数理解析研究所講究録
1158 巻 2000 年 180-189

Plana's Summation Formula for Holomorphic Functions of Exponential Type

吉野 邦生 (Kunio Yoshino)
諏訪 将範 (Masanori Suwa)

Department of Mathematics, Sophia University

Abstract

In this paper we derive Plana's summation formula for holomorphic functions of exponential type by using theory of analytic functionals with unbounded carrier.

In §1, we will recall the definitions and properties of analytic functionals with unbounded carrier.

In §2, we derive Plana's summation formula for holomorphic functions of exponential type.

In §3, we will give some examples of Plana's summation formula.

1 The definitions of analytic functionals with unbounded carrier and their properties.

Let L and L_ε be following strip regions:

$$\begin{aligned} L &= (-\infty, a] + \imath[-b, b], \\ \text{For } \varepsilon > 0, \quad L_\varepsilon &= (-\infty, a+\varepsilon] + \imath[-b-\varepsilon, b+\varepsilon]. \end{aligned}$$

For $\varepsilon > 0$, $\varepsilon' > 0$ and $k' \in \mathbb{R}$, we put

$$\begin{aligned} &Q_b(L_\varepsilon : k' + \varepsilon') \\ &:= \left\{ f(\zeta) \in \mathcal{H}(\mathring{L}_\varepsilon) \cap \mathcal{C}(L_\varepsilon) : \sup_{\zeta \in L_\varepsilon} |f(\zeta) e^{k'\xi + \varepsilon'|\xi|}| < \infty, \quad \zeta = \xi + \imath\eta \right\}. \end{aligned}$$

$\mathcal{H}(\overset{\circ}{L_\varepsilon})$ is the space of holomorphic functions defined on $\overset{\circ}{L_\varepsilon}$, (interior of L_ε). $\mathcal{C}(L_\varepsilon)$ is the space of continuous functions defined on L_ε. We put

$$Q(L:k') = \varinjlim_{\varepsilon\to 0, \varepsilon'\to 0} Q_b(L_\varepsilon : k' + \varepsilon'),$$

where $\varinjlim$ means inductive limit. If $z \in (-k', \infty) + \imath\mathbb{R}$, then the function $e^{\zeta z}$ of ζ belongs to $Q(L:k')$. We denote by $Q'(L:k')$ the dual space of $Q(L:k')$. The elements of $Q'(L:k')$ is called analytic functionals with carrier L and of type k'.
We define the Fourier-Borel transform $\tilde{T}(z)$ of $T \in Q'(L:k')$ as follows:

$$\tilde{T}(z) =< T_\zeta, e^{\zeta z} > .$$

$\tilde{T}(z)$ is holomorphic function on the right half plane $(-k', \infty) + \imath\mathbb{R}$ and satisfies following estimate:

$$\forall \varepsilon > 0, \varepsilon' > 0, \exists C_{\varepsilon,\varepsilon'} \geq 0 \text{ such that}$$

$$|\tilde{T}(z)| \leq C_{\varepsilon,\varepsilon'} e^{ax+b|y|+\varepsilon|z|}, \quad (\,\mathrm{Re} z \geq -k' + \varepsilon', \quad z = x + \imath y). \tag{1}$$

$\mathrm{Exp}((-k', \infty) + \imath\mathbb{R} : L)$ denotes the space of holomorphic functions defined on the right half plane $(-k', \infty) + \imath\mathbb{R}$ and satisfy the estimates (1). Following theorem characterizes the Fourier-Borel transform of $Q'(L:k')$.

Theorem 1.1 ([3]). *Fourier-Borel transform is a linear topological isomorphism from* $Q'(L:k')$ *onto* $\mathrm{Exp}((-k', \infty) + \imath\mathbb{R} : L)$.

2 Plana's summation formula for holomorphic functions of exponential type.

In this section, we will derive Plana's summation formula for holomorphic functions of exponential type by using the theory of analytic functionals with unbounded carrier.

Proposition 2.1. *Let $T \in Q'(L:k')$. If $k' > 0$, $0 \leq b < 2\pi$, $\mathrm{Re}s > a$ and $|\mathrm{Im}s| + b < 2\pi$, then we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \tilde{T}(n)e^{-sn} \\
&= \frac{1}{2}\tilde{T}(0) + \int_0^{\infty} \tilde{T}(x)e^{-sx}dx + \imath \int_0^{\infty} \frac{\tilde{T}(\imath x)e^{-\imath sx} - \tilde{T}(-\imath x)e^{\imath sx}}{e^{2\pi x} - 1} dx.
\end{aligned}$$

In order to prove proposition 2.1, we prepare the following lemmas.

Lemma 2.2. *Assume that* $\mathrm{Re}s > a$. *Then the function* $\dfrac{1}{1 - e^{\zeta - s}}$ *of* ζ *belongs to* $Q(L:k')$ *and*

$$\lim_{N\to\infty} \sum_{n=0}^{N} e^{(\zeta - s)n} = \frac{1}{1 - e^{\zeta - s}} \quad \text{in } Q(L:k').$$

Lemma 2.3. *If* $|\mathrm{Im}(\zeta - s)| < 2\pi$, *we have*

$$\frac{1}{e^{\zeta - s} - 1} = \frac{-1}{\zeta - s} - \frac{1}{2} + 2\int_0^{\infty} \frac{\sin(\zeta - s)x}{e^{2\pi x} - 1} dx.$$

Proof. Following equality is well known:

$$\frac{1}{e^{\zeta - s} - 1} = \frac{1}{2\imath} \sum_{n=1}^{\infty} \left\{ \frac{1}{2\pi n - \imath(\zeta - s)} - \frac{1}{2\pi n + \imath(\zeta - s)} \right\}.$$

Therefore we have

$$\begin{aligned}
\frac{1}{e^{\zeta - s} - 1} &= \frac{1}{2\imath} \sum_{n=1}^{\infty} \left\{ \frac{1}{2\pi n - \imath(\zeta - s)} - \frac{1}{2\pi n + \imath(\zeta - s)} \right\} \\
&= \frac{1}{2\imath} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\{2\pi n - \imath(\zeta - s)\}x} - e^{-\{2\pi n + \imath(\zeta - s)\}x} dx \\
&= \lim_{N\to\infty} \int_0^{\infty} \sum_{n=1}^{N} e^{-2\pi n x} \sin(\zeta - s)x dx. \qquad (2)
\end{aligned}$$

Proof. By proposition 2.4 and Fubini's theorem, we obtain the lemma 2.5. ∎

Lemma 2.6.

$$\left\langle T_\zeta, \int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x}-1}dx \right\rangle = \int_0^\infty \frac{\tilde{T}(-\imath x)e^{\imath sx} - \tilde{T}(\imath x)e^{-\imath sx}}{e^{2\pi x}-1}dx.$$

Proof. By proposition 2.4 and Fubini's theorem, we obtain the lemma 2.6. ∎

Proof of Proposition 2.1. Since $\tilde{T}(n) = < T_\zeta, e^{\zeta n} >$,

$$\begin{aligned} \sum_{n=0}^{\infty} \tilde{T}(n)e^{-sn} &= \sum_{n=0}^{\infty} < T_\zeta, e^{\zeta n} > e^{-sn} \\ &= \left\langle T_\zeta, \frac{1}{1-e^{\zeta-s}} \right\rangle, \end{aligned} \tag{3}$$

By lemma 2.3, we have

$$\frac{1}{1-e^{\zeta-s}} = \frac{1}{2} + \frac{1}{\zeta-s} + \imath\int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x}-1}dx.$$

By lemma 2.6, we have

$$\begin{aligned} (3) &= \left\langle T_\zeta, \frac{1}{2} + \frac{1}{\zeta-s} + \imath\int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x}-1}dx \right\rangle \\ &= \left\langle T_\zeta, \frac{1}{2} \right\rangle + \left\langle T_\zeta, \frac{1}{\zeta-s} \right\rangle \\ &\quad + \imath\int_0^\infty \left\{ \frac{< T_\zeta, e^{-\imath(\zeta-s)x} > - < T_\zeta, e^{\imath(\zeta-s)x} >}{e^{2\pi x}-1} \right\} dx \\ &= \frac{1}{2}\tilde{T}(0) + \int_0^\infty \tilde{T}(x)e^{-sx}dx \\ &\quad + \imath\int_0^\infty \left\{ \frac{< T_\zeta, e^{-\imath(\zeta-s)x} > - < T_\zeta, e^{\imath(\zeta-s)x} >}{e^{2\pi x}-1} \right\} dx. \\ &= \frac{1}{2}\tilde{T}(0) + \int_0^\infty \tilde{T}(x)e^{-sx}dx + \imath\int_0^\infty \frac{\tilde{T}(-\imath x)e^{\imath sx} - \tilde{T}(\imath x)e^{-\imath sx}}{e^{2\pi x}-1}dx. \end{aligned}$$

∎

We put

$$\begin{aligned} f_N(x) &= \sum_{n=1}^{N} e^{-2\pi n x}\sin(\zeta - s)x, \\ f(x) &= \left|\frac{\sin(\zeta - s)x}{e^{2\pi x} - 1}\right|, \end{aligned}$$

then we have

$$|f_N(x)| \leq |f(x)| \qquad (x \geq 0),$$

$$f(x) \in L^1([0,\infty)).$$

Therefore by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} (2) &= \int_0^\infty \sum_{n=1}^{\infty} e^{-2\pi n x}\sin(\zeta - s)x dx \\ &= \int_0^\infty \frac{\sin(\zeta - s)x}{e^{2\pi x} - 1} dx. \qquad \blacksquare \end{aligned}$$

Proposition 2.4 ([4]). *Let $T \in Q'(L : k')$. For $\forall \varepsilon > 0$ and $\forall \varepsilon' > 0$, there exists a finite measure μ on L_ε such that*

$$< T, \varphi > = \int_{L_\varepsilon} \varphi(\zeta) e^{k'\xi + \frac{\varepsilon'}{2}|\xi|} d\mu(\zeta) \qquad (\zeta = \xi + \imath\eta),$$

$$\textit{for } \varphi(\zeta) \in Q_b(L_\varepsilon : k' + \varepsilon').$$

If $k' > 0$ and $s \notin L$, then the function $\dfrac{1}{\zeta - s}$ of ζ belongs to $Q(L : k')$. So we can define the Cauchy-Hilbert transform $\check{T}(s)$ of $T \in Q'(L : k')$ as follows:

$$\check{T}(s) = \left\langle T_\zeta, \frac{1}{\zeta - s} \right\rangle.$$

Lemma 2.5. *If* Re$s > a$, *we have*

$$\check{T}(s) = \int_0^\infty \tilde{T}(x) e^{-xs} dx.$$

Theorem 2.7. *Suppose that* $f(z) \in \mathrm{Exp}((-k',\infty)+\imath\mathbb{R}:L)$. *If* $0 \le b < 2\pi$, $|\mathrm{Im}s| + b < 2\pi$, $\mathrm{Re}s > a$ *and* $k' > 0$, *then following equality holds:*

$$\sum_{n=0}^{\infty} f(n)e^{-sn}$$
$$= \frac{1}{2}f(0) + \int_0^{\infty} f(x)e^{-sx}dx + \imath\int_0^{\infty} \frac{f(\imath x)e^{-\imath sx} - f(-\imath x)e^{\imath sx}}{e^{2\pi x}-1}dx. \qquad (4)$$

Proof. By theorem 1.1, there exists $T \in Q'(L:k')$ such that $f(z) = \tilde{T}(z)$. By proposition 2.1, we obtain (4). ∎

Corollary 2.8. *Suppose that* $f(z)$ *is a holomorphic function on the right half plane* $(-k',\infty)+\imath\mathbb{R}$, $k' > 0$ *and satisfies*

$$\forall\varepsilon' > 0,\ \exists C_{\varepsilon'} \ge 0, \quad |f(z)| \le C_{\varepsilon'}\frac{e^{b|\mathrm{Im}z|}}{(1+|x|)^m}, \quad (\mathrm{Re}z \ge -k'+\varepsilon', \quad z = x+\imath y).$$

If $m > 1$ *and* $0 \le b < 2\pi$, *then we have*

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(t)dt + \imath\int_0^{\infty} \frac{f(\imath t) - f(-\imath t)}{e^{2\pi t}-1}dt. \qquad (5)$$

Proof. From the assumption $m > 1$, $\sum_{n=0}^{\infty} f(n)$ converges absolutely and $f(t) \in L^1[0,\infty)$. So we can apply Lebesgue dominated convergence theorem. Letting $s \to 0$ in (4), we obtain (5). ∎

Corollary 2.9. *Let* $f(z)$ *be a holomorphic function and* $\mathrm{Im}f(z) \ge 0$ *on* $(-k',\infty)+\imath\mathbb{R}$, $k' > 0$. *If* $\mathrm{Re}s > 0$ *and* $|\mathrm{Im}s| < 2\pi$, *we have*

$$\sum_{n=0}^{\infty} f(n)e^{-ns} = \frac{1}{2}f(0) + \int_0^{\infty} f(x)e^{-xs}dx + \imath\int_0^{\infty} \frac{f(\imath x)e^{-\imath sx} - f(-\imath x)e^{\imath sx}}{e^{2\pi x}-1}dx.$$

Proof. In order to prove corollary 2.9, we use the following lemma:

Lemma 2.10 ([5]). *Let* $f(z)$ *be a holomorphic and* $\mathrm{Im}f(z) \ge 0$ *on the upper half plane* $\mathbb{R}+\imath\mathbb{R}^+ := \{z \in \mathbb{C} : z = x+\imath y, \quad y > 0\}$. *Then it satisfies the estimate*

$$|f(z)| \le \sqrt{2}|f(\imath)|\frac{1+|z|^2}{|y|} \quad z \in \mathbb{R}+\imath\mathbb{R}^+. \qquad (6)$$

Therefore by (6) and theorem 2.7, we obtain the corollary 2.9. ∎

Proposition 2.11. *Let* $T \in Q'(L : k')$. *If* $k' > 0$, $0 \le b < 2\pi$, $\mathrm{Re}s > a$ *and* $|\mathrm{Im}s| + b < 2\pi$, *then we have*

$$\sum_{n=0}^{\infty} \tilde{T}(n)e^{-sn}$$
$$= \frac{1}{2}\tilde{T}(0) + \int_0^{\infty} \tilde{T}(x)e^{-sx}dx + \sum_{n=1}^{\infty}(\check{T}(s+2\pi\imath n) - \check{T}(s-2\pi\imath n)).$$

Proof. By proposition 2.1 and lemma 2.5, we obtain proposition 2.11. ∎

3 Examples and applications.

Plana's summation formula is very useful in the theory of the special functions [2], [7]. In this section we will give some examples.

Example 3.1. For

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(t)dt + \imath\int_0^{\infty} \frac{f(\imath t) - f(-\imath t)}{e^{2\pi t} - 1}dt,$$

we put $f(z) = e^{-Az}$ $(\mathrm{Re}A > 0,\ |\mathrm{Im}A| < 2\pi)$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-An} &= \frac{1}{2} + \int_0^{\infty} e^{-At}dt + \imath\int_0^{\infty} \frac{e^{-\imath At} - e^{\imath At}}{e^{2\pi t} - 1}dt \\ \frac{1}{1-e^{-A}} &= \frac{1}{2} + \frac{1}{A} + \imath\int_0^{\infty} \frac{e^{-\imath At} - e^{\imath At}}{e^{2\pi t} - 1}dt \\ &= \frac{1}{2} + \frac{1}{A} + 2\int_0^{\infty} \frac{\sin At}{e^{2\pi t} - 1}dt. \end{aligned}$$

Here we have

$$\begin{aligned} \int_0^{\infty} \frac{e^{-\imath At} - e^{\imath At}}{e^{2\pi t} - 1}dt &= \int_0^{\infty} (e^{-\imath At} - e^{\imath At})\sum_{n=1}^{\infty} e^{-2\pi nt}dt \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi n + \imath A} - \frac{1}{2\pi n - \imath A}\right). \end{aligned}$$

Hence we have

$$\frac{1}{1-e^{-A}} = \frac{1}{2}+\frac{1}{A}+\sum_{n=1}^{\infty}\left(\frac{1}{A-2\imath\pi n}+\frac{1}{A+2\imath\pi n}\right).$$

Example 3.2. Let $f(z)\equiv 1$, i.e. $f(z) =< \delta_\zeta, e^{\zeta z} >$. Then

$$\sum_{n=0}^{\infty} f(n)e^{-sn} = \frac{1}{2}f(0)+\int_0^{\infty} f(t)e^{-st}dt+\imath\int_0^{\infty}\frac{f(\imath t)e^{-\imath st}-f(-\imath t)e^{\imath st}}{e^{2\pi t}-1}dt$$

becomes

$$\sum_{n=0}^{\infty} e^{-sn} = \frac{1}{2}+\int_0^{\infty} e^{-st}dt+\imath\int_0^{\infty}\frac{e^{-\imath st}-e^{\imath st}}{e^{2\pi t}-1}dt$$

$$\frac{1}{e^s-1}+\frac{1}{2}-\frac{1}{s} = 2\int_0^{\infty}\frac{\sin st}{e^{2\pi t}-1}dt.$$

Example 3.3 (Schlömilch expansion ([6])). Let $f(z) = J_0(za)$ ($J_0(z)$ is Bessel function with degree 0, $a>0$, $s>0$). Then we have

$$\sum_{n=1}^{\infty} J_0(na)e^{-ns}$$

$$= \frac{1}{\sqrt{a^2+s^2}}-\frac{1}{2}+\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{a^2+(s+2\imath\pi n)^2}}-\frac{1}{\sqrt{a^2+(s-2\imath\pi n)^2}}\right\}.$$

Proof. We have

$$\int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} J_0(ta)e^{-st}dt = \frac{1}{\sqrt{s^2+a^2}}.$$

By proposition 2.11, we obtain example 3.3. ∎

Example 3.4. Let $l \in \mathbb{N}\backslash\{0\}$ and $f(z) = J_l(za)$ ($J_l(z)$ is Bessel function with degree l, $a > 0$, $s > 0$). Then we have

$$\sum_{n=1}^{\infty} J_l(na)e^{-sn} = \frac{(-s+\sqrt{a^2+s^2})^l}{a^l\sqrt{a^2+s^2}} + \sum_{n=1}^{\infty}\left\{\frac{\left(-s-2\pi\imath n+\sqrt{a^2+(s+2\pi\imath n)^2}\right)^l}{a^l\sqrt{a^2+(s+2\pi\imath n)^2}} - \frac{\left(-s+2\pi\imath n-\sqrt{a^2+(s-2\pi\imath n)^2}\right)^l}{a^l\sqrt{a^2+(s-2\pi\imath n)^2}}\right\}.$$

Proof. We have

$$\int_0^{\infty} J_l(ta)e^{-st}dt = \frac{\left(-s+\sqrt{a^2+s^2}\right)^l}{a^l\sqrt{a^2+s^2}}.$$

We obtain example 3.4. ∎

Example 3.5 (Hermite's Formula for Hurwitz Zeta Function ([2])). Let

$$f_a(\zeta) = \frac{1}{(\zeta+a)^z}, \qquad a \text{ is a positive constant}, \quad \mathrm{Re}z > 1.$$

Then we have

$$\sum_{n=0}^{\infty}\frac{1}{(n+a)^z} = \frac{1}{2a^z}+\frac{a^{1-z}}{z-1}+2\int_0^{\infty}\frac{\sin\left(z\arctan(\frac{t}{a})\right)}{\left(e^{2\pi t}-1\right)\left(a^2+t^2\right)^{\frac{z}{2}}}dt \qquad (\mathrm{Re}z > 1).$$

Proof. We have

$$\imath\int_0^{\infty}\frac{(-\imath t+a)^z-(\imath t+a)^z}{(e^{2\pi t}-1)(a^2+t^2)^z}dt = 2\int_0^{\infty}\frac{\sin(z\arctan(\frac{t}{a}))}{(e^{2\pi t}-1)(a^2+t^2)^{\frac{z}{2}}}dt.$$

Therefore we have example 3.5. ∎

Example 3.6 (Appell's Function ([2])).

$$f(\zeta) = \frac{b^\zeta}{(\zeta + a)^z}, \qquad (a, b \in \mathbb{R}, \quad 0 < b < 1, \quad \mathrm{Re} z > 1, \quad a > 0).$$

Then we have

$$\sum_{n=0}^{\infty} \frac{b^n}{(n+a)^z}$$
$$= \frac{1}{2a^z} + \int_0^\infty \frac{b^t}{(t+a)^z} dt + 2\int_0^\infty \frac{\sin(z \arctan\left(\frac{t}{a}\right) - t\log b)}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt.$$

Proof. We have

$$\imath \int_0^\infty \frac{f(\imath t) - f(-\imath t)}{e^{2\pi t} - 1} dt = 2\int_0^\infty \frac{\sin(z \arctan\left(\frac{t}{a}\right) - t\log b)}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt.$$

Therefore we have example 3.6. ∎

References

[1] R.P.Boas : *Entire Functions*, Academic Press, (1954).

[2] A.Erdely : *Higher Trancsendental Funtions Vol.1*, Mcgrawhill, (1953).

[3] M.Morimoto : *Analytic functionals with non-compact carriers*, Tokyo J.Math.vol.1, (1978), 77-103.

[4] P.Sargos and M.Morimoto : *Transformation des fonctionnelles analytiques à porteurs non compacts*, Tokyo J.Math.vol.4,(1981), 457-492.

[5] V.S.Vladimirov : *Holomorphic functions with nonnegative imaginary part in a tubular domain over a cone*, Math.USSR Sbornik Vol.8, (1969), 125-146.

[6] G.N.Watson : *Theory of Bessel functions*, Cambridge University Press, (1966)

[7] E.T.Whittaker and G.N.Watson : *Modern Analysis*, Cambridge University Press, (1927)