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Formal Gevrey class of formal power series solution for singular first order linear partial differential operators

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1 Introduction

In this paper, we will study the following first order partial differential equation

$$Lu(z)=\sum_{i=1}^{n}a_{i}(z)\partial_{z_{i}}u(z)=F(z,u(z)) \tag{1}$$

where $z=(z_{1},\cdots,z_{n})\in \mathbf{C}^{n}$ and $\partial_{z_{i}}=\partial/\partial z_{i}$ for $i=1,\cdots,n$. We assume the following conditions through this paper. The functions $a_{i}(z)$ and $F(z,u)$ are holomorphic functions in a neighborhood of the origin in $\mathbf{C}^{n}$ and $\mathbf{C}^{n+1}$ respectively, and $a_{i}(z)$ satisfies $a_{i}(0)=0$ for $i=1,\cdots,n$.

There are many results for (1). Oshima [O] and Kaplan [K] studied the existence of holomorphic solutions under some conditions.

We treat a formal power series solution for (1). If the solution converges, then our result becomes that of [O] and [K]. Our purpose in this paper is to give precise estimates of (1) in a formal Gevrey class via an appropriate coordinates change for (1).

We consider three examples in case $n=2$. We put

$$P_{1}=(z_{1}\partial_{z_{1}}+1)-z_{1}^{2}\partial_{z_{2}},$$

$$P_{2}=(z_{1}\partial_{z_{1}}+1)-z_{2}^{2}\partial_{z_{2}}$$

and

$$P_{3}=(z_{1}\partial_{z_{1}}+1)-(z_{1}^{2}+z_{2}^{2})\partial_{z_{2}}.$$

The operator P_{1} satisfies the conditions of [O] and [K] and $P_{1}u(z)=F(z,u)$ has a unique holomorphic solution.

Next we consider P_{2} and P_{3}. They do not satisfy the conditions of [O] and [K], while the equation

$$P_{2}u(z)=\frac{z_{2}}{1-z_{1}} \tag{2}$$

has a formal power series solution $u(z)=\sum u_{\beta_{1},\beta_{2}}z_{1}^{\beta_{1}}z_{2}^{\beta_{2}}$ with

$$u_{\beta_{1},\beta_{2}}=\frac{(\beta_{2}-1)!}{(\beta_{1}+1)^{\beta_{2}-1}}. \tag{3}$$

We find that the solution diverges with respect to a variable z_2, while

$$\sum u_{\beta_1,\beta_2}\frac{z_1^{\beta_1}z_2^{\beta_2}}{\beta_2!}$$

converges in a neighborhood of the origin by (3).

Our motivation comes from the following example. We consider

$$P_3u(z)=\frac{z_2}{1-z_1}. \tag{4}$$

We expect that (4) has a formal power series solution with similar property as in (2). But we obtain that (4) has a formal power series solution with

$$u_{\beta_1,\beta_2}\geq\frac{([\beta_1/2]+\beta_2)!([\beta_1/2]+\beta_2-1)![\beta_1/2]!}{(\beta_1+1)!\beta_2}.$$

We find that this solution diverges with respect to the both variables (z_1,z_2).

We consider the following equation

$$z_1\frac{d\phi(z_1)}{dz_1}=z_1^2+(\phi(z_1))^2. \tag{5}$$

This equation has a holomorphic solution $\phi(z_1)$ in a neighborhood of the origin with $\phi(z_1)\equiv O(z_1^2)$. For the solution $\phi(z_1)$, we change the coordinate

$$x=z_1 \text{ and } t=z_2+\phi(z_1). \tag{6}$$

Then the solution $u(z)=v(x(z),t(z))=\sum v_{\beta_1,\beta_2}x^{\beta_1}t^{\beta_2}$ has that

$$\sum v_{\beta_1,\beta_2}\frac{x^{\beta_1}t^{\beta_2}}{\beta_2!} \tag{7}$$

converges in a neighborhood of the origin.

In this paper, we find a good coordinate as (6) and give an estimate as (7) for (1).

2 Notations and Main result

The sets $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{N}$ denote the set of all real numbers, complex numbers and nonnegative integers respectively. Let $z\in\mathbf{C}^n$, $x\in\mathbf{C}^{n_0}$, $t\in\mathbf{C}^{n_1-n_0}$, and $y\in\mathbf{C}^{n-n_1}$. The set $\mathbf{C}\{y\}[[x,t]]$ denotes the set of all formal power series $\sum_{|k|+|l|\geq 0}u_{k,l}(y)x^kt^l$ with coefficients $\{u_{k,l}(y)\}$ holomorphic functions in a common neighborhood of the origin.

Definition 2.1 *Let* $u(x,t,y) = \sum_{|k|+|l|\geq 0} u_{k,l}(y)x^k t^l \in \mathbf{C}\{y\}[[x,t]]$. *If* $\sum_{|k|+|l|\geq 0} u_{k,l}(y)\frac{x^k t^l}{|l|!^d}$ *is a convergent power series for* $d \geq 0$, *then we say that* $u(x,t,y)$ *belongs to a formal Gevrey space* $G_t^{\{d\}}(x,t,y)$.

We say that d is a formal Gevrey index and t is Gevrey variables with respect to d.

We give the following two notations for the operator L.

(1)

$$S = \{z \in \mathcal{U}; a_i(z) = 0 \text{ for } i = 1, 2, \cdots, n\} \tag{8}$$

where $\mathcal{U}$ is a neighborhood of the origin in $\mathbf{C}^n$.

(2) The matrix $\left(\dfrac{\partial a}{\partial z}(0)\right)$ denotes the Jacobian matrix of $a := (a_1(z), \cdots, a_n(z))$ at the origin.

We assume that (**??**) satisfies the following conditions (A.1)–(A.4).

(**A.1**) S is a complex submanifold of codimension n_1 in $\mathcal{U}$ $(1 \leq n_1 \leq n)$.

If we assume (A.1), then there exist n_1-holomorphic functions $\zeta_i = \zeta_i(z)$ with $\zeta_i(0) = 0$ $(i = 1, 2, \cdots, n_1)$ that are functional independent each other such that

$$S = \{z \in \mathcal{U}; \zeta_i(z) = 0 \text{ for } i = 1, 2, \cdots, n_1\}. \tag{9}$$

(**A.2**) The function $F(z,u)$ is a holomorphic function in a neighborhood of the origin of $\mathbf{C}^n \times \mathbf{C}$ with $F(z,0) \equiv 0$ for $z \in S$.

(**A.3**) Jordan normal form of $\left(\dfrac{\partial a}{\partial z}(0)\right)$ is

$$J(\lambda,\mu) = \begin{pmatrix}
\lambda_1 & 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\mu_1 & \lambda_2 & 0 & \cdots & \cdots & 0 & & & \vdots \\
0 & \mu_2 & \ddots & \ddots & & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & & 0 \\
0 & \cdots & \cdots & 0 & \mu_{n_0-1} & \lambda_{n_0} & 0 & & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & & 0 \\
\vdots & & & & & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{pmatrix} \tag{10}$$

where λ_i is the nonzero eigenvalues for $i = 1, 2, \cdots, n_0$ and $\mu_i = 0$ or 1 for $i = 1, 2, \cdots, n_0 - 1$ with $1 \leq n_0 \leq n_1$.

We will define the following $\mathcal{M}$ by using $\zeta_1(z), \ldots, \zeta_{n_1}(z)$ in (9). Let $\mathbf{C}\{z\}$ be the ring of convergent power series at the origin in the variables $\{z\}$. Then we define

$$\mathcal{M} := \sum_{i=1}^{n_1} \mathbf{C}\{z\}\zeta_i(z). \tag{11}$$

Therefore by (A.1), we have $a_i(z) \in \mathcal{M}$ for $i = 1, \cdots, n$.

We define an ideal that is constructed by some elements of $\mathcal{M}$. Let m be any positive integer and set $\{g_1(z), \cdots, g_m(z)\} \subset \mathcal{M}$. Then we define

$$\mathcal{I}\{g_1, \cdots, g_m\} := \sum_{i=1}^{m} \mathbf{C}\{z\}g_i. \tag{12}$$

By (A.3), we can take n_0-functions $\left\{a_{i_j}\right\}_{j=1}^{n_0}$ that are functional independent each other. If we assume (A.1) and (A.3), then we have

$$\mathcal{M} \supset \mathcal{M}^2 \supset \mathcal{M}^3 \supset \cdots \text{ and } \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \subset \mathcal{M}. \tag{13}$$

Hence there exists δ_i such that $\delta_i := \sup\left\{d; a_i \in \mathcal{M}^d \textit{ mod } \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\}\right\}$ for each $i = 1, \cdots, n$. If $a_i \in \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\}$, then we define $\delta_i := \infty$. Then we can define the following multiplicity δ

$$\delta := \min\{\delta_1, \delta_2, \cdots, \delta_n\}$$

and we have

$$a_i \in \mathcal{M}^\delta \textit{ mod } \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \text{ for } i = 1, \cdots, n. \tag{14}$$

We assume condition (A.4).

(A.4)

$$\delta \geq 2.$$

Our main result in this paper is the following.

Theorem 2.2 *Assume (A.1), (A.2), (A.3) and (A.4). Further assume that there exists a positive constant σ such that*

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i - c\right| \geq \sigma(|k| + 1) \textit{ for } \forall k = (k_1, k_2, \cdots, k_{n_0}) \in \mathbf{N}^{n_0}$$

where $|k| = k_1 + k_2 + \cdots + k_{n_0}$ *and* $c = \dfrac{\partial F}{\partial u}(0,0)$. *Then we have the following two results.*

1) *There exists a unique formal power series solution* $u(z)$ *such that* (??).

2) *There exist local coordinates* $(x(z), t(z), y(z)) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1 - n_0} \times \mathbf{C}^{n-n_1}$ *in a neighborhood of the origin such that*

$$S = \{z \in \mathbf{C}^n; x(z) = 0, t(z) = 0\}$$

and

$$u(z) = U(x(z), t(z), y(z)) \in G_t^{\{\frac{1}{\delta - 1}\}}(x(z), t(z), y(z)).$$

If $\delta = \infty$ then we have $n_0 = n_1$. We remark that the case $\delta = \infty$ are treated in [O] and [K].

3 Properties of multiplicity and Estimates of Gamma function

In this section, we give some lemmas that are needed to prove Theorem 2.2.

3.1 Properties of multiplicity δ

We assume conditions (A.1) and (A.3), and under two conditions we show that multiplicity δ is invariant under a coordinate change and independent of a choice of n_0 independent functions from $\{a_1, \cdots, a_n\}$. Hence we may assume that $\{a_1, \cdots, a_{n_0}\}$ are functional independent by rewiting number. Then we put

$$\delta := \min\{\delta_1, \delta_2, \cdots, \delta_n\} \text{ with } \delta_i := \sup\left\{d; a_i \in \mathcal{M}^d \; mod \; \mathcal{I}\{a_1, \cdots, a_{n_0}\}\right\}.$$

Lemma 3.1 *Assume* (*A*.1) *and* (*A*.3). *Then the number* δ *is independent of a choice of* n_0 *independent functions from* $\{a_1, \cdots, a_n\}$.

Lemma 3.2 *Assume* (*A*.1) *and* (*A*.3). *Then the number* δ *is invariant under the coordinate change* $(Z_1, \cdots, Z_n)$.

3.2 Estimates of Gamma function

Here we show some lemmas needed in Section 4 in order to estimate formal power series solutions.

Let p, q, r, k_i, and $l_i \in \mathbf{N}$ for $i = 1, 2, \cdots, r$, $\delta \geq 2$ and $x! := \Gamma(x+1)$ for $x \geq 0$.

Lemma 3.3 *Let* $p + \sum_{i=1}^{r} k_i = k$, *and* $q + \sum_{i=1}^{r} l_i = l$. *Then we have*

$$\prod_{i=1}^{r} \frac{\left(k_i + \frac{1}{\delta-1} l_i\right)!}{k_i!} \leq \frac{\left(k + \frac{1}{\delta-1} l\right)!}{k!}.$$

Lemma 3.4 *Let* $p + k_1 = k$ *and* $q + l_1 = l$. *Further if* $p = 0$, *assume* $q \geq \delta$. *Then we have*

$$\frac{\left(k_1 + 1 + \frac{1}{\delta-1} l_1\right)!}{k_1!} \leq (k+1) \frac{\left(k + \frac{1}{\delta-1} l\right)!}{k!}.$$

Lemma 3.5 *Let* $p + k_1 = k$, $q + l_1 = l$ *and* $q > 0$. *Further if* $p = 0$, *assume* $q \geq \delta$. *Then we have*

$$(l_1 + 1) \frac{\left(k_1 + \frac{1}{\delta-1}(l_1 + 1)\right)!}{k_1!} \leq (\delta - 1)(k+1) \frac{\left(k + \frac{1}{\delta-1} l\right)!}{k!}.$$

Lemma 3.6 *Let* $p + k_1 = k$ *and* $q + l_1 = l$. *Further if* $p = 0$, *assume* $q \geq \delta$. *Then we have*

$$\frac{\left(k_1 + \frac{1}{\delta-1} l_1\right)!}{k_1!} \leq (k+1) \frac{\left(k + \frac{1}{\delta-1} l - 1\right)!}{k!}.$$

4 Gevrey estimates

In this section, we will study a particular equation that satisfies the assumptions of Theorem 2.2. We show that this equation has a formal power series solution that belongs to a Gevrey class. In Section 6, we reduce (??) to (16) by coodinates change and we can prove Theorem 2.2.

Let $x = (x_1, x_2, \cdots, x_{m_0}) \in \mathbf{C}^{m_0}$, $t = (t_1, t_2, \cdots, t_{m_1}) \in \mathbf{C}^{m_1}$ and $y = (y_1, y_2, \cdots, y_{m_2}) \in \mathbf{C}^{m_2}$, where $m_0 \geq 1$. We consider the following equation

$$Lu = F(x, t, y, u(x, t, y)), \tag{15}$$

where

$$L = \sum_{i=1}^{m_0} \{\lambda_i x_i + \mu_{i-1} x_{i-1} + a_i(x, t, y)\} \partial_{x_i} + \sum_{i=1}^{m_1} b_i(x, t, y) \partial_{t_i} + \sum_{i=1}^{m_2} c_i(x, t, y) \partial_{y_i} \tag{16}$$

with

$$a_{i_0}(x,t,y) \equiv O((|x|+|t|+|y|)^2), c_{i_2}(x,t,y) \equiv O((|x|+|t|+|y|)^2),$$
$$a_{i_0}(0,t,y) \equiv b_{i_1}(0,t,y) \equiv c_{i_2}(0,t,y) \equiv O(|t|^\delta)\ \delta \geq 2, \quad b_{i_1}(x,0,y) \equiv 0 \tag{17}$$

for $i_0 = 1,2,\cdots,m_0$, $i_1 = 1,2,\cdots,m_1$ and $i_2 = 1,2,\cdots,m_2$. It follows from (17) that

$$\begin{aligned} a_{i_0}(x,t,y) &= \sum_{|p|+|q|\geq 1} a_{i_0,p,q}(y)x^p t^q, \ b_{i_1}(x,t,y) = \sum_{|p|+|q|\geq 2} b_{i_1,p,q}(y)x^p t^q, \\ c_{i_2}(x,t,y) &= \sum_{|p|+|q|\geq 1} c_{i_2,p,q}(y)x^p t^q \end{aligned}$$

with

$$\begin{aligned} a_{i_0,0,q}(y) &\equiv b_{i_1,0,q}(y) \equiv c_{i_2,0,q}(y) \equiv 0 \text{ for } |q| = 1,\cdots,\delta-1, \\ a_{i_0,p,0}(0) &= c_{i_2,p,0}(0) = 0 \text{ for } |p|=1, \quad b_{i_1,p,0}(y) \equiv 0 \text{ for } \forall p \in \mathbf{N}^{m_0}. \end{aligned}$$

The function $F(x,t,y,u)$ is a holomorphic function in a neighborhood of the origin such that

$$F(0,0,y,0) \equiv 0.$$

Theorem 4.1 *Assume that there exists a positive constant σ such that*

$$\left|\sum_{i=1}^{m_0} \lambda_i k_i - c\right| \geq \sigma(|k|+1) \text{ for } \forall k = (k_1,k_2,\cdots,k_{m_0}) \in \mathbf{N}^{m_0} \tag{18}$$

for (15) *where* $|k| = k_1 + k_2 + \cdots + k_{m_0}$ *and* $c = \dfrac{\partial F}{\partial u}(0,0,0,0)$. *Then equation* (15) *has a unique formal power series solution* $u(x,t,y)$ *which belongs to* $G_t^{\{\frac{1}{\delta-1}\}}(x,t,y)$.

Proof. Put

$$u(x,t,y) = \sum_{|k|+|l|\geq 1} u_{k,l}(y)x^k t^l, \ F(x,t,y,u) = \sum_{|p|+|q|+r\geq 1} F_{p,q,r}(y)x^p t^q u^r \tag{19}$$

and

$$u_{k,l}(y) = \frac{\left(|k| + \frac{1}{\delta-1}|l|\right)!}{|k|!} v_{k,l}(y). \tag{20}$$

Then we consider a formal power series $v(x,t,y) = \sum_{|k|+|l|\geq 1} v_{k,l}(y)x^k t^l$. In order to prove Theorem 4.1, we will show that a formal power series $v(x,t,y)$ exists and it converges in a neighborhood of the origin. Therefore $u(x,t,y)$ belongs to $G_t^{\{\frac{1}{\delta-1}\}}(x,t,y)$ by (20).

We define

$$\begin{aligned} e(n,1) &= (1,0,\cdots,0),\cdots,e(n,n)=(0,\cdots,0,1)\in \mathbf{N}^n \text{ for } \forall n=1,2,\cdots, \\ k_{(i)} &= (k_1^i,k_2^i,\cdots,k_{m_0}^i)\in \mathbf{N}^{m_0},\ l_{(i)}=(l_1^i,l_2^i,\cdots,l_{m_1}^i)\in \mathbf{N}^{m_1}, \\ k_{\{r\}} &= \left(\sum_{i=1}^r k_1^i,\cdots,\sum_{i=1}^r k_{m_0}^i\right) \text{ and } l_{\{r\}}=\left(\sum_{i=1}^r l_1^i,\cdots,\sum_{i=1}^r l_{m_1}^i\right). \end{aligned}$$

By substituting (19) and (20) into (15), we have the following recurrence relations

$$\begin{aligned} &\lambda_i v_{e(m_0,i),0}(y)+\mu_i v_{e(m_0,i+1),0}(y)+\sum_{j=1}^{m_0} a_{j,e(m_0,i),0}(y)v_{e(m_0,j),0}(y) \\ &\quad = F_{e(m_0,i),0,0}(y)+F_{0,0,1}(y)v_{e(m_0,i),0}(y) \text{ for } i=1,2,\cdots,m_0, \end{aligned} \tag{21}$$

$$0=F_{0,e(m_1,i),0}(y)+F_{0,0,1}(y)(1/(\delta-1))!v_{0,e(m_1,i)}(y) \text{ for } i=1,2,\cdots,m_1 \tag{22}$$

and for $|k|+|l|\geq 2$

$$\begin{aligned} v_{k,l}(y) &+ \sum_{i=1}^{m_0}\frac{\mu_{i-1}(k_i+1)}{(\sum_{i=1}^{m_0}\lambda_i k_i-F_{0,0,1}(y))}v_{k+e(m_0,i)-e(m_0,i-1),l}(y) \\ &+ \sum_{i=1}^{m_0}\sum_{\substack{p+k(1)=k\\|p|=1}}\frac{(k_i^1+1)}{(\sum_{i=1}^{m_0}\lambda_i k_i-F_{0,0,1}(y))}a_{i,p,0}(y)v_{k_{(1)}+e(m_0,i),l}(y) \\ &= I_1-I_2-I_3-I_4 \end{aligned} \tag{23}$$

where

$$\begin{aligned} I_1 &= \frac{1}{(\sum_{i=1}^{m_0}\lambda_i k_i-F_{0,0,1}(y))}\frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{\substack{p+k_{\{r\}}=k\\ q+l_{\{r\}}=l\\ |p|+|q|+r\geq 1\\ (p,q,r)\neq(0,0,1)}} F_{p,q,r}(y)\prod_{i=1}^r\frac{(|k_{(i)}|+\frac{1}{\delta-1}|l_{(i)}|)!}{|k_{(i)}|!}v_{k_{(i)},l_{(i)}}(y), \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{(\sum_{i=1}^{m_0}\lambda_i k_i-F_{0,0,1}(y))}\frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{i=1}^{m_0}\sum_{\substack{p+k_{(1)}=k\\ q+l_{(1)}=l\\ |p|+|q|\geq 2}}(k_i^1+1)a_{i,p,q}(y)\frac{(|k_{(1)}|+1+\frac{1}{\delta-1}|l_{(1)}|)!}{(|k_{(1)}|+1)!}v_{k_{(1)}+e(m_0,i),l}(y), \end{aligned}$$

$$I_3 = \frac{1}{(\sum_{i=1}^{m_0} \lambda_i k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k| + \frac{1}{\delta-1}|l|)!}$$
$$\times \sum_{i=1}^{m_1} \sum_{\substack{p+k_{(1)}=k \\ q+l_{(1)}=l \\ |p|+|q|\geq 2}} (l_i^1 + 1) b_{i,p,q}(y) \frac{(|k_{(1)}| + \frac{1}{\delta-1}(|l_{(1)}|+1))!}{|k_{(1)}|!} v_{k_{(1)},l_{(1)}+e(m_1,i)}(y)$$

and

$$I_4 = \frac{1}{(\sum_{i=1}^{m_0} \lambda_i k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k| + \frac{1}{\delta-1}|l|)!}$$
$$\times \sum_{i=1}^{m_2} \sum_{\substack{p+k_{(1)}=k \\ q+l_{(1)}=l \\ |p|+|q|\geq 1}} c_{i,p,q}(y) \frac{(|k_{(1)}| + \frac{1}{\delta-1}|l_{(1)}|)!}{|k_{(1)}|!} \partial_{y_i} v_{k_{(1)},l_{(1)}}(y).$$

Let us show that $\{v_{k,l}(y)\}_{|k|+|l|\geq 1}$ are inductively determined.
For $v(x,t,y) = \sum_{|k|+|l|\geq 1} v_{k,l}(y) x^k t^l$, we define

$$(v)_m = \sum_{|k|+|l|=m} v_{k,l}(y) x^k t^l \text{ and } ||(v)_m||_{r_0} = \sum_{|k|+|l|=m} \max_{|y|\leq r_0} |v_{k,l}(y)|$$

The system equation (21) and (22) have a holomorphic solution $\{v_{k,l}(y)\}_{|k|+|l|=1}$ for sufficiently small $|y|$ by the conditions $a_{j,e(m_0,i),0}(0) = 0$ and (18). In a word, we have $(v)_1$.

Next we consider $(v)_m$ for $m \geq 2$. For (23) we define

$$(Lv)_m := (v)_m + \sum_{|k|+|l|=m} \left\{ \sum_{i=1}^{m_0} \{\mu_{i-1} \frac{(k_i+1)}{\sum_{j=1}^{m_0} \lambda_j k_j - F_{0,0,1}(y)} v_{k+e(m_0,i)-e(m_0,i-1),l} \right.$$
$$\left. + \sum_{\substack{p+k_{(1)}=k \\ |p|=1}} \frac{(k_i^1+1)}{\sum_{j=1}^{m_0} \lambda_j k_j - F_{0,0,1}(y)} a_{i,p,0} v_{k_{(1)}+e(m_0,i),l}\} \right\} x^k t^l.$$

Then (23) becomes
$$(Lv)_m = \{(v)_{m'};\ m' < m\}. \tag{24}$$

For $(Lv)_m$, we have the following lemma.

Lemma 4.2 *Assume* (18). *Then there exist positive constants* σ_1 *and* r_0 *such that*
$$||(Lv)_m||_{r_0} \geq \sigma_1 ||(v)_m||_{r_0}. \tag{25}$$

For $m' < m$, we assume that $(v)_{m'}$ is determined. By (24) and Lemma 4.2, we have $(v)_m$. Therefore $(v)_m$ is inductively determined for all $m \geq 1$. In a word, equation (15) has a unique formal power series solution.

Next we show that $v(x,t,y)$ converges. So we will give an estimate of $v_{k,l}(y)$. By Lemma 3.3, 3.4, 3.5,3.6 and 4.2, we obtain the following inequality from (23)

$$
\begin{aligned}
\sigma_3 ||(v)_m||_{r_0} \leq & \sum_{\substack{|p|+|q|+m_{\{r\}}=m \\ |p|+|q|+r\geq 1 \\ (p,q,r)\neq(0,0,1)}} \max_{|y|\leq r_0} |F_{p,q,r}(y)| \prod_{i=1}^{r} ||(v)_{m_{(i)}}||_{r_0} \\
& + \sum_{i=1}^{m_0} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 2}} \max_{|y|\leq r_0} |a_{i,p,q}(y)| ||(v)_{m_{(1)}+1}||_{r_0} \\
& + (\delta-1) \sum_{i=1}^{m_1} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 2}} \max_{|y|\leq r_0} |b_{i,p,q}(y)| ||(v)_{m_{(1)}+1}||_{r_0} \\
& + \sum_{i=1}^{m_2} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 1}} \frac{1}{|k|+\frac{1}{\delta-1}|l|} \max_{|y|\leq r_0} |c_{i,p,q}(y)| ||(\partial_{y_i} v)_{m_{(1)}}||_{r_0},
\end{aligned} \tag{26}
$$

where $m_{\{r\}} = \sum_{i=1}^{r} m_i$ and $\sigma_3 = \sigma_1/\sigma_2$.
We define $F_{p,q,r}(R_0)$, $a_{i,p,q}(R_0)$, $b_{i,p,q}(R_0)$ and $c_{i,p,q}(R_0)$ as follows;

$$
\begin{aligned}
& F_{0,0,1}(R_0) := 0, \ F_{p,q,0}(R_0) := \frac{\sigma_3 \max_{i=1,\cdots,m_2} \{||(v)_1||_{R_0}, ||(\partial_{y_i} v)_1||_{R_0}\}}{m_0+m_1} \quad (|p|+|q|=1), \\
& F_{p,q,r}(R_0) := \max_{|y|\leq R_0} |F_{p,q,r}(y)| \ (|p|+|q|+r\geq 2), \ a_{i,p,q}(R_0) := \max_{|y|\leq R_0} |a_{i,p,q}(y)|, \\
& b_{i,p,q}(R_0) := \max_{|y|\leq R_0} |b_{i,p,q}(y)|, \ c_{i,p,q}(R_0) := \max_{|y|\leq R_0} |c_{i,p,q}(y)|.
\end{aligned} \tag{27}
$$

Let $0 < r_0 < R_0 < 1$. We consider the following equation

$$
\begin{aligned}
\sigma_3 Y = & \frac{1}{R_0-r_0} \sum_{|p|+|q|+r\geq 1} \frac{F_{p,q,r}(R_0)}{(R_0-r_0)^{|p|+|q|-1}} X^{|p|+|q|} Y^r \\
& + \frac{1}{R_0-r_0} \sum_{i=1}^{m_0} \sum_{|p|+|q|\geq 2} \frac{a_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-2}} X^{|p|+|q|-1} Y \\
& + \frac{\delta-1}{R_0-r_0} \sum_{i=1}^{m_1} \sum_{|p|+|q|\geq 2} \frac{b_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-2}} X^{|p|+|q|-1} Y \\
& + \frac{(\delta-1)e}{R_0-r_0} \sum_{i=1}^{m_2} \sum_{|p|+|q|\geq 1} \frac{c_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-1}} X^{|p|+|q|} Y.
\end{aligned} \tag{28}
$$

By the condition $F_{0,0,1}=0$ and implicit function theorem at $Y=X=0$, equation (28) admits a holomorphic solution $Y(X)$.

Proposition 4.3 *We obtain that* (28) *has a holomorphic solution* $\sum_{m\geq 1} Y_m(r_0)X^m$ *with the estimates*

$$\begin{aligned} &||(v)_m||_{r_0} \leq Y_m(r_0) \\ &||(\partial_{y_i}v)_m||_{r_0} \leq emY_m(r_0) \ \textit{for}\ i=1,\cdots,m_2. \end{aligned} \tag{29}$$

We use the following lemma in order to prove Proposition 4.3.

Lemma 4.4 *If* $(v)_m$ *satisfies*

$$||(v)_m||_{r_0} \leq \frac{C}{(R_0-r_0)^p} \quad \textit{for} \quad 0<r_0<R_0$$

for some $p \geq 0$ *and* $C>0$*, then we have*

$$||(\partial_{y_i}v)_m||_{r_0} \leq \frac{(p+1)eC}{(R_0-r_0)^{p+1}} \quad \textit{for} \quad i=1,2,\cdots,m_2$$

where $||(v)_m||_{r_0}=\sum_{|k|+|l|=m}\max_{|y|\leq r_0}|v_{k,l}(y)|$.

Proof of Proposition 4.3. By substituting $\sum_{m\geq 1} Y_m(r_0)X^m$ into (28), we have the following recurrence relations

$$\sigma_3 Y_1 = \sum_{|p|+|q|=1} F_{p,q,0}(R_0) \text{ for } m=1 \tag{30}$$

and for $m \geq 2$

$$\begin{aligned} \sigma_3(R_0-r_0)Y_m \ &= \sum_{\substack{|p|+|q|+m_{\{r\}}=m \\ |p|+|q|+r\geq 1}} \frac{F_{p,q,r}(R_0)}{(R_0-r_0)^{|p|+|q|+r-2}} \prod_{i=1}^{r} Y_{m_{(i)}} \\ &+ \sum_{i=1}^{m_0} \sum_{\substack{|p|+|q|-1+m_{(1)}=m \\ |p|+|q|\geq 2}} \frac{a_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-2}} Y_{m_{(1)}} \\ &+ (\delta-1)\sum_{i=1}^{m_1} \sum_{\substack{|p|+|q|-1+m_{(1)}=m \\ |p|+|q|\geq 2}} \frac{b_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-2}} Y_{m_{(1)}} \\ &+ (\delta-1)e\sum_{i=1}^{m_2} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 1}} \frac{c_{i,p,q}(R_0)}{(R_0-r_0)^{|p|+|q|-1}} Y_{m_{(1)}}. \end{aligned} \tag{31}$$

Then $Y_m(r_0)$ is inductively determined for $m \geq 1$ by (30) and (31) as in the case of $(v)_m$, and we obtain that Y_m becomes a form $C_m/(R_0 - r_0)^{m-1}$ with $C_m \geq 0$ by easy calculation. By (27) and (30), we obtain (29) for $m = 1$.

Next we assume (29) for $m' < m$ $(m \geq 2)$. By (26) and (31), we obtain

$$||(v)_m||_{r_0} \leq (R_0 - r_0)Y_m(r_0) \leq Y_m(r_0). \tag{32}$$

By $||(v)_m||_{r_0} \leq (R_0 - r_0)Y_m = C_m/(R_0 - r_0)^{m-2}$ and Lemma 4.4, we have

$$||(\partial_{y_i} v)_m||_{r_0} \leq \frac{e(m-1)C_m}{(R_0 - r_0)^{m-1}} \leq emY_m(r_0). \tag{33}$$

Hence we obtain Proposition 4.3 for $m \geq 1$. Q.E.D.

By Proposition 4.3, we have that $v(x, t, y)$ converges. Hence this completes the proof of Theorem 4.1. Q.E.D.

5 Holomorphic solution of system equation

In this section, we consider the existence of a holomorphic solution for a nonlinear first order partial differential equation. By the result, we obtain the existence of coordinates change for main theorem to be reduced to the form studied in Section 4. In fact, we prove Main theorem by using the coordinate change in the next section.

Let $w = (w_1, \ldots, w_n) = (w_1, \cdots, w_{n_0}, w_{n_0+1}, \cdots, w_n) = (w', w'') \in \mathbf{C}^n$, $p \in \mathbf{N}^{n_0}$ and $q \in \mathbf{N}^m$, and $b_{j,l}(w, \Phi)$, $c_j(w, \Phi)$ are convergent power series in a neighborhood of the origin in $\mathbf{C}^n \times \mathbf{C}^m$ where $\Phi = (\Phi_1, \cdots, \Phi_m)$ for $j = 1, \cdots, m$ and $l = 1, \cdots, n$. We assume that $b_{j,l}(w, \Phi)$, $c_j(w, \Phi)$ have the following expansion

$$\begin{aligned} b_{j,l}(w, \Phi) &= \sum_{|p|+|q|\geq 1} b_{j,l,p,q}(w'')\{w'\}^p\{\Phi\}^q \\ c_j(w, \Phi) &= \sum_{|p|+|q|\geq 1} c_{j,p,q}(w'')\{w'\}^p\{\Phi\}^q \end{aligned}$$

where $b_{j,l,p,q}(0) = c_{j,p,q}(0) = 0$ $(|p| + |q| = 1)$.

We consider the following system equation

$$\sum_{i=1}^{n_0}(\lambda_i w_i + \mu_{i-1}w_{i-1})\partial_{w_i}\Phi_j = \sum_{l=1}^{n} b_{j,l}(w, \Phi)\partial_{w_l}\Phi_j + c_j(w, \Phi) \tag{34}$$

with $j = 1, \cdots, m$.

Then we have the following proposition.

Proposition 5.1 *Assume that there exists a positive constant* σ_4 *such that*

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i\right| \geq \sigma_4 |k| \ for \ \forall k = (k_1, k_2, \cdots, k_{n_0}) \in \mathbf{N}^{n_0}.$$

Then we obtain that (34) *has a tuple of unique holomorphic solution* $(\Phi_1(w), \cdots, \Phi_m(w))$ *in a neighborhood of the origin with* $\Phi_j(0, w'') \equiv 0$ *for* $j = 1, 2, \cdots, m$.

We can prove Proposition 5.1 as in Theorem 4.1. We omit a proof.

6 Proof of Theorem

In this section, we transform equation (??) to the one studied in Section 4 (Theorem 4.1) via a coordinate change. Hence Main theorem is completely proved by Theorem 4.1.

Suppose that $\eta(z) = (\eta_1(z), \cdots, \eta_n(z))$ is a local coordinate in a neighborhood of the origin. Then by $\eta = \eta(z)$, the operator L becomes

$$L = \sum_{i=1}^{n} a'_i(\eta)\partial_{\eta_i} \tag{35}$$

where

$$\sum_{j=1}^{n} a_j(z)\partial_{z_j}\eta_i(z) = a'_i(\eta(z)) \tag{36}$$

for $i = 1, \cdots, n$.

Lemma 6.1 *Assume* (*A*.1) *and* (*A*.3). *There exist some coordinate changes* η *such that*

$$\begin{array}{lll} 1. & a'_i(\eta) = \lambda_i\eta_i + \mu_{i-1}\eta_{i-1} + b_i(\eta) & for \ \ i = 1, \cdots, n_0 \\ & a'_i(\eta) = b_i(\eta) & for \ \ i = n_0 + 1, \cdots, n. \\ 2. & b_i(\eta) = O(|\eta|^2) & for \ \ i = 1, \cdots, n. \\ 3. & b_i(0, \cdots, 0, \eta_{n_1+1}, \eta_{n_1+2}, \cdots, \eta_n) \equiv 0 & for \ \ i = 1, 2, \cdots, n. \end{array} \tag{37}$$

Proof. We omit a proof. Q.E.D.

By Lemma 6.1 we may assume that L is in the form

$$L = \sum_{i=1}^{n_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + b_i(z))\, \partial_{z_i} + \sum_{i=n_0+1}^{n} b_i(z)\partial_{z_i}, \tag{38}$$

where

$$b_i(0, \cdots, 0, z_{n_1+1}, \cdots, z_n) \equiv 0 \text{ and } b_i(z) = O(|z|^2)$$

for $i = 1, 2, \cdots, n$. Put $z' = (z_1, \cdots, z_{n_0})$, $z'' = (z_{n_0+1}, \cdots, z_{n_1})$ and $z''' = (z_{n_1+1}, \cdots, z_n)$.

Lemma 6.2 *Assume that* (38) *satisfies that there exists a positive constant* σ *such that*

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i - c\right| \geq \sigma(|k| + 1) \text{ for some } c \text{ and all } k \in \mathbf{N}^{n_0}. \tag{39}$$

Then there exists a tuple of holomorphic function $(\Phi_1(z', z'''), \cdots, \Phi_{n_1-n_0}(z', z'''))$ *with* $\Phi_j(0, z''') \equiv 0$ *for* $j = 1, \cdots, n_1 - n_0$ *such that*

$$L(z_{n_0+j} - \Phi_j(z', z''')) = \sum_{i=1}^{n_1-n_0} E_{i,j}(z)(z_{n_0+i} - \Phi_i(z', z''')) \tag{40}$$

where $E_{i,j}(z)$ *is a holomorphic function in a neighborhood of the origin with* $E_{i,j}(0) = 0$ *for* $i, j = 1, \cdots, n_1 - n_0$.

Proof. We consider the following equation in order to prove Lemma 6.2

$$\begin{aligned}&\sum_{i=1}^{n_0} \{\lambda_i z_i + \mu_{i-1} z_{i-1} + b_i(z', \Phi, z''')\} \partial_{z_i} \Phi_j(z', z''') \\ &+ \sum_{i=n_1+1}^{n} b_i(z', \Phi, z''') \partial_{z_i} \Phi_j(z', z''') = b_{n_0+j}(z', \Phi, z''')\end{aligned}$$

for $j = 1, \cdots, n_1 - n_0$, where $\Phi = (\Phi_1, \cdots, \Phi_{n_1-n_0})$. By condition (39), there exists a positive constant σ_4 such that

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i\right| \geq \sigma_4 |k| \text{ for } k \in \mathbf{N}^{n_0}.$$

We have that (41) satisfies the assumptions of Proposition 5.1 by putting $m = n_1 - n_0$, $z' \mapsto w'$ and $z''' \mapsto w''$, where m, w' and w'' in Section 5. Therefore we obtain that (41) has a tuple of holomorphic solution $\{\Phi_j(z', z''')\}_{j=1}^{n_1-n_0}$ with $\Phi_j(0, z''') \equiv 0$ for $j = 1, \cdots, n_1 - n_0$.

Next put $\tau_j = z_{n_0+j} - \Phi_j(z', z''')$. Then we have

$$L\tau_j = \sum_{i=1}^{n} (b_i(z', \Phi, z''') - b_i(z)) \partial_{z_i} \Phi_j(z', z''') + b_{n_0+j}(z) - b_{n_0+j}(z', \Phi, z'''). \tag{41}$$

Further we can put

$$b_j(z', \Phi, z''') - b_j(z) = \sum_{i=1}^{n_1-n_0} e_{i,j}(z)(z_{n_0+i} - \Phi_i) = \sum_{i=1}^{n_1-n_0} e_{i,j}(z)\tau_i \tag{42}$$

for holomorphic functions $e_{i,j}(z)$. Therefore we have the desired result. Q.E.D.

By Lemma 6.2 and the coordinate change $\tau_j = z_{n_0+j} - \Phi_j(z', z''')$ with $j = 1, \cdots, n_1 - n_0$, L becomes the following form

$$L = \sum_{i=1}^{n_0} \{\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i(z', \tau, z''')\} \partial_{z_i} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}(z', \tau, z''') \partial_{\tau_i} + \sum_{i=n_1+1}^{n} c_i(z', \tau, z''') \partial_{z_i}, \tag{43}$$

where $c_{i_0}(0, 0, z''') \equiv c_{i_1}(z', 0, z''') \equiv 0$ for $i_0 = 1, \cdots, n_0 . n_1 + 1, \cdots, n$, $i_1 = n_0 + 1, n_0 + 2, \cdots, n_1$ and $c_i(z', \tau, z''') \equiv O((|z'| + |\tau| + |z'''|)^2)$ for $i = 1, \cdots, n$.

In the following lemma, we seek multiplicity δ. So we refer to multiplicity. We have

$$c_i(z', \tau, z''') = \sum_{j=1}^{n_0} d_{i,j}(\lambda_j z_j + \mu_{j-1} z_{j-1} + c_j) + O(|\tau|^\delta) \tag{44}$$

for $i = n_0 + 1, \cdots, n$ by (14), Lemma 3.1 and 3.2, where $d_{i,j} = d_{i,j}(z', \tau, z''')$ is a holomorphic function. Then we obtain the following result.

Lemma 6.3 *There exist local coordinates $(x, t, y) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1-n_0} \times \mathbf{C}^{n-n_1}$ such that* (43) *becomes the following form*

$$L = \sum_{i=1}^{n_0} \{\lambda_i x_i + \mu_{i-1} x_{i-1} + A_i(x, t, y)\} \partial_{x_i} + \sum_{i=1}^{n_1-n_0} A_{n_0+i}(x, t, y) \partial_{t_i} + \sum_{i=1}^{n-n_1} A_{n_1+i}(x, t, y) \partial_{y_i},$$

where $A_{i_0}(0, t, y) \equiv O(|t|^\delta)$ and $A_{i_1}(0, 0, y) \equiv A_{i_2}(x, 0, y) \equiv 0$ for $i_0 = 1, \cdots, n$, $i_1 = 1, \cdots, n_0, n_1 + 1, \cdots, n$ and $i_2 = n_0 + 1, n_0 + 2, \cdots, n_1$.

Proof. Let $x_{i_0} = \lambda_{i_0} z_{i_0} + \mu_{i_0-1} z_{i_0-1} + c_{i_0}(z', \tau, z''')$, $t_{i_1} = \tau_{i_1}$ and $y_{i_2} = z_{i_2+n_1}$ for $i_0 = 1, 2, \cdots, n_0$, $i_1 = 1, 2, \cdots, n_1 - n_0$ and $i_2 = 1, 2, \cdots, n - n_1$. Then we have

$$L = \sum_{i_0=1}^{n_0} (L x_{i_0}) \partial_{x_{i_0}} + \sum_{i_1=1}^{n_1-n_0} (L t_{i_1}) \partial_{t_{i_1}} + \sum_{i_2=1}^{n-n_1} (L y_{i_2}) \partial_{y_{i_2}}.$$

For $x_i = \lambda_i z_i + \mu_{i-1} z_{i-1} + c_i(z', t, y)$ $(i = 1, 2, \cdots, n_0)$ by implicit function theorem at $x = z' = t = 0$, we obtain n_0-holomorphic functions $z' = (z_1(x, t, y), z_2(x, t, y), \cdots, z_{n_0}(x, t, y))$ with $z_i(0, 0, y) \equiv 0$ for $i = 1, 2, \cdots, n_0$. By (44), we have

$$c_i(z'(0, t, y), t, y) \equiv O(|t|^\delta) \tag{45}$$

for $i = n_0+1, \cdots, n$. Put

$$A_{i_0}(x,t,y) = \left\{ \sum_{i=1}^{n_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i)\partial_{z_i} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}\partial_{\tau_i} + \sum_{i=n_1+1}^{n} c_i \partial_{z_i} \right\} c_{i_0} \Bigg|_{z'=z'(x,t,y)}$$

and $A_{i_1}(x,t,y) = c_{i_1}|_{z'=z'(x,t,y)}$ for $i_0 = 1, 2, \cdots, n_0$ and $i_1 = n_0+1, n_0+2, \cdots, n$. Then by (45) we have

$$A_i(0,t,y) \equiv O(|t|^\delta)$$

for $i = 1, \cdots, n$. Since we have

$$\begin{aligned} Lx_{i_0} &= \lambda_{i_0}(\lambda_{i_0} z_{i_0} + \mu_{i_0-1} z_{i_0-1} + c_{i_0}) + \mu_{i_0-1}(\lambda_{i_0-1} z_{i_0-1} + \mu_{i_0-2} z_{i_0-2} + c_{i_0-1}) \\ &+ \sum_{i=1}^{n_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i)\partial_{z_i} c_{i_0} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}\partial_{\tau_i} c_{i_0} + \sum_{i=n_1+1}^{n} c_i \partial_{z_i} c_{i_0}, \\ Lt_{i_1} &= c_{n_0+i_1} \text{ and } Ly_{i_2} = c_{n_1+i_2} \end{aligned}$$

for $i_0 = 1, 2, \cdots, n_0$, $i_1 = 1, 2, \cdots, n_1 - n_0$ and $i_2 = 1, 2, \cdots, n - n_1$, we obtain the desired result. Q.E.D.

By Lemma 6.3, we find that (1) becomes (16) by putting $m_0 = n_0$, $m_1 = n_1 - n_0$ and $m_2 = n - n_1$. Hence this completes the proof of Theorem 2.2 by Theorem 4.1.

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