta) < \infty. \tag{2.6}$$

Let $(Q_t^\circ)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ to $M(E)^\circ$. Denote by $\mathcal{K}(Q^\circ)$ the set of entrance laws K for $(Q_t^\circ)_{t\geq 0}$ satisfying (2.4). We can also give a general characterization for $\mathcal{K}(Q^\circ)$ as follows. See also Dynkin (1989).

THEOREM 2.3. *Suppose that $(Q_t)_{t\geq 0}$ is the semigroup of the (ξ,ϕ)-superprocess. Then an arbitrary entrance law $H \in \mathcal{K}(Q^\circ)$ may be represented as*

$$\begin{aligned}\int_{M(E)^\circ} &\left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) \\ &= S_t(\kappa, f) + \int_{\mathcal{K}(P)} (1 - \exp\{-S_t(\eta, f)\})\, F(\mathrm{d}\eta)\end{aligned} \tag{2.7}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (2.6). If, in addition,

$$\int_a^\infty \left[\sup_{x\in E} |\phi(x,z)^{-1}|\right]\mathrm{d}z < \infty \tag{2.8}$$

for some constant $a > 0$, then (2.7) defines an entrance law $H \in \mathcal{K}(Q^\circ)$ for any $\kappa \in \mathcal{K}(P)$ and σ-finite measure F on $\mathcal{K}(P)$ satisfying (2.6).

PROOF: If $H \in \mathcal{K}(Q^\circ)$, then $(K)_{t>0} = I(0, H_t)_{t>0}$ defines an infinitely divisible probability entrance law $K \in \mathcal{K}^1(Q)$. Thus the representation (2.7) follows by Theorem 2.2. If (2.8) holds, there is a family of σ-finite measures $\{L_t(x,\cdot) : t > 0, x \in E\}$ on $M(E)^\circ$ such that $Q_t(\delta_x, \cdot) = I(0, L_t(x,\cdot))$; see Dawson (1993; pp195-196). Using this one can show that an arbitrary infinitely divisible probability entrance law $K \in \mathcal{K}^1(Q)$ may be given as $(K)_{t>0} = I(0, H_t)_{t>0}$ for some $H \in \mathcal{K}(Q^\circ)$. Then (2.7) defines the entrance law $H \in \mathcal{K}(Q^\circ)$ by Theorem 2.2. ▮

It follows by Theorems 2.1 and 2.2 that, under the first moment condition, the transition semigroup of a general immigration process associated with the (ξ, ϕ)-superprocess is given by

$$\begin{aligned}\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^N(\mu, \mathrm{d}\nu) = \exp\bigg\{ &- \mu(V_t f) - \int_0^t S_r(\kappa, f)\mathrm{d}r \\ &- \int_0^t \mathrm{d}r \int_{\mathcal{K}(P)} (1 - \exp\{-S_r(\eta, f)\})\, F(\mathrm{d}\eta)\bigg\},\end{aligned} \tag{2.9}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (2.6). Let us look at two classical examples of the immigration process; some other examples will be given in section 4.

Example 2.1. Let $a > 0$ and $d \geq 0$ be real constants. We consider the one-dimensional stochastic differential equation

$$\mathrm{d}Y_t = \sqrt{2a|Y_t|}\, \mathrm{d}B_t + d\mathrm{d}t, \tag{2.10}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion starting from zero. The equation defines a unique conservative diffusion process Y on $I\!R^+$ with generator $L^{a,d}$ such that

$$L^{a,d} f(x) = ax \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + d \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

and $\mathcal{D}(L^{a,d}) = C_0^2(I\!R^+)$, twice continuously differentiable functions on $I\!R^+$ vanishing at infinity. Indeed, Y is an MBI-process with the underlying space E degenerating to a single-point-set, which is a special kind of *continuous state branching process with immigration* (CBI-process); see e.g. Ikeda and Watanabe (1989; p235) and Kawazu and Watanabe (1971). Let $\{Y_t(d) : t \geq 0\}$ be the solution to (2.10) with $a = 2$. Then $\{Y_t(d)^{1/2} : t \geq 0\}$ is a *Bessel diffusion process* with parameter d. That is, the Bessel diffusion is essentially a particular case of the CBI-process. This connection between the Bessel diffusion and the immigration process was first noticed by Shiga and Watanabe (1973).

Example 2.2. Let us recall the Ray-Knight theorem on Brownian local times. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, B_t, \mathbf{P}_x)$ is a one dimensional Brownian motion with the local times $\{l(t,x) : t \geq 0, x \in I\!R\}$, which is a continuous two parameter process such that a.s.

$$2\int_A l(t,x)\mathrm{d}x = \int_0^t 1_A(B_s)\mathrm{d}s, \quad t \geq 0, A \in \mathcal{B}(I\!R).$$

For $b \geq 0$ and $\alpha \geq 0$, let $T_\alpha(-b) = \inf\{t >: l(t,-b) > \alpha\}$. Then $\{l(T_\alpha(-b), x) : x \in I\!R\}$ under $\mathbf{P}_0$ is an inhomogeneous Markov process with continuous paths and $l(T_\alpha(-b), -b) = \alpha$. There are three homogeneity intervals; $\{l(T_\alpha(-b), x) : x \geq 0\}$ and

$\{l(T_\alpha(-b), -b-x) : x \geq 0\}$ have the same generator $L^{1,0}$ and $\{l(T_\alpha(-b), -b+x) : 0 \leq x \leq b\}$ has the generator $L^{1,1}$. See e.g. Knight (1981; p137).

3. Construction of immigration processes

Let us review some facts in potential theory; see e.g. Dellacherie et al. (1992) and Getoor (1990). A family of σ-finite measures $(J_t)_{t\in I\!R}$ is called an *entrance rule* for $(Q_t^\circ)_{t\geq 0}$ if $J_s Q_{t-s}^\circ \leq J_t$ for $t > s \in I\!R$ and $J_s Q_{t-s}^\circ \uparrow J_t$ as $s \uparrow t$. An *entrance law* at $r \in I\!R$ is an entrance rule $(J_t)_{t\in I\!R}$ so that $J_t = 0$ for $t \leq r$ and $J_s Q_{t-s}^\circ = J_t$ for all $t > s > r$. (In particular, if $r = 0$ we simply say that $(J_t)_{t\in I\!R}$ an entrance law for $(Q_t)_{t\geq 0}$.) Let $W(M(E))$ denote the space of paths $\{w_t : t \in I\!R\}$ that are $M(E)^\circ$-valued and right continuous on an open interval $(\alpha(w), \beta(w))$ and take the value of the null measure elsewhere. The path $[0]$ constantly equal to 0 corresponds to (α, β) being empty. Set $\alpha([0]) = +\infty$ and $\beta([0]) = -\infty$. Let $(\mathcal{H}^\circ, \mathcal{H}_t^\circ)_{t\in I\!R}$ be the natural σ-algebras on $W(M(E))$ generated by the coordinate process. The shift operators $\{\sigma_t : t \in I\!R\}$ on $W(M(E))$ are defined by $\sigma_t w_s = w_{t+s}$.

To an entrance rule $(J_t)_{t\in I\!R}$ for $(Q_t^\circ)_{t\geq 0}$, there corresponds a unique σ-finite measure $\mathbf{Q}^J$ on $(W(M(E)), \mathcal{H}^\circ)$ under which the coordinate process $\{w_t : t \in I\!R\}$ is a Markov process with one-dimensional distributions $(J_t)_{t\in I\!R}$ and semigroup $(Q_t^\circ)_{t\geq 0}$. That is, for any $t_1 < \cdots < t_n \in I\!R$, and $\nu_1, \cdots, \nu_n \in M(E)^\circ$,

$$\begin{aligned} &\mathbf{Q}^J\{\alpha < t_1, w_{t_1} \in \mathrm{d}\nu_1, w_{t_2} \in \mathrm{d}\nu_2, \cdots, w_{t_n} \in \mathrm{d}\nu_n, t_n < \beta\} \\ &= J_{t_1}(\mathrm{d}\nu_1) Q_{t_2-t_1}^\circ(\nu_1, \mathrm{d}\nu_2) \cdots Q_{t_n-t_{n-1}}^\circ(\nu_{n-1}, \mathrm{d}\nu_n). \end{aligned} \tag{3.1}$$

The existence of this measure was proved by Kuznetsov (1974); see also Getoor and Glover (1987). The system $(W(M(E)), \mathcal{H}^\circ, \mathcal{H}_t^\circ, w_t, \mathbf{Q}^J)$ is now commonly called the *Kuznetsov process* determined by $(J_t)_{t\in I\!R}$, and $\mathbf{Q}^J$ is called the *Kuznetsov measure.* The entrance rule $(J_t)_{t\in I\!R}$ may be represented as

$$J_t = \int_{I\!R} J_t^s \rho(\mathrm{d}s), \quad t \in I\!R, \tag{3.2}$$

where $(J_t^s)_{t\in I\!R}$ is an entrance law at $s \in I\!R$ and $\rho(\mathrm{d}s)$ is a σ-finite measure on $I\!R$. The representation (3.2) yields

$$\mathbf{Q}^J(\mathrm{d}w) = \int_{I\!R} {}^s\mathbf{Q}(\mathrm{d}w)\rho(\mathrm{d}s), \tag{3.3}$$

where ${}^s\mathbf{Q}(\mathrm{d}w)$ is the Kuznetsov measure determined by $(J_t^s)_{t\in I\!R}$. See e.g. Getoor and Glover (1987). If $(J_t)_{t\in I\!R}$ is an entrance law at $r \in I\!R$, then $\mathbf{Q}^J$ is supported by $W_r(M(E))$, the subset of $W(M(E))$ comprising paths $\{w_t : t \in I\!R\}$ such that $\alpha(w) = r$. If F is an excessive measure for $(Q_t^\circ)_{t\geq 0}$ and $J_t \equiv F$, then $\mathbf{Q}^J$ is stationary, that is, $\mathbf{Q}^J \circ \sigma_t^{-1} = \mathbf{Q}^J$ for all $t \in I\!R$. Let $\mathcal{H}^J$ be the $\mathbf{Q}^J$-completion of $\mathcal{H}^\circ$ and let $\mathcal{H}_t^J$ be the σ-algebra generated by $\mathcal{H}_t^\circ$ and the ideal of $\mathbf{Q}^J$ null sets in $\mathcal{H}^J$. Note $\mathcal{H} = \cap\mathcal{H}^J$ and $\mathcal{H}_t = \cap\mathcal{H}_t^J$ where the intersection is over all excessive measures.

Now we take an entrance rule $(J_t)_{t\in\mathbb{R}}$ for $(Q_t^\circ)_{t\geq 0}$ and suppose that $N^J(\mathrm{d}w)$ is a Poisson random measure on $W(M(E))$ with intensity $\mathbf{Q}^J(\mathrm{d}w)$. Define

$$Y_t^J = \int_{W(M(E))} w_t\, N^J(\mathrm{d}w), \quad t \in \mathbb{R}. \tag{3.4}$$

PROPOSITION 3.1. *In the situation described above, $\{Y_t^J : t \in \mathbb{R}\}$ is a Markov process in $M(E)$ having one-dimensional distributions $I(0, J_t)_{t\in\mathbb{R}}$ and non-homogeneous transition semigroup $(R_t^r)_{r\leq t\in\mathbb{R}}$ given by*

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} R_t^r(\mu, \mathrm{d}\nu)\\ &= \exp\Big\{ -\mu(V_t f) - \int_{(r,t]}\int_{M(E)^\circ} \Big(1-\mathrm{e}^{-\nu(f)}\Big)\, J_t^s(\mathrm{d}\nu)\rho(\mathrm{d}s)\Big\}.\end{aligned}$$

PROOF: It is easy to see that $\{Y_t^J : t \in \mathbb{R}\}$ has one-dimensional distributions $I(0, J_t)_{t\in\mathbb{R}}$. By (3.3), for any $r < t \in \mathbb{R}$ and non-negative, bounded Borel function F on $M(E)$ with $F(0) = 0$,

$$\mathbf{Q}^J\{F(w_t); r < \alpha \leq t\} = \int_{(r,t]} J_t^s(F)\, \rho(\mathrm{d}s). \tag{3.5}$$

Then the desired results follow from (3.5) and the Markov property of $\mathbf{Q}^J$. ∎

From Theorem 2.1 it follows that, if $(N_t)_{t\geq 0}$ is a skew convolution semigroup, then $N_0 = \delta_0$ and each N_t is infinitely divisible. The next theorem shows that a general immigration process started with the null measure may be decomposed into two parts, one part is deterministic and the other part can be constructed from a Kuznetsov process.

THEOREM 3.2. *Suppose that $(N_t)_{t\geq 0}$ is a skew convolution semigroup with $N_t = I(\gamma_t, G_t)$. Define $G_t = 0$ for $t < 0$. Then $(G_t)_{t\in\mathbb{R}}$ is an entrance rule for $(Q_t^\circ)_{t\geq 0}$. Let Y_t^G be given by (3.4) with $J = G$ and let $Y_t = \gamma_t + Y_t^G$. Then $\{Y_t : t \geq 0\}$ is an immigration process with one-dimensional distributions $(N_t)_{t\geq 0}$ and transition semigroup $(Q_t^N)_{t\geq 0}$.*

PROOF: Recall that $Q_t(\mu, \cdot)$ is an infinitely divisible probability measure on $M(E)$ for all $t \geq 0$ and $\mu \in M(E)$. Suppose $Q_t(\delta_x, \cdot) = I(\lambda_t(x, \cdot), L_t(x, \cdot))$. By (1.4) we have for $t > r > 0$

$$G_t = G_{t-r} + G_r Q_{t-r}^\circ + \int_E \gamma_r(\mathrm{d}x) L_{t-r}(x, \cdot),$$

and hence $G_r Q_{t-r} \leq G_t$. Suppose $(N_t)_{t\geq 0}$ is given by (2.3) with $K_t = I(\eta_t, H_t)$. Since

$$G_t Q_{t-r}^\circ = \int_0^t H_s Q_{t-r}^\circ \mathrm{d}s = G_r Q_{t-r}^\circ + \int_r^t H_s Q_{t-r}^\circ \mathrm{d}s,$$

we have $G_r Q^\circ_{t-r} \uparrow G_t$ as $r \uparrow t$. Thus $(G_t)_{t\in\mathbb{R}}$ is an entrance rule. Suppose $(G_t)_{t\in\mathbb{R}}$ is represented by (3.2) with G^s_t in place of J^s_t. Then for $t \geq r \geq 0$,

$$\begin{aligned}\int_{(r,t]} G^s_t\,\rho(\mathrm{d}s) &= \int_{(0,t]} G^s_t\,\rho(\mathrm{d}s) - \int_{(0,r]} G^s_r Q^\circ_{t-r}\rho(\mathrm{d}s)\\ &= G_t - G_r Q^\circ_{t-r} = \int_0^t H_s\mathrm{d}s - \int_0^r H_s Q^\circ_{t-r}\mathrm{d}s.\end{aligned}\tag{3.6}$$

The relation $K_{s+t} = K_s Q_t$ yields

$$\eta_{s+t} = \int_E \eta_s(\mathrm{d}x)\lambda_t(x,\cdot), \quad H_{s+t} = \int_E \eta_s(\mathrm{d}x)L_t(x,\cdot) + H_s Q^\circ_t.\tag{3.7}$$

It follows that

$$\int_0^r H_{s+t-r}\mathrm{d}s - \int_0^r H_s Q^\circ_{t-r}\mathrm{d}s = \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x)L_{t-r}(x,\cdot).$$

Substituting this into (3.6) gives

$$\begin{aligned}\int_{(r,t]} G^s_t\,\rho(\mathrm{d}s) &= \int_0^{t-r} H_s\mathrm{d}s + \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x)L_{t-r}(x,\cdot)\\ &= G_{t-r} + \int_E \gamma_r(\mathrm{d}x)L_{t-r}(x,\cdot).\end{aligned}\tag{3.8}$$

Since $\{\gamma_t : t \geq 0\}$ is deterministic, $\{Y_t : t \geq 0\}$ is a Markov process with one-dimensional distributions $(N_t)_{t\geq 0}$. By Proposition 3.1 we have

$$\begin{aligned}&\mathbf{E}\left[\exp\{-Y_t(f)\}\big|Y_s : 0 \leq s \leq r\right]\\ &= \exp\bigg\{ - Y^G_r(V_{t-r}f) - \gamma_t(f)\\ &\qquad - \int_{(r,t]} \rho(\mathrm{d}s) \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) G^s_t(\mathrm{d}\nu)\bigg\}.\end{aligned}\tag{3.9}$$

Then we appeal (3.7) to see that

$$\begin{aligned}\gamma_t &= \int_0^{t-r} \eta_s\mathrm{d}s + \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x)\lambda_{t-r}(x,\cdot)\\ &= \gamma_{t-r} + \int_E \gamma_r(\mathrm{d}x)\lambda_{t-r}(x,\cdot).\end{aligned}\tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we get

$$\begin{aligned}&\mathbf{E}\left[\exp\{-Y_t(f)\}\big|Y_s : 0 \leq s \leq r\right]\\ &= \exp\bigg\{ - Y_r(V_{t-r}f) - \gamma_{t-r}(f) - \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) G_{t-r}(\mathrm{d}\nu)\bigg\},\end{aligned}$$

that is, $\{Y_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t^N)_{t\geq 0}$. The theorem is proved. ∎

Let $H \in \mathcal{K}(Q^\circ)$ and let $\mathbf{Q}^H$ be corresponding the Kuznetsov measure on $W(M(E))$. Set $G_t^H = \int_0^t H_s \mathrm{d}s$. If $N(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0,\infty) \times W(M(E))$ with intensity $\mathrm{d}s \times \mathbf{Q}^H(\mathrm{d}w)$, then

$$Y_t = \int_{[0,t]} \int_{W_0(M(E))} w_{t-s} N(\mathrm{d}s, \mathrm{d}w), \quad t \geq 0, \tag{3.11}$$

defines an immigration process corresponding to the skew convolution semigroup $(N_t)_{t\geq 0}$ with $N_t = I(0, G_t^H)$. This type of constructions for immigration processes have been considered in Li (1996b), Li and Shiga (1995) and Shiga (1990). We mention that the general immigration process can only be constructed as in Theorem 3.2 not in the form (3.11). Another related work is Evans (1993), where a conditioned (ξ,ϕ)-superprocess was constructed by adding up masses thrown off by an "immortal particle" moving around as a copy of ξ.

The construction using Kuznetsov process makes it possible to generalize some existing results for (ξ,ϕ)-superprocess to the immigration process. As an example let us give a characterization for the "weighted occupation time" of the immigration process. For simplicity we only consider a special case. Recall that if X is a (ξ,ϕ)-superprocess, for any $\mu \in M(E)$ and $f, g \in B(E)^+$ we have

$$\mathbf{Q}_\mu \exp\bigg\{ - X_t(f) - \int_0^t X_s(g)\mathrm{d}s \bigg\} = \exp\{-\mu(V_t(f,g))\}, \tag{3.12}$$

where $V_t(f,g)(x) \equiv u_t(x)$ is the unique bounded, positive solution to

$$u_t(x) + \int_0^t \mathrm{d}s \int_E \phi(x, u_s(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x) + \int_0^t P_s g(x)\mathrm{d}s. \tag{3.13}$$

See e.g. Fitzsimmons (1988) and Iscoe (1986). The formulas (3.12) and (3.13) characterize the joint distribution of X_t and the weighted occupation time $\int_0^t X_s \mathrm{d}s$. By (2.9) we know that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\bigg\{ - \mu(V_t f) - \int_0^t S_r(\kappa, f)\mathrm{d}r \bigg\} \tag{3.14}$$

defines the transition semigroup $(Q_t^\kappa)_{t\geq 0}$ of an immigration process associated with the (ξ,ϕ)-superprocess. Let $h = \int_0^1 P_s 1 \mathrm{d}s \in B(E)^+$. From Li (1996b) we know that $(Q_t^\kappa)_{t\geq 0}$ has a realization $(W, \mathcal{G}, \mathcal{G}_t, Y_t, \mathbf{Q}_\mu^\kappa)$ such that for any $g \in B(E)^+$ the path $\{Y_t(g \wedge h) : t \geq 0\}$ is a.s. measurable and locally bounded, hence $\int_0^t Y_s(g)\mathrm{d}s$ can be defined a.s. by increasing limits.

THEOREM 3.3. *Suppose that condition (2.8) holds. Let $(W, \mathcal{G}, \mathcal{G}_t, Y_t, \mathbf{Q}_\mu^\kappa)$ be the realization of $(Q_t^\kappa)_{t\geq 0}$ described above. Then for any $\mu \in M(E)$ and $f, g \in B(E)^+$,*

$$\mathbf{Q}_\mu^\kappa \exp\Big\{ -Y_t(f) - \int_0^t Y_s(g)\mathrm{d}s\Big\} = \exp\Big\{ -\mu(u_t) - \int_0^t S_r(\kappa, f, g)\mathrm{d}r\Big\},$$

where $u_t(x)$ is defined by (3.13) and

$$S_t(\kappa, f, g) = \kappa_t(f) + \int_0^t \kappa_s(g)\mathrm{d}s - \int_0^t \kappa_{t-s}(\phi(u_s))\mathrm{d}s, \quad t > 0. \tag{3.15}$$

PROOF: Under the hypothesis, (4.9) defines an entrance law $L\kappa \in \mathcal{K}(Q^\circ)$. For any $t > 0$ and $f, g \in B(E)^+$,

$$\begin{aligned}
&\mathbf{Q}^{L\kappa}\Big(1 - \exp\Big\{ -w_t(f) - \int_0^t w_s(g)\mathrm{d}s\Big\}\Big)\\
=&\lim_{r\downarrow 0} \mathbf{Q}^{L\kappa}\Big(1 - \mathbf{Q}_{w_r}\exp\Big\{ -X_{t-r}(f) - \int_0^{t-r} X_s(g)\mathrm{d}s\Big\}\Big)\\
=&\lim_{r\downarrow 0} \mathbf{Q}^{L\kappa}\left(1 - \exp\{-w_r(u_{t-r})\}\right)\\
=&S_t(\kappa, f, g),
\end{aligned}$$

where we have appealed (4.9), (3.13) and (3.15) for the last equality. Then using the construction (3.11) we get

$$\begin{aligned}
&\mathbf{Q}_0^\kappa \exp\Big\{ -Y_t(f) - \int_0^t Y_s(g)\mathrm{d}s\Big\}\\
=&\exp\Big\{ -\int_0^t \mathbf{Q}^{L\kappa}\Big(1 - \exp\Big\{ -w_{t-r}(f) - \int_r^t w_{s-r}(g)\mathrm{d}s\Big\}\Big)\mathrm{d}r\Big\}\\
=&\exp\Big\{ -\int_0^t S_{t-r}(\kappa, f, g)\mathrm{d}r\Big\},
\end{aligned}$$

and the desired result follows by the relation $\mathbf{Q}_\mu^\kappa = \mathbf{Q}_\mu * \mathbf{Q}_0^\kappa$. ∎

4. Almost sure behavior of Kuznetsov processes

In this section we study the behavior of Kuznetsov processes near their birth times. The discussion is of interest in providing insights into the trajectory structures of the immigration process. In particular, this will answer the problem posed in the introduction. Let us consider a (ξ, ϕ)-superprocess X in $M(E)$. We shall need to consider two topologies on the space E: the original topology and the Ray topology of ξ. We write E_r for the set E furnished with the Ray topology of ξ. The notation $M(E_r)$ is self-explanatory.

Recall that if $H \in \mathcal{K}(Q^\circ)$, the Kuznetsov measure $\mathbf{Q}^H$ is supported by $W_0(M(E))$. Let $(P_t^b)_{t\geq 0}$ be the semigroup of bounded kernels on E defined by

$$P_t^b f(x) = \mathbf{P}_x f(\xi_t) \exp\left\{ - \int_0^t b(\xi_s) \mathrm{d}s \right\}. \tag{4.1}$$

For any $H \in \mathcal{K}(Q^\circ)$, it is clear that

$$\gamma_t(f) = \int_{M(E)^\circ} \nu(f) H_t(\mathrm{d}\nu) \tag{4.2}$$

defines an entrance law $\gamma = (\gamma_t)_{t>0}$ for $(P_t^b)_{t\geq 0}$. Now fix $x \in E$ and suppose

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) L_t(x, \mathrm{d}\nu) = V_t f(x) \tag{4.3}$$

defines an entrance law $L(x) \in \mathcal{K}(Q^\circ)$. Clearly, $(P_t^b(x, \cdot))_{t>0}$ is a minimal entrance law for $(P_t^b)_{t\geq 0}$, which may be given by (4.2) with $H_t(\mathrm{d}\nu)$ replaced by $L_t(x, \mathrm{d}\nu)$. From those facts it can be deduced easily that $L(x) \in \mathcal{K}(Q^\circ)$ is minimal.

THEOREM 4.1. *Let $\mathbf{Q}^{L(x)}$ denote the Kuznetsov measure on $W(M(E))$ determined by $L(x) \in \mathcal{K}(Q^\circ)$. Then we have $w_t(E) \to 0$ and $w_t(E)^{-1} w_t \to \delta_x$ in $M(E_r)$ as $t \downarrow 0$ for $\mathbf{Q}^{L(x)}$-a.a. paths $w \in W(M(E))$.*

PROOF: The results were proved in Li and Shiga (1995) for the case where $(P_t)_{t\geq 0}$ is Feller and $\phi(x, z) \equiv z^2/2$ by a theorem of Perkins (1992) which asserts that a conditioned (ξ, ϕ)-superprocess is a generalized Fleming-Viot superprocess. The calculations in Li and Shiga (1995) are complicated and cannot be generalized to the present situation. We here give a proof of the theorem based on an h-transform of the (ξ, ϕ)-superprocess. We shall assume $(P_t)_{t\geq 0}$ is conservative. The proof for a non-conservative underlying semigroup can be reduced to this case as in Li and Shiga (1995).

Let $\mathcal{R}$ be a countable Ray cone for ξ as constructed in Sharpe (1988) and let $\bar{E}$ be the corresponding Ray-Knight compactification of E with the Ray topology. Note that each $f \in \mathcal{R}$ is continuous on E_r and admits a unique continuous extension $\bar{f}$ to $\bar{E}$. We regard $M(E_r)$ as a topological subspace of $M(\bar{E})$ in the usual way. Since $\bar{E}$ is a compact metric space, $M(\bar{E})$ is locally compact and separable. For any fixed $u > 0$,

$$R_t^r(\mu, \mathrm{d}\nu) = \mu(P_{u-r}^b 1)^{-1} \nu(P_{u-t}^b 1) Q_{t-r}(\mu, \mathrm{d}\nu), \quad 0 \leq r \leq t \leq u, \tag{4.4}$$

defines an inhomogeneous transition semigroup (R_t^r) on $M(E)^\circ$. We define the probability measure $\mathbf{R}_u^{L(x)}(\mathrm{d}w)$ on $W(M(E))$ by

$$\mathbf{R}_u^{L(x)}(\mathrm{d}w) = P_u^b 1(x)^{-1} w_u(1) \mathbf{Q}^{L(x)}(\mathrm{d}w).$$

Then $\{w_t : 0 < t \leq u\}$ under $\mathbf{R}_u^{L(x)}$ is a Markov process with semigroup (R_t^r) and one-dimensional distributions

$$H_t(x, \mathrm{d}\nu) := P_u^b 1(x)^{-1} \nu(P_{u-t}^b 1) L_t(x, \mathrm{d}\nu), \quad 0 < t \leq u. \tag{4.5}$$

Since $L(x) \in \mathcal{K}(Q^\circ)$ is minimal, $(H_t(x,\cdot))_{0<t\le u}$ is a minimal (probability) entrance law for (R_t^r). Take $f \in \mathcal{R}$. By (4.3) – (4.5) and the martingale convergence theorem we have $\mathbf{R}_u^{L(x)}$-a.s.

$$\begin{aligned} V_t f(x) &= \int_{M(E)^\circ} \left(1-\mathrm{e}^{-\nu(f)}\right) \nu(P_{u-t}^b 1)^{-1} H_t(x,\mathrm{d}\nu) P_u^b 1(x) \\ &= \lim_{r\downarrow 0} \int_{M(E)^\circ} \left(1-\mathrm{e}^{-\nu(f)}\right) \nu(P_{u-t}^b 1)^{-1} R_t^r(w_r,\mathrm{d}\nu) P_u^b 1(x) \\ &= \lim_{r\downarrow 0} w_r(P_{u-r}^b 1)^{-1} \left(1-\exp\{-w_r(V_{t-r}f)\}\right) P_u^b 1(x). \end{aligned} \tag{4.6}$$

By (4.1) and (4.6) it follows that $\mathbf{R}_u^{L(x)}$-a.s.

$$V_t f(x) \le \liminf_{r\downarrow 0} w_r(P_{u-r}^b 1)^{-1} P_u^b 1(x) \le \liminf_{r\downarrow 0} \mathrm{e}^{2\|b\|u} w_r(1)^{-1}.$$

Note that $V_t f(x)$ is right continuous in $t \ge 0$. Then letting $t \downarrow 0$ and $f \uparrow \infty$ in the above inequality yields that $\mathbf{R}_u^{L(x)}$-a.s. $w_t(1) \to 0$ as $t \downarrow 0$. Since for each $u > 0$ the measures $\mathbf{R}_u^{L(x)}$ and $\mathbf{Q}^{L(x)}$ are mutually absolutely continuous on $\{w \in W_0(M(E)) : w_u(1) > 0\}$, we obtain the first assertion. By the same reasoning as (4.6) we have $\mathbf{R}_t^{L(x)}$-a.s.

$$\begin{aligned} P_t^b f(x) &= \int_{M(E)^\circ} \nu(f)\nu(1)^{-1} H_t(x,\mathrm{d}\nu) P_t^b 1(x) \\ &= \lim_{r\downarrow 0} w_r(P_{t-r}^b 1)^{-1} w_r(P_{t-r}^b f) P_t^b 1(x). \end{aligned} \tag{4.7}$$

Clearly, $\mathbf{R}_u^{L(x)}$ is absolutely continuous relative to $\mathbf{R}_t^{L(x)}$ for $u \ge t > 0$. Since $f \in \mathcal{R}$ is an α-excessive function for $(P_t)_{t\ge 0}$ for some $\alpha = \alpha(f) \ge 0$, from (4.1) and (4.7) it follows that $\mathbf{R}_u^{L(x)}$-a.s.

$$\mathrm{e}^{-\|b\|t} P_t f(x) \le \liminf_{r\downarrow 0} \mathrm{e}^{(3\|b\|+\alpha)t} w_r(1)^{-1} w_r(f).$$

Take $w \in W_0(M(E))$ along which the above inequality holds for all $f \in \mathcal{R}$ and all rationals $t \in (0,u]$. Let $r_k = r_k(w)$ be a sequence such that $r_k \downarrow 0$ and $w_{r_k}(1)^{-1} w_{r_k} \to \hat{w}_0$ in $M(\bar{E})$ as $k \to \infty$, where $\hat{w}_0$ is a probability measure on $\bar{E}$. Then we have

$$\mathrm{e}^{-\|b\|t} P_t f(x) \le \mathrm{e}^{(3\|b\|+\alpha)t} \hat{w}_0(\bar{f}).$$

Letting $t \downarrow 0$ gives $f(x) \le \hat{w}_0(\bar{f})$, so we have $\hat{w}_0 = \delta_x$. Those clearly imply $w_r(1)^{-1} w_r \to \delta_x$ in $M(E_r)$ as $r \downarrow 0$, and the second assertion follows immediately. ∎

Now we consider an h-transform of the underlying semigroup. Set $h(x) = \int_0^1 P_s 1(x)\mathrm{d}s$ for $x \in E$. Since $h \in B(E)^+$ is an excessive function for $(P_t)_{t\ge 0}$,

$$T_t f(x) = h(x)^{-1} \int_E f(y) h(y) P_t(x,\mathrm{d}y) \tag{4.8}$$

defines a Borel right semigroup $(T_t)_{t\geq 0}$ on E. See e.g. Sharpe (1988). Let $(T_t^\partial)_{t\geq 0}$ be a conservative extension of $(T_t)_{t\geq 0}$ to $E^\partial := E \cup \{\partial\}$, where ∂ is the cemetery point. Let E_D^∂ denote the entrance space of $(T_t^\partial)_{t\geq 0}$ with the Ray topology. Let $E_D^T = E_D^\partial \setminus \{\partial\}$ and let $(\bar{T}_t)_{t\geq 0}$ be the Ray extension of $(T_t^\partial)_{t\geq 0}$ to E_D^T. Then $(\bar{T}_t)_{t\geq 0}$ is also a Borel right semigroups.

Let $\kappa \in \mathcal{K}(P)$ be non-trivial and assume

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = S_t(\kappa, f) \tag{4.9}$$

defines an entrance law $H := L\kappa \in \mathcal{K}(Q^\circ)$. Let $\mathbf{Q}^{L\kappa}$ denote the corresponding Kuznetsov measure on $W(M(E))$. Then we have

THEOREM 4.2. *For $w \in W(M(E))$ define the $M(E_D^T)$-valued path $\{h\bar{w}_t : t \in \mathbb{R}\}$ by*

$$h\bar{w}_t(E_D^T \setminus E) = 0 \text{ and } h\bar{w}_t(\mathrm{d}x) = h(x)w_t(\mathrm{d}x) \text{ for } x \in E. \tag{4.10}$$

Then for $\mathbf{Q}^{L\kappa}$-a.a. $w \in W(M(E))$, $\{h\bar{w}_t : t > 0\}$ is right continuous in the topology of $M(E_D^T)$ and $h\bar{w}_t \to 0$ as $t \downarrow 0$. Moreover, for $\mathbf{Q}^{L\kappa}$-a.a. $w \in W(M(E))$ we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some $x(w) \in E_D^T$ as $t \downarrow 0$.

PROOF: By the results in Fitzsimmons (1988), if $f \in B(E)$ is finely continuous relative to $(P_t)_{t\geq 0}$, then $\{w_t(f) : t > 0\}$ is right continuous for a.a. $w \in W(M(E))$. Since the excessive function $h \in B(E)^+$ is finely continuous, so is fh for any bounded continuous function f on E. It follows that $\{hw_t : t > 0\}$ is right continuous for a.a. $w \in W(M(E))$. We may define a cumulant semigroup $(U_t)_{t\geq 0}$ by $U_t f = h^{-1}V_t(hf)$. Then $\{hw_t : t > 0\}$ is a Markov process with Borel right transition semigroup given by (1.3) with $(V_t)_{t\geq 0}$ replaced by $(U_t)_{t\geq 0}$. Let E_r^T denote the set E furnished with the relative topology from E_D^T. Applying the results in Fitzsimmons (1988) again we conclude that $\{hw_t : t > 0\}$ is right continuous in $M(E_r^T)$ for a.a. $w \in W(M(E))$. Therefore, $\{h\bar{w}_t : t > 0\}$ is right continuous in $M(E_D^T)$ for a.a. $w \in W(M(E))$. Note that

$$\begin{cases} \bar{\psi}(x,z) = h(x)^{-1}\phi(x, h(x)z), & x \in E, \\ \bar{\psi}(x,z) = 0, & x \in E_D^T \setminus E, \end{cases}$$

defines a branching mechanism $\bar{\psi}(\cdot,\cdot)$ on E_D^T. Let $(\bar{U}_t)_{t\geq 0}$ be the cumulant semigroup given by

$$\bar{U}_t\bar{f}(x) + \int_0^t \mathrm{d}s \int_{E_D^T} \bar{\psi}(y, \bar{U}_s\bar{f}(y))\bar{T}_{t-s}(x, \mathrm{d}y) = \bar{T}_t\bar{f}(x), \quad t \geq 0, x \in E_D^T. \tag{4.11}$$

Then $(\bar{U}_t)_{t\geq 0}$ corresponds to Borel right transition semigroup $(\bar{Q}_t)_{t\geq 0}$ on $M(E_D^T)$. For any $t > 0$ and $x \in E_D^T$, the measure $\bar{T}_t(x,\cdot)$ is supported by E, so $\bar{T}_t\bar{f}(x)$ and $\bar{U}_t\bar{f}(x)$ are independent of the values of $\bar{f}$ on $E_D^T \setminus E$. Indeed, if $f = \bar{f}|_E$ for $f \in B(E_D^T)^+$, then $\bar{U}_t\bar{f}(x) = U_t f(x)$ for all $x \in E$. We may write $\bar{T}_t f$ and $\bar{U}_t f$ instead of $\bar{U}_t\bar{f}$ and $\bar{U}_t\bar{f}$ respectively. Clearly, the definitions of $\bar{T}_t f$ and $\bar{U}_t f$ can be extended to all non-negative Borel functions f on E by increasing limits. As shown in Li (1996b), there exists a measure $\rho \in M(E_D^T)$ such that $\kappa_t(f) = \rho(\bar{T}_t(h^{-1}f))$ and $S_t(\kappa, f) = \rho(\bar{U}_t(h^{-1}f))$. Then $\{h\bar{w}_t : t > 0\}$ is a Markov process with transition semigroup $(\bar{Q}_t)_{t\geq 0}$ and

$$\mathbf{Q}^{L\kappa}\left(1 - e^{-h\bar{w}_t(\bar{f})}\right) = \rho(\bar{U}_t\bar{f}).$$

Now the results follow by Theorem 4.1 applied to $(\bar{U}_t)_{t\geq 0}$ and $(\bar{T}_t)_{t\geq 0}$. ∎

By Theorem 2.2 we have an entrance law $K := l\kappa \in \mathcal{K}^1(Q)$ given by

$$\int_{M(E)} e^{-\nu(f)} K_t(d\nu) = \exp\{-S_t(\kappa, f)\}. \tag{4.12}$$

Obviously, $K = l\kappa$ determines a unique entrance law in $\mathcal{K}(Q^\circ)$. Let $\mathbf{Q}^{l\kappa}$ denote the corresponding Kuznetsov measure on $W(M(E))$.

THEOREM 4.3. *For $\mathbf{Q}^{l\kappa}$-a.a. $w \in W(M(E))$, $\{h\bar{w}_t : t > 0\}$ is right continuous and $h\bar{w}_t \to \rho$ for some $\rho \in M(E_D^T)$ as $t \downarrow 0$.*

PROOF: We use the notation introduced in the proof of Theorem 4.2. Clearly, $\{h\bar{w}_t : t > 0\}$ under $\mathbf{Q}^{l\kappa}$ is a Markov process with transition semigroup $(\bar{Q}_t)_{t\geq 0}$ and

$$\mathbf{Q}^{l\kappa} \exp\{-h\bar{w}_t(\bar{f})\} = \exp\{-\rho(\bar{U}_t\bar{f})\}.$$

Thus the assertions hold by the uniqueness of transition probability. ∎

Finally, we consider the path behavior of the Kuznetsov process determined by a general entrance rule. Let $(J_t)_{t\in \mathbb{R}}$ be an entrance rule for $(Q_t^\circ)_{t\geq 0}$ satisfying

$$\int_r^t ds \int_{M(E)^\circ} \nu(E) J_s(d\nu) < \infty, \quad r < t \in \mathbb{R}. \tag{4.13}$$

We may assume $(J_t)_{t\in\mathbb{R}}$ is given by (3.2) with the entrance laws $\{(J_{s+t}^s)_{t>0} : s \in \mathbb{R}\}$ taken from $\mathcal{K}(Q^\circ)$.

THEOREM 4.4. *In the situation described above, for $\mathbf{Q}^J$-a.a. paths $w \in W(M(E))$ the process $\{h\bar{w}_t : t \in \mathbb{R}\}$ defined by (4.10) is right continuous in $M(E_D^T)^\circ$ on the interval $(\alpha(w), \beta(w))$ and $h\bar{w}_t \to h\bar{w}_\alpha$ for some $h\bar{w}_\alpha \in M(E_D^T)$ as $t \downarrow \alpha(w)$. Moreover, for $\mathbf{Q}^J$-a.a. paths $w \in W(M(E))$ with $h\bar{w}_\alpha = 0$, we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some $x(w) \in E_D^T$ as $t \downarrow \alpha(w)$.*

PROOF: Let $\mathbf{Q}^H$ be the Kuznetsov measure on $W(M(E))$ corresponding to the entrance law $H \in \mathcal{K}(Q^\circ)$ represented by (2.7). Then

$$\mathbf{Q}^H(dw) = \mathbf{Q}^{L\kappa}(dw) + \int_{\mathcal{K}(P)} \mathbf{Q}^{l\eta}(dw) F(d\eta).$$

By Theorems 4.2 and 4.3, for $\mathbf{Q}^H$-a.a. $w \in W(M(E))$ the process $\{h\bar{w}_t : t > 0\}$ is right continuous in $M(E_D^T)$ and $h\bar{w}_t \to h\bar{w}_0$ for some $h\bar{w}_0 \in M(E_D^T)$ as $t \downarrow 0$. Furthermore, for $\mathbf{Q}^H$-a.a. $w \in W(M(E))$ with $h\bar{w}_0 = 0$, we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some

$x(w) \in E_D^T$ as $t \downarrow 0$. Then the desired result holds by the representation (3.3) of the measure $\mathbf{Q}^J(\mathrm{d}w)$. ∎

Clearly, (4.13) is satisfied by the entrance rule $(G_t)_{t\in I\!R}$ mentioned in Theorem 3.2. By Theorem 4.4, $\mathbf{Q}^G$-a.a. paths $w \in W(M(E))$ start propagation in E_D^T. Combining this with the construction of the immigration process given by Theorem 3.2 answers the problem posed in the introduction.

Example 4.1. Suppose that ξ is the minimal Brownian motion in $H := (0,\infty)$ with transition semigroup given by

$$P_t f(x) = \int_H [g_t(x-y) - g_t(x+y)]\, f(y)\mathrm{d}y,$$

where $g_t(x) = \exp\left\{-x^2/2t\right\}/\sqrt{2\pi t}$. In this case, we may identify E_D^T as $I\!R^+$. Let $\kappa \in \mathcal{K}(P)$ be defined by $\kappa_t(f) = \partial_0 P_t f$, where ∂_0 denotes the upward derivative at the origin. Then $S_t(\kappa, f) = \partial_0 V_t f$. Define $({}^\kappa Q_t)_{t\geq 0}$ by

$$\begin{aligned}&\int_{M(H)} \mathrm{e}^{-\nu(f)}\ {}^\kappa Q_t(\mu, \mathrm{d}\nu)\\ &=\exp\left\{ -\mu(V_t f) - \int_0^t (1-\exp\{-\partial_0 V_s f\})\,\mathrm{d}s\right\}.\end{aligned}$$

Let $G_t = \int_0^t l\kappa_s \mathrm{d}s$. Then (3.4) defines an immigration process with transition semigroup $({}^\kappa Q_t)_{t\geq 0}$. Using the notation introduced in the proof of Theorem 4.2 we have $S_t(\kappa, f) = h'(0^+)\bar{U}_t(h^{-1}f)(0)$. By the proofs of Theorems 4.3 and 4.4, $h\bar{w}_t \to h'(0^+)\delta_0$ and hence $w_t(H) \to \infty$ as $t \downarrow \alpha$ for $\mathbf{Q}^G$-a.a. $w \in W(M(H))$. Intuitively, the immigration process is generated by cliques of immigrants with infinite mass coming in H from the original. The semigroup $({}^\kappa Q_t)_{t\geq 0}$ has no right continuous realization; see Li (1996b). This example also shows that the transformation $w_t \mapsto h\bar{w}_t$ is necessary if one hopes to get the limit $\lim_{t\downarrow\alpha} w_t$ for $w \in W(M(H))$.

Example 4.2. Under the conditions of the previous example, assume further that $\phi(x,z) \equiv z^2/2$. We define $(Q_t^\kappa)_{t\geq 0}$ by

$$\int_{M(H)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\left\{ -\mu(V_t f) - \int_0^t \partial_0 V_s f \mathrm{d}s\right\}.$$

By the results in Li and Shiga (1995), the semigroup $(Q_t^\kappa)_{t\geq 0}$ corresponds to an immigration diffusion process $(W, \mathcal{G}, \mathcal{G}_t, Y_t, \mathbf{Q}_\mu^\kappa)$ such that $\{Y_t(\mathrm{d}x) : t > 0\}$ is $\mathbf{Q}_\mu^\kappa$-a.s. absolutely continuous relative to the Lebesgue measure on H having continuous density $\{Y_t(x) : t > 0, x > 0\}$ which satisfies $Y_t(0^+) \equiv 2$ and solves the following stochastic partial differential equation with singular drift term:

$$\frac{\partial}{\partial t} Y_t(x) = \sqrt{Y_t(x)}\ \dot{W}_t(x) + \frac{1}{2}\Delta Y_t(x) + d_0,$$

where $\dot{W}_t(x)$ is a time-space white noise, Δ denotes the Laplacian on H with Dirichlet boundary condition, and $-d_0$ is the derivative of the Dirac function at the origin. More precisely, $\langle d_0, f\rangle = f'(0^+)$ for $f \in C_0^2(I\!R^+)$, twice continuously differentiable functions on $I\!R^+$ vanishing at zero and infinity. The immigration process may be constructed by a special form of (3.11). Using a similar argument as in the previous example one sees that the process is generated by infinitesimal masses entering from the original.

5. Stationary immigration processes

An interesting feature of the associated immigration processes is their stationary distributions may be represented by excessive measures of the original MB-process. Based on this fact and the construction using Kuznetsov processes, we may give some interpretations of the results of Fitzsimmons and Maisonneuve (1986) in terms of stationary immigration processes.

Suppose that $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \theta_t, \mathbf{Q}_\mu)$ is a Borel right MB-process with transition semigroup $(Q_t)_{t\geq 0}$, where W is the space of paths $\{\omega_t : t \geq 0\}$ that are $M(E)^\circ$-valued and right continuous on the interval $[0, \beta(w))$ and take the null measure 0 elsewhere. The path $[0]$ constantly equal to 0 corresponds to $\beta([0]) = 0$. Given two probability measures F_1 and F_2 on $M(E)$, we write $F_1 \preceq F_2$ if there is a probability G such that $F_1 * G = F_2$. Let $\mathcal{E}^*(Q)$ denote the set of all probability measures F on $M(E)$ such that

$$\int_{M(E)^\circ} \nu(1) F(\mathrm{d}\nu) < \infty \tag{5.1}$$

and $FQ_t \preceq F$ for all $t \geq 0$. We write $F \in \mathcal{E}_i^*(Q)$ if $F \in \mathcal{E}^*(Q)$ is a stationary distribution for $(Q_t)_{t\geq 0}$, and write $F \in \mathcal{E}_p^*(Q)$ if $F \in \mathcal{E}^*(Q)$ and $\lim_{t\to\infty} FQ_t = \delta_0$. Clearly $\delta_0 \in \mathcal{E}_i^*(Q)$. (The MB-process may have other non-trivial stationary distributions.) The following theorem shows that $\mathcal{E}^*(Q)$ is identical with the totality of stationary distributions of immigration processes associated with X.

THEOREM 5.1. *Let $F \in \mathcal{E}^*(Q)$. Then it may be written uniquely as $F = F_i * F_p$, where $F_i = \lim_{t\to\infty} FQ_t \in \mathcal{E}_i^*(Q)$ and $F_p \in \mathcal{E}_p^*(Q)$. Moreover, there is a unique skew convolution semigroup $(N_t)_{t\geq 0}$ such that $\lim_{t\to\infty} N_t = F_p$.*

PROOF: Let $(N_t)_{t\geq 0}$ be the distributions on $M(E)$ satisfying $F = (FQ_t) * N_t$. By the branching property of the semigroup $(Q_t)_{t\geq 0}$ one checks for any $r \geq 0$ and $t \geq 0$,

$$\begin{aligned}(FQ_{r+t}) * N_{r+t} =& F = (FQ_t) * N_t = \{[(FQ_r) * N_r]Q_t\} * N_t \\ =& (FQ_{r+t}) * (N_r Q_t) * N_t.\end{aligned} \tag{5.2}$$

It follows that $(N_t)_{t\geq 0}$ satisfies the relation (1.4), so it is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$. By the definition of $\mathcal{E}^*(Q)$, we have $FQ_{r+t} \preceq FQ_t$, so the following limits exist and give the Laplace functionals of two probability measures F_i and F_p:

$$L_{F_i}(f) = \uparrow \lim_{t\uparrow\infty} L_{FQ_t}(f), \quad L_{F_p}(f) = \downarrow \lim_{t\uparrow\infty} L_{N_t}(f). \tag{5.3}$$

Clearly, $F_i \in \mathcal{E}_i^*(Q)$ and $F = F_i * F_p$. On the other hand,

$$F_i * F_p = F = (FQ_t) * N_t = F_i * (F_p Q_t) * N_t,$$

so $F_p = (F_p Q_t) * N_t$. Therefore $F_p \in \mathcal{E}^*(Q)$ and $\lim_{t\to\infty} F_p Q_t = \delta_0$. The uniqueness of the decomposition is immediate. ∎

Let $\mathcal{E}(Q^\circ)$ denote the class of all excessive measures F for $(Q_t^\circ)_{t\geq 0}$ satisfying (5.1). Let $\mathcal{E}_i(Q^\circ)$ be the subset of $\mathcal{E}(Q^\circ)$ comprising invariant measures, and $\mathcal{E}_p(Q^\circ)$ the subset of purely excessive measures. The classes $\mathcal{E}(Q^\circ)$ and $\mathcal{E}^*(Q)$ are closely related. Indeed, $F \in \mathcal{E}^*(Q)$ is infinitely divisible if and only if $F = I(\rho, J)$ for $\rho \in M(E)$ and $J \in \mathcal{E}(Q^\circ)$ satisfying

$$\int_E \rho(\mathrm{d}x)\lambda_t(x,\cdot) \leq \rho \text{ and } \int_E \rho(\mathrm{d}x)L_t(x,\cdot) + JQ_t^\circ \leq J. \tag{5.4}$$

Under the condition (2.8), $F \in \mathcal{E}^*(Q)$ is infinitely divisible if and only if $F = I(0, J)$ for some $J \in \mathcal{E}(Q^\circ)$.

Let $J \in \mathcal{E}(Q^\circ)$. Then $I(0, J) \in \mathcal{E}^*(Q)$. The corresponding stationary immigration process may be constructed as follows. It is well-known that J has the Riesz type decomposition $J = J_i + J_p$, where $J_i \in \mathcal{E}_i(Q^\circ)$ and $J_p \in \mathcal{E}_p(Q^\circ)$. Moreover, J_p may be represented as $J_p = \int_0^\infty H_t \,\mathrm{d}t$, where $H \in \mathcal{K}(Q^\circ)$ is given by

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = \mathbf{Q}^J \left\{1 - \mathrm{e}^{-w_{\alpha+t}(f)}; 0 < \alpha < 1\right\}. \tag{5.5}$$

See Fitzsimmons and Maisonneuve (1986). Let $\mathbf{Q}^J$ be the Kuznetsov measure on $W(M(E))$ determined by J. Let $N^J(\mathrm{d}w)$ be a Poisson random measure with intensity $\mathbf{Q}^J(\mathrm{d}w)$ and define

$$Y_t^J = \int_{W(M(E))} w_t \, N^J(\mathrm{d}w), \quad t \in I\!R. \tag{5.6}$$

As for the proof of Theorem 3.2 one may show that $\{Y_t^J : t \in I\!R\}$ is a stationary immigration process having one-dimensional distribution $I(0, J)$ and transition semigroup $(Q_t^H)_{t\geq 0}$ given by

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^H(\mu, \mathrm{d}\nu)\\ &= \exp\left\{ -\mu(V_t f) - \int_0^t \mathrm{d}s \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu)\right\}.\end{aligned} \tag{5.7}$$

The Kuznetsov measures determined by J_i and J_p are restrictions of $\mathbf{Q}^J$ to $\{w \in W(M(E)) : \alpha(w) = -\infty\}$ and $\{w \in W(M(E)) : \alpha(w) > -\infty\}$ respectively; see Fitzsimmons and Maisonneuve (1986). Then

$$Y_t^{(p)} = \int_{W(M(E))} w_t 1_{\{\alpha > -\infty\}} N^J(\mathrm{d}w), \quad t \in I\!R,$$

defines a stationary immigration process having transition semigroup $(Q_t^H)_{t\geq 0}$ and one-dimensional distribution $I(0, J_p)$. Intuitively, $\{Y_t^{(p)} : t \in I\!R\}$ is the "purely immigrative" part of the population. On the contrary,

$$Y_t^{(i)} = \int_{W(M(E))} w_t 1_{\{\alpha=-\infty\}} N^J(\mathrm{d}w), \quad t \in I\!R,$$

is a stationary Markov process with semigroup $(Q_t)_{t\geq 0}$ and one-dimensional distribution $I(0, J_i)$, which represents the "native" part of the population.

Suppose that T is an *exact terminal time* for X. That is, T is a $(\mathcal{G}_t)$-stopping time such that (i) $t+T(\theta_t\omega) = T$ for all t and ω with $t < T(\omega)$ and (ii) $T(\omega) =\downarrow \lim_{t\downarrow 0}[t+T(\theta_t\omega)]$ for every $\omega \in W$. Assume that $T([0]) = \infty$. For $J \in \mathcal{E}(Q^\circ)$, we may define the *balayage* $R_T J \in \mathcal{E}(Q^\circ)$ as in Getoor (1990). The *birthing* and *truncated shift* operators $\{b_t : t \in I\!R\}$ and $\{\theta_t : t \in I\!R\}$ on $W(M(E))$ are defined by

$$(5.8) \qquad \begin{cases} b_t w_s = w_s \text{ if } s > t \text{ and } b_t w_s = 0 \text{ if } s \leq t; \\ \theta_t w_s = w_{t+s} \text{ if } s > 0 \text{ and } \theta_t w_s = 0 \text{ if } s \leq 0. \end{cases}$$

Clearly, W is isomorphic to $\{w \in W_0(M(E)) : w_{\alpha+} \text{ exists in } M(E)\}$. Using this isomorphism, we define a $(\mathcal{H}_t)$-stopping time T on $W(M(E))$ by

$$T(w) =\downarrow \lim_{t\downarrow\alpha}[t + T(\theta_t w_+)],$$

where $\theta_t w_+ \in W$ is defined by $\theta_t w_{+s} = \theta_t w_{s+}$. Then the pure excessive part $(R_T J)_p$ of $R_T J$ may be represented as $(R_T J)_p = \int_0^\infty R_T H_t \mathrm{d}t$ with

$$R_T H_t(F) = \mathbf{Q}^J\{F(w_{T+t}); 0 < T < 1\}.$$

By Getoor (1990; 7.5, 7.9) we have

$$(5.9) \qquad \mathbf{Q}^{R_T J}\{G(w)\} = \mathbf{Q}^J\{G \circ b_T(w); T < \beta\}.$$

The next theorem follows from (5.7) and (5.9) immediately.

THEOREM 5.2. *Define* $\{b_T Y_t^J : t \in I\!R\}$ *by*

$$(5.10) \qquad b_T Y_t^J = \int_{W(M(E))} b_T w_t \, N^J(\mathrm{d}w), \quad t \in I\!R.$$

Then $\{b_T Y_t^J : t \in I\!R\}$ *is a stationary immigration process having one-dimensional distribution* $I(0, R_T J)$ *and the transition semigroup* $(Q_t^T)_{t\geq 0}$ *whose Laplace functional is given by the right hand side of (5.7) with* $R_T H_t$ *in place of* H_t.

Imagine a situation in which the realistic immigration is given by $\{Y_t^J : t \in I\!R\}$. Suppose that the immigrant $w \in W(M(E))$ is first found not at the entering time $\alpha(w)$ but only at the random time $T(w) \geq \alpha(w)$. Then Theorem 5.2 asserts that the observed process $\{Y_t^J : t \in I\!R\}$ is still a stationary Markov process.

The immigration processes defined by (5.6) or (5.10) are usually not right continuous, but they may have right continuous modifications. For $w \in W(M(E))$, we set

$$\begin{cases} w_{t+} = \lim_{s\downarrow t} w_s & \text{if the limit exists in } M(E), \\ w_{t+} = 0 & \text{if the above limit does not exist in } M(E), \end{cases}$$

and define the process $\{{}^0Y_t^J : t \in \mathbb{R}\}$ by

$$ {}^0Y_t^J = \int_{W(M(E))} w_{t+} N^J(\mathrm{d}w), \quad t \in \mathbb{R}, \tag{5.11}$$

Since $\mathbf{Q}^J\{w_{t+} \neq w_t\} = \mathbf{Q}^J\{\alpha = t\} = 0$, we have ${}^0Y_t^J = Y_t^J$ a.s. for every $t \in \mathbb{R}$. In other words, $\{{}^0Y_t^J : t \in \mathbb{R}\}$ is a modification for $\{Y_t^J : t \in \mathbb{R}\}$.

THEOREM 5.3. *Suppose that X is a (ξ, ϕ)-superprocess.*
(i) If $J \in \mathcal{E}_i(Q^\circ)$, then $\{{}^0Y_t^J \equiv Y_t^J : t \in \mathbb{R}\}$ is a.s. right continuous.
(ii) If $J \in \mathcal{E}_p(Q^\circ)$ is a measure potential, that is,

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) J(\mathrm{d}\nu) = \int_0^\infty \mathrm{d}s \int_{M(E)^\circ} (1 - \exp\{-\nu(V_s f)\})\, G(\mathrm{d}\nu),$$

where $\nu(1)G(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$, then $\{{}^0Y_t^J : t \in \mathbb{R}\}$ is a.s. right continuous.

PROOF: Since (i) is easy, we only give the proof of (ii) here. For $k = 1, 2, \cdots$ let

$$W_k(M(E)) = \{w \in W(M(E)) : w_{\alpha+}(E) \geq 1/k\}.$$

By Fitzsimmons and Maisonneuve (1986) the path $\{w_{t+} : t \in \mathbb{R}\}$ is right continuous for $\mathbf{Q}^J$-a.a. $w \in W(M(E))$ and $\mathbf{Q}^J([\cup_{k=1}^\infty W_k(M(E))]^c) = 0$. Define

$${}^0Y_t^{(k)} = \int_{W_k(M(E))} X_t(w, \cdot) 1_{\{\alpha(w) \geq -k\}} N^J(\mathrm{d}w), \quad t \in \mathbb{R}.$$

Clearly, $\{{}^0Y_t^{(k)} : t \geq -k\}$ is a Markov process in $M(E)$ with semigroup given by

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^{(k)}(\mu, \mathrm{d}\nu) = \exp\Big\{ - \mu(V_t f) - \int_0^t \mathrm{d}s \int_{M(E)} \left(1 - \mathrm{e}^{-\nu(V_s f)}\right) 1_{\{\nu(E) \geq 1/k\}} G(\mathrm{d}\nu) \Big\}.$$

Observe that for each $l > -k$ the process $\{{}^0Y_t^{(k)} : -k \leq t \leq l\}$ is a.s. a finite sum of right continuous paths and ${}^0Y_t^{(k)} \to {}^0Y_t^J$ increasingly as $k \to \infty$, so the result follows as Theorem 4.1 in Li (1996b). ∎

THEOREM 5.4. *Suppose that X is a (ξ,ϕ)-superprocess. Let $J \in \mathcal{E}(Q^\circ)$ and let $\{Y_t^J : t \in I\!R\}$ be defined by (5.6). For $r > 0$ set*

$$ {}^rY_t^J = \int_{W(M(E))} w_t 1_{\{t \geq \alpha + r\}} N(\mathrm{d}w), \quad t \in I\!R. $$

Then $\{{}^rY_t^J : t \in I\!R\}$ is an a.s. right continuous stationary immigration process and ${}^rY_t^J \to Y_t^J$ increasingly a.s. as $r \downarrow 0$ for every $t \in I\!R$.

PROOF: Clearly, $JQ_r^\circ \in \mathcal{E}(Q^\circ)$ and

$$ JQ_r^\circ = J_i + J_p Q_r^\circ = J_i + \int_0^\infty H_r Q_s^\circ \mathrm{d}s. $$

Using (3.1) one may check that the Kuznetsov measure on $W(M(E))$ determined by $JQ_r^\circ \in \mathcal{E}(Q^\circ)$ is the image of $\mathbf{Q}^J$ under the mapping $\{w_t : t \in I\!R\} \mapsto \{w_t 1_{\{t > \alpha + r\}} : t \in I\!R\}$. It follows that $\{{}^rY_t^J : t \in I\!R\}$ is a stationary immigration process corresponding to JQ_r°. By Theorem 5.3, $\{{}^rY_t^J : t \in I\!R\}$ is a.s. right continuous. Now the desired result is immediate. ∎

References

Athreya, K.B. and Ney, P.E. (1972), *Branching Processes*, Springer-Verlag, Berlin.

Dawson, D.A. (1992), *Infinitely Divisible Random Measures and Superprocesses*, In: Procedings of 1990 Workshop on Stochastic Analysis and Related Topics, Silivri, Turkey.

Dawson, D.A. (1993), *Measure-valued Markov Processes*, Ecole d'Eté de Probabilités de Saint-Flour XXI-1991, Lecture Notes Math. **1541**, 1-260, Springer-Verlag.

Dawson, D.A. and Ivanoff, D. (1978), *Branching diffusions and random measures*, Advances in Probability and Related Topics **5**, A. Joffe and P. Ney eds., 61-103.

Dellacherie, C., Maisonneuve, B. and Meyer, P.A. (1992), *Probabilitiés et Potential*, Ch. 17-24, Hermann.

Dynkin, E.B. (1989), *Three classes of infinite dimensional diffusion processes*, J. Funct. Anal. **86**, 75-110.

Evans, S. (1992), *The entrance space of a Markov branching process conditioned on non-extinction*, Can. Math. Bull **35**, 70-74.

Evans, S. (1993), *Two representations of conditioned superprocess*, Proceedings of Royal Society of Edinburgh **123A**, 959-971.

Fitzsimmons, P.J. (1988), *Construction and regularity of measure-valued Markov branching processes*, Israel J. Math. **64**, 337-361.

Fitzsimmons, P.J. (1992), *On the martingale problem for measure-valued Markov branching processes*, In: Seminar on Stoch. Proc. 1991, Cinlar, E. et al eds., Birkhäuser, Basel.

Fitzsimmons, P.J. and Maisonneuve, B. (1986), *Excessive measures and Markov processes with random birth and death*, Probab. Theory Related Fields **18**, 571-575.

Getoor, R.K. (1990), *Excessive Measures*, Birkhäuser, Basel.

Getoor, R.K. and Glover, J. (1987), *Constructing Markov processes with random times of birth and death*, In: Seminar on Stochastic Processes 1986, Cinlar, E. et al eds., Birkhäuser, Basel.

Ikeda, N. and Watanabe, S. (1989), *Stochastic Differential Equations and Diffusion Processes*, 2nd Ed., North-Holland, Amsterdam.

Iscoe, I. (1986), *A weighted occupation time for a class of measure-valued branching processes*, Probab. Theory Related Fields **71**, 85-116.

Kallenberg, O. (1975), *Random measures*, Academic Press, New York.

Kawazu, K. and Watanabe, S. (1971), *Branching processes with immigration and related limit theorems*, Theory Probab. Appl. **16**, 34-51.

Kuznetsov, S. E. (1974), *Construction of Markov processes with random times of birth and death*, Theory Probab. Appl. **18**, 571-575.

Li, Z.H. (1996a), *Convolution semigroups associated with measure-valued branching processes*, Chinese Sci. Bull. (English Edition) **41**, 276-280.

Li, Z.H. (1996b), *Immigration structures associated with Dawson-Watanabe superprocesses*, Stochastic Process. Appl. **62**, 73-86.

Li, Z.H. and Shiga, T. (1995), *Measure-valued branching diffusions: immigrations, excursions and limit theorems*, J. Math. Kyoto Univ. **35**, 233-274.

Perkins, E.A. (1992), *Conditional Dawson-Watanabe processes and Fleming-Viot processes*, In: Seminar on Stoch. Proc. 1991, Cinlar, E. et al eds., Birkhäuser, Basel.

Sharpe, M.J. (1988), *General Theory of Markov Processes*, Academic Press, New York.

Shiga, T. (1990), *A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes*, J. Math. Kyoto Univ. **30**, 245-279.

Shiga, T. and Watanabe, S. (1973), *Bessel diffusions as a one-parameter family of diffusion processes*, Z. Wahrsch. verw. Geb. **27**, 37-46.

Silverstein, M.L. (1969), *Continuous state branching semmigroups*, Z. Wahrsch. verw. Geb. **9**, 235-257.

Watanabe, S. (1968), *A limit theorem of branching processes and continuous state branching processes*, J. Math. Kyoto Univ. **8**, 141-167.