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Kyoto University
Microlocalization of the Topological Boundary Value Morphism for Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) \(\mathcal{D}\)-Module, here in this article, we shall write Module with a capital letter, instead of sheaf of modules). If the system is regular-specializable, we can define the nearby-cycle of the system in the theory of \(\mathcal{D}\)-Modules. The definitions of regular-specializable \(\mathcal{D}\)-Module and its nearby-cycle are initiated by Kashiwara [Kas], Kashiwara and Kawai [K-K 1] and Malgrange [Mal] for regular-holonomic cases. These definitions extended to the specializable \(\mathcal{D}\)-Module (see Laurent [L], Laurent and Malgrange [L-Ma] and Mebkhout [Me]). After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 3], [Sc 4], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism (called the topological boundary value morphism) which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the induced system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the near-hyperbolicity). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

In this section, we shall fix the notation used in later sections. We denote the set of integers, of real numbers and of complex numbers by \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) respectively as usual. Moreover we set \( \mathbb{N} := \{ n \in \mathbb{Z}; n \geq 1 \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

In this article, all the manifolds are assumed to be paracompact. Let \( M \) be an \((n+1)\)-dimensional real analytic manifold and \( N \) a one-codimensional closed real analytic submanifold of \( M \). Let \( X \) and \( Y \) be complexifications of \( M \) and \( N \) respectively such that \( Y \) is a closed submanifold of \( X \) and that \( Y \cap M = N \). Moreover in this paper, we assume the existence of a partial complexification of \( M \) in \( X \); that is, there exists a \((2n+1)\)-dimensional real analytic submanifold \( L \) of \( X \) containing both \( M \) and \( Y \) such that the triplet \((N, M, L)\) is locally isomorphic to \((\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R})\) by a local coordinate system \((z, \tau) = (x + \sqrt{-1}y, t + \sqrt{-1}s)\) of \( X \) around each point of \( N \). We say such a coordinate system admissible. We shall mainly follow the notation in Kashiwara-Schapira [K-S 2]; we denote the normal deformations of \( N \) and \( Y \) in \( M \) and \( L \) by \( \overline{M}_N \) and \( \widetilde{L}_Y \) respectively and regard \( \overline{M}_N \) as a closed submanifold of \( \widetilde{L}_Y \). We have the following commutative diagram:

![Diagram](image)

and by admissible coordinates we have locally the following relation:

\[
\begin{align*}
N &= \mathbb{R}^n_x \times \{0\} \\
\xrightarrow{i} & M = \mathbb{R}^n_x \times \mathbb{R}_t \\
\xrightarrow{i} & Y = \mathbb{C}^n_x \times \{0\} \\
\xrightarrow{i} & L = \mathbb{C}^n_x \times \mathbb{R}_t
\end{align*}
\]

With these coordinates, we often identify \( T_Y X \) and \( T_Y L \) with \( X \) and \( L \) respectively. The projection \( \tau_Y: T_Y L \rightarrow Y \) induces natural mappings:

\[
T_N^*Y \xrightarrow{\tau_Y*} T_N^*M \times T_N^*Y \xrightarrow{i} T_N^*M T_Y L,
\]

and by \( \tau_Y^* \) we identify \( T_{T_N^*M T_Y L} \) with \( T_N^*M \times T_N^*Y \). Similarly by natural mappings

\[
T_{\widetilde{L}_Y}^* \xrightarrow{s_{\widetilde{L}_Y}*} T_N^*M \times T_{\widetilde{L}_Y}^* \xrightarrow{i} T_{T_N^*M T_Y L}.
\]
we identify $T_N M \times T_Y L$ with $T_{T_N M}^* T_Y L$.

$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z, t) \in T_Y L; t > 0\}.$$ 

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Note that to define $T_Y L^+$ (or $T_N M^+$) by means of admissible coordinates is equivalent to determining a local isomorphism $\sigma_{Y/L} \simeq \mathbb{Z}_Y$ (or equivalently $\sigma_{N/M} \simeq \mathbb{Z}_N$). Here $\sigma_{Y/L}$ denotes the relative orientation sheaf.

Define open embeddings $f$ and $f_N$ by:

$$T_Y L^+ \overset{f}{\to} T_Y L$$

$$T_N M^+ \overset{f_N}{\to} T_N M.$$ 

Thus we regard $T_N M^+ \times T_Y L$ as an open set of $T_{T_N M}^* T_Y L$. Moreover $f$ induces mappings:

$$T_{T_N M}^* T_Y L \overset{f}{\to} T_{T_N M}^* T_Y L$$

$$T_N M^+ \times T_Y L \overset{f_N \times \text{id}}{\to} T_N M^+ \times T_Y Y.$$ 

Hence we identify $T_{T_N M}^* T_Y L^+$ with $T_N M^+ \times T_Y Y$, and $f_\pi$ with $f_N \times \text{id}$.

## 2 Several Sheaves Attached to the Boundary

In this section, we recall several sheaves attached to the boundary due to Oaku [Oa 2]. These sheaves will play essential roles for our boundary value problem. We remark that in Oaku [Oa 2] these sheaves are defined on cosphere bundles. So we shall present equivalent but slightly different definitions on cotangent bundles along the line of Oaku-Yamazaki [O-Y]. We refer to Oaku [Oa 2] or Oaku-Yamazaki [O-Y] for the proofs. Note that although the higher-codimensional case is treated in Oaku-Yamazaki [O-Y], the same proofs also work as in the one-codimensional case.

As usual, we denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of holomorphic functions on $X$, of hyperfunctions on $M$ and of microfunctions on $T_M^* X$ respectively. Further, we denote by $\mathcal{B}_L$ the sheaf of hyperfunctions with holomorphic parameters on $L$; that is,

$$\mathcal{B}_L := \mathcal{H}_1^L(\mathcal{O}_X) \otimes \sigma_{L/X} \simeq i_L^! \mathcal{O}_X \otimes \sigma_{L/X}[1].$$

We denote as usual by $\nu$ and $\mu$ the Sato specialization and microlocalization functors respectively.
2.1 Definition. We set:
\[
C_{N|M} := s_{L \pi}^{-1} \mathcal{H}^{n} (\mu_{M}(j_{L} \ast \overline{p}_{LL}^{-1} B L_{Y})) \otimes \omega_{N/L},
\]
\[
B_{N|M} := C_{N|M}|_{T_{N} M}.
\]
We denote by \(\pi_{N|M}\) the natural projection from \(T_{N}^{*} M\) to \(T_{N} M\). Let \(\pi_{N|M}\) be the restriction of \(\pi_{N|M}\) to \(T_{N}^{*} M \setminus T_{N}^{*} M N\) as usual. By virtue of the following proposition, we can regard \(\mathcal{G}_{N|M}\) as a microlocalization of \(\nu_{N}(B_{M})\):

2.2 Proposition. There exists the following exact sequence on \(T_{N} M\):
\[
0 \longrightarrow \nu_{Y} (B L_{Y})|_{T_{N} M} \longrightarrow B_{N|M} \longrightarrow \pi_{N|M*} C_{N|M} \longrightarrow 0.
\]
Moreover, an isomorphism \(\nu_{N}(B_{M}) \simeq B_{N|M}\) holds.

2.3 Definition. We set:
\[
\tilde{C}_{N|M} := \mathcal{H}^{n} (\mu_{T_{N} M}(\nu_{Y}(B L_{Y}))) \otimes \omega_{N/Y},
\]
\[
\tilde{B}_{N|M} := \tilde{C}_{N|M}|_{T_{N} M} \simeq \mathcal{H}^{n}_{T_{N} M}(\nu_{Y}(B L_{Y})) \otimes \omega_{N/Y}.
\]
By the following fact, we can regard \(C_{N|M}\) as a subsheaf of \(\tilde{C}_{N|M}\):

2.4 Proposition. There exists a natural monomorphism \(C_{N|M} \longrightarrow \tilde{C}_{N|M}\).

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable \(\mathcal{D}\)-Module and its nearby-cycle.

As usual, we denote by \(\mathcal{D}_{X}\) the sheaf on \(X\) of holomorphic differential operators, and by \(\{\mathcal{D}_{X}^{(m)}\}_{m \in \mathbb{N}_{0}}\) the usual order filtration on \(\mathcal{D}_{X}\). First, let us recall the definition of the \(V\)-filtration:

3.1 Definition. Denote by \(J_{Y}\) the defining Ideal of \(Y\) in \(O_{X}\) with a convention that \(J_{Y}^{j} = O_{X}\) for \(j \leq 0\). The \(V\)-filtration \(\{V_{Y}^{k}(\mathcal{D}_{X})\}_{k \in \mathbb{Z}}\) (along \(Y\)) is a filtration on \(\mathcal{D}_{X}|_{Y}\) defined by
\[
V_{Y}^{k}(\mathcal{D}_{X}) := \bigcap_{j \in \mathbb{Z}} \{ P \in \mathcal{D}_{X}|_{Y}; P^{j} \subset \mathcal{J}_{Y}^{j-k}\}.
\]

It is easy to see that by admissible coordinates, this filtration written as
\[
V_{Y}^{k}(\mathcal{D}_{X}) = \{ \sum_{j-i \leq k} P_{ij}(z; \partial) \tau^{i} \partial^{j} \in \mathcal{D}_{X}|_{Y}\}.
\]
For the fundamental properties of this filtration, we refer to Björk [Bj], Sabbah [Sab] and Schapira [Sc 2]).

Let us denote by \(\vartheta\) the Euler operator. Note that \(\vartheta \in V_{Y}^{0}(\mathcal{D}_{X}) \setminus V_{Y}^{-1}(\mathcal{D}_{X})\) and that \(\vartheta\) can be represented by \(\tau \partial_{\tau}\) by admissible coordinates.
3.2 Definition. A coherent $\mathcal{D}_X$-Module $\mathcal{M}$ defined on a neighborhood of $Y$ is said to be regular-specializable (along $Y$) if there exist locally a coherent $\mathcal{O}_X$-sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

1. $\mathcal{M}_0$ generates $\mathcal{M}$ over $\mathcal{D}_X$; that is, $\mathcal{M} = \mathcal{D}_X \mathcal{M}_0$;
2. $b(\theta) \mathcal{M}_0 \subset (\mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)) \mathcal{M}_0$, where $m$ is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase “along $Y$” since $Y$ is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module for which $Y$ is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K1] proved that every regular-holonomic $\mathcal{D}_X|_Y$-Module is regular-specializable.

3.4 Proposition. If $\mathcal{M}$ is a regular-specializable $\mathcal{D}_X$-Module, then each cohomology of $R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mu_Y(\mathcal{O}_X))$ and $R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \nu_Y(\mathcal{O}_X))$ is a locally $\mathbb{C}^\times$-conic sheaf.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module. Recall that a $V$-filtration $\{F^k \mathcal{M}\}_{k \in \mathbb{Z}}$ is said to be good if there exist locally a system of generators $\{u_j\}_{j=1}^m$ and $k_j \in \mathbb{Z}$ such that for any $k \in \mathbb{Z}$

$$F^k \mathcal{M} = \sum_{j=1}^m V_Y^{k-k_j}(\mathcal{D}_X) u_j$$

holds. The following theorem is proved by Kashiwara [Kas] (cf. also Björk [Bj]):

3.5 Theorem. Set $G := \{\alpha \in \mathbb{C}; 0 \leq \text{Re}\alpha < 1\}$. Then, for any regular-specializable $\mathcal{D}_X$-Module $\mathcal{M}$, there exist a unique good $V$-filtration $\{V_G^k(\mathcal{M})\}_{k \in \mathbb{Z}}$ on $\mathcal{M}$ and a non-zero polynomial $b_G(\alpha) \in \mathbb{C}[\alpha]$ such that $b_G^{-1}(0) \subset G$ and that for any $k \in \mathbb{Z}$ the following holds:

$$b_G(\theta + k) V_G^k(\mathcal{M}) \subset V_G^{k-1}(\mathcal{M}).$$

3.6 Definition. Under the notation of Theorem 3.5, we set:

$$\Psi_Y(\mathcal{M}) := V_G^0(\mathcal{M})/V_G^{-1}(\mathcal{M}),$$
$$\Phi_Y(\mathcal{M}) := V_G^1(\mathcal{M})/V_G^0(\mathcal{M}),$$

and call $\Psi_Y(\mathcal{M})$ the nearby-cycle of $\mathcal{M}$ and $\Phi_Y(\mathcal{M})$ the vanishing-cycle of $\mathcal{M}$ respectively.

3.7 Remark. Laurent [L] extended the definitions of nearby and vanishing cycles to the derived category of bounded complexes with (regular) specializable cohomology by using the theory of second microlocalization.
Let $\iota: Y \rightarrow X$ be the natural inclusion. Then the induced system, or the inverse image in the sense of $\mathcal{D}$-Modules is defined by

$$D\iota^* M := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \iota^{-1} M.$$ 

Then we have (cf. Laurent [L], Mebkhout [Me] or Sabbah [Sab]):

3.8 Proposition. If $\mathcal{D}_X$-Module $M$ is regular-specializable, then $\Psi_Y(M)$, $\Phi_Y(M)$ and each cohomology of $D\iota^* M$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle:

$$\Phi_Y(M) \rightarrow \Psi_Y(M) \rightarrow D\iota^* M \rightarrow +1.$$ 

As usual, we denote by $\mathcal{C}_{Y|X}^\mathbb{R} := \mu_Y(\mathcal{O}_X)[1]$ the sheaf of real holomorphic microfunctions on $T^*_Y X$. Set $\hat{T}_Y X := T_Y X \setminus T_Y Y$ as usual (the definition of $\hat{T}_Y X$ is similar). Using an admissible coordinate system we define a continuous section $\sigma: Y \rightarrow \hat{T}_Y X$ by $z \mapsto (z, 1)$. Similarly we define $\hat{\sigma}: Y \rightarrow \hat{T}_Y X$ by $z \mapsto (z, 1)$. Denote by $N_{X|Y}$ the sheaf of Nilsson class functions on $X$ along $Y$ and regard as a sheaf on $Y$. Then the following theorem is proved by Laurent [L] (cf. also Kashiwara-Kawai [K-K2]):

3.9 Theorem. Let $M$ be a regular-specializable $\mathcal{D}_X$-Module. Then, there exists the following isomorphism of distinguished triangles:

$$R \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)|_Y \rightarrow R \mathcal{H}om_{\mathcal{D}_X}(M, \nu_Y(\mathcal{O}_X)) \rightarrow R \mathcal{H}om_{\mathcal{D}_X}(M, \nu_Y(\mathcal{O}_X)) \rightarrow +1,$$

$$R \mathcal{H}om_{\mathcal{D}_Y}(D\iota^* M, \mathcal{O}_Y) \rightarrow R \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), \mathcal{O}_Y) \rightarrow R \mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(M), \mathcal{O}_Y) \rightarrow +1.$$

Moreover, a natural morphism $N_{X|Y} \rightarrow \nu_Y(\mathcal{O}_X)$ induces an isomorphism:

$$R \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), N_{X|Y}) \simeq R \mathcal{H}om_{\mathcal{D}_X}(M, \nu_Y(\mathcal{O}_X)).$$

3.10 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$R \mathcal{H}om_{\mathcal{D}_Y}(D\iota^* M, \mathcal{O}_Y) \simeq R \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.
4 Boundary Value Morphism

In this section, we shall define our injective boundary value morphism. Recall the mappings $f_{\pi}$ and $\tau_{Y_{\pi}}$ defined in Section 1.

4.1 Theorem. For any regular-specializable $D_{X}$-Module $M$, there exists the following isomorphism:

$$f_{\pi}^{-1}R\text{Hom}_{D_{X}}(M, C_{N|M}) \cong f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}R\text{Hom}_{D_{Y}}(\Psi_{Y}(M), C_{N}).$$

The proof is based on Proposition 3.4 and Theorem 3.9.

4.2 Definition. For any regular-specializable $D_{X}$-Module $M$, we define by virtue of Proposition 2.4 and Theorem 4.1:

$$\beta: f_{\pi}^{-1}R\text{Hom}_{D_{X}}(M, C_{N|M}) \rightarrow f_{\pi}^{-1}R\text{Hom}_{D_{X}}(M, \widetilde{C}_{N|M})$$

$$\rightarrow f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}R\text{Hom}_{D_{Y}}(\Psi_{Y}(M), C_{N}).$$

By the construction, we can obtain the following Holmgren type theorem:

4.3 Theorem. (1) The morphism $\beta$ gives a monomorphism

$$\beta^{0}: f_{\pi}^{-1}\text{Hom}_{D_{X}}(M, C_{N|M}) \rightarrow f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}\text{Hom}_{D_{Y}}(\Psi_{Y}(M), C_{N}).$$

(2) The restriction of $\beta^{0}$ to the zero-section $T_{N}M^{+}$ coincides with the topological boundary value morphism in the sense of Monteiro Fernandes [MF 1].

4.4 Remark. (1) For a general Fuchsian system in the sense of Tahara [T], Oaku [Oa 2] defined an injective boundary value morphism under additional conditions of characteristic exponents by using a detailed study due to Tahara [T].

(2) Let $C_{N|M}^{F} \subset C_{N|M}$ be the subsheaf consisting of $F$-mild microfunctions, and $\widetilde{C}_{N|M}^{A} := \mu_{N}(O_{X}|_{Y}) \otimes\mathcal{O}_{N/Y}[n]$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $M$ be a regular-specializable $D_{X}$-Module and set $\mathcal{M}_{Y} := \mathcal{H}^{0}(D_{*}M) = \mathcal{O}_{Y} \otimes_{\mathcal{O}_{X}} M$. Since $M$ is a Fuchsian system, by the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$f_{\pi}^{-1}\text{Hom}_{D_{X}}(M, \mathcal{C}_{N|M}^{F}) \rightarrow f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}\text{Hom}_{D_{X}}(M, \widetilde{C}_{N|M}^{A}) \rightarrow f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}\text{Hom}_{D_{Y}}(\mathcal{M}_{Y}, C_{N})$$

that is, the boundary value morphism

$$\gamma^{F}: f_{\pi}^{-1}\text{Hom}_{D_{X}}(M, \mathcal{C}_{N|M}^{F}) \rightarrow f_{\pi}^{-1}\tau_{Y_{\pi}}^{-1}\text{Hom}_{D_{Y}}(\mathcal{M}_{Y}, C_{N})$$

for $F$-mild microfunctions and $\beta^{0}$ are compatible.
5 Solvability

In this section, we shall state the solvability theorem under a kind of hyperbolicity condition. First, let us recall the following (Laurent-Monteiro Fernandes [L-MF 2]):

5.1 Definition. Let \( M \) be a coherent \( D_X \)-Module on a neighborhood of \( Y \). Then we say \( M \) is near-hyperbolic at \( x_0 \in N \) (in \( dt \)-codirection) if there exist positive constants \( C \) and \( \epsilon_1 \) such that

\[
\text{char}(M) \cap \{(z, \tau; z^* \tau)^*; |z - x_0| < \epsilon_1, \text{Re} \tau > 0\} \\
\subset \{(z, \tau; z^* \tau)^*; |\text{Re} \tau^*| < C(|\text{Im} z|(|\text{Im} z| + |\text{Im} \tau|) + |\text{Re} z^*|)\}
\]

holds by an admissible coordinate system.

5.2 Remark. As is shown by Laurent-Monteiro Fernandes [L-MF 2, Lemma 1.3.2], the near-hyperbolicity condition is weaker than the hyperbolicity condition (see also Bony-Schapira [B-S]).

5.3 Theorem. Let \( M \) be a regular-specializable \( D_X \)-Module. Assume that \( M \) is near-hyperbolic at \( x_0 \in N \). Then, for any \( p^* = (x_0, t_0; \sqrt{-1} \langle \xi_0, dx \rangle) \in T_{T_N^*M}^{*}L^+ \)

\[
\beta: R\mathcal{H}om_{D_X}(M, C_{N|M})_{p^*} \rightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(M), C_N)_{\tau(p^*)}
\]

is an isomorphism.

5.4 Remark. (1) Let \( M \) be a coherent \( D_X \)-Module for which \( Y \) is non-characteristic. Then, it is known that \( \Psi_Y(M) \Rightarrow D^\gamma M \simeq M_Y \). Moreover by virtue of the commutative diagram in Remark 4.4, we see that \( \beta^0 \) is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa2]). In particular, the restriction of \( \beta^0 \) to the zero-section \( T_N^*M^+ \) is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Moreover, if each \( \pm dt \in T_N^*M \) is hyperbolic for \( M \), then the nearly-hyperbolic condition is satisfied (cf. Kashiwara-Schapira [K-S 1]).

(2) Assume that \( X = \mathbb{C}^{n+1} \) and so on by taking an admissible coordinate system. Let \( b(\alpha) \) be a non-zero polynomial with degree \( m \), and \( Q \in D_X^{(m)} \cap V_Y^{-1}(D_X) \) and set \( M := D_X / D_X(b(\alpha) + Q) \). Then \( M \) is regular-specializable. For simplicity, assume that

\[
b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j} \quad (\alpha_i - \alpha_j \notin \mathbb{Z} \text{ for } 1 \leq i \neq j \leq \mu)
\]

(note that \( \sum_{j=1}^{\mu} \nu_j = m \)). Then a direct calculation shows that \( \Psi_Y(M) \simeq D_Y^{\oplus m} \), and \( \beta^0 \) is equivalent to \( \gamma \) in Oaku [Oa2]: Let \( p^* = (x_0, t_0; \sqrt{-1} \langle \xi_0, dx \rangle) \) be a point of \( T_{T_N^*M^+}^*L^+ \), and \( f(x, t) \) a germ of \( \mathcal{H}om_{D_X}(M, C_{N|M}) \) at \( p^* \). Then, since \( R\mathcal{H}om_{D_X}(M, N_{X|Y}) \simeq \)
by virtue of Theorem 3.9, we can see that $f(x, t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z, \tau) \tau^{\alpha_j}(\log \tau)^{k-1}$$

as a germ of $\text{Hom}_{D_X}(M, C_{N|M})$ at $p^*$. Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(z, 0) \in X; |x_0 - z| < \epsilon, \text{Im} \ z \in \Gamma\}$ with a positive constant $\epsilon$ and an open convex cone $\Gamma$ such that $\xi_0 \in \text{Int}(\Gamma^o)$ (the interior of the dual cone $\Gamma^o$ of $\Gamma$). Then, $\beta^0(f)$ is equivalent to $\{(\text{sp}_N(F_{jk}(x + \sqrt{-1} \Gamma 0, \theta)); 1 \leq k \leq \nu_j, 1 \leq j \leq \mu\}$.

5.5 Example. Assume that $X = \mathbb{C}^{n+1}$. Take an operator $A(z; \partial_z) \in D_Y^{(1)}$ at the origin and set $A^0 := \text{id}$ and $A^{(j)} := \frac{1}{j!} A \circ A^{(j-1)} \in D_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0, 1; \sqrt{-1} \langle \xi, dx \rangle)$ be a point of $T^{*}_{Y, \nu} M^{+} L^{+}$ and set $p_0 := (0; \sqrt{-1} \langle \xi, dx \rangle) \in T^{*}_{Y}$. Consider the following differential equations:

$$M_1 := D_X / D_X (\theta (\theta - 1) - \tau A(z; \partial_z) \theta),$$

$$M_2 := D_X / D_X ((\theta - 1)^2 - \tau A(z; \partial_z) \theta),$$

$$M_3 := D_X / D_X ((\theta - 1)(\theta - 2) - \tau A(z; \partial_z) \theta).$$

Let $f_i(x, t)$ be a germ of $\text{Hom}_{D_X}(M, C_{N|M})$ at $p^*$. Then:

1. $f_1(x, t)$ has the following defining function as a germ of $\text{Hom}_{D_X}(M, C_{N|M})$ at $p^*$:

$$F_1(z, \tau) = U_0(z) + \sum_{j=0}^{\infty} A^{(j)} U_1(z) \tau^{j+1}.$$ 

In this case, $f_1(x, t)$ is always $F$-mild. Hence $\beta^0(f_1(x, t))$ is given by $\gamma^F(f_1(x, t)) = \{(\partial^l f_1)(x, +0)\}_{l=0,1} = \{\text{sp}_N(U_l(x))\}_{l=0,1}$ at $p_0$. Indeed if $\tau \neq 0$, $M_1$ is isomorphic to $D_X / D_X (\partial^2_{\tau} - \partial_{\tau} A(z; \partial_z))$ for which $Y$ is non-characteristic.

2. $f_2(x, t)$ has the following defining function as a germ of $\text{Hom}_{D_X}(M, C_{N|M})$ at $p^*$:

$$F_2(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_1(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^{j} A^{(j)} U_0(z) \frac{1}{k} \tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)} U_0(z) \tau^{j+1} \log \tau,$$

and $\beta^0(f_1(x, t))$ is given by $\{\text{sp}_N(U_l(x))\}_{l=0,1}$ at $p_0$. Further if $f_1(x, t)$ is $F$-mild, then $U_0(z) = 0$ and $\gamma^F(f_2(x, t)) = \{(\partial^l f_2)(x, +0)\}_{l=0,1} = \{\text{sp}_N(U_l(x))\}$ at $p_0$. 

...
$f_3(x, t)$ has the following defining function as a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{G}_{N|M})$ at $p^*$:

$$F_3(z, \tau) = \sum_{j=0}^{\infty} A^{(j)}U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{jA^{(j)}U_1(z)}{k} \tau^{j+1}$$

$$+ \left( A\tau^2 + \sum_{j=2}^{\infty} jA^{(j)}U_1(z) \tau^{j+1} \right) \log \tau,$$

and $\beta^0(f_3(x, t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=1, 2}$ at $p_0$. In the case where $f_3(x, t)$ is $F$-mild, we must impose the condition $A\tau = 0$. Under this condition, $\gamma^F(f_3(x, t))$ is given by $\gamma^F(f_3(x, t)) = \{(\partial_t f_3)(x, +0)\}_{0 \leq t \leq 2} = \{0, \text{sp}_N(U_1)(x), 2\text{sp}_N(U_2)(x)\}$ at $p_0$ with $A(\partial_t f_3)(x, +0) = A\text{sp}_N(U_1)(x) = 0$.

### References


