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Computing point residues for a shape basis case via differential operators

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1 Introduction

In this paper, we study computational aspects of point residues. We concentrate on a shape basis case and we present algorithms which compute point residues for this generic case.

In 1987, Gianni and Mora ([2]) proved the following result:

(Shape lemma) Let \( I \) be a radical 0-dimensional ideal in \( \mathbb{Q}[z] \), regular in \( z_1 \). Then there are \( g_1(z_1), \ldots, g_n(z_1) \in \mathbb{Q}[z_1] \) such that \( g_1 \) is squarefree, \( \deg(g_i) < \deg(g_1) \) for \( i > 1 \) and the Gröbner basis of the ideal \( I \) w.r.t. the lexicographical order \( \succ \) with \( z_1 \succ \cdots \succ z_n \) is of the form

\[
\{g_1(z_1), z_2 - g_2(z_1), \ldots, z_n - g_n(z_1)\}.
\]

On the other hand, if the reduced Gröbner basis of \( I \) w.r.t. \( \succ \) is of this form, then \( I \) is a radical 0-dimensional ideal.

Furthermore, it is known that for "almost every" system of algebraic equations with finitely many solutions, after a suitable linear coordinate transformation, the reduced Gröbner basis of the transformed ideal will be in this simple form even though the system does not coincide with its radical ([5], [6], [7], [15]). The basis of the form (1.1) is called the shape basis of \( I \).

We study the algebraic local cohomology class associated with the shape basis of a given 0-dimensional ideal \( I \). We explicitly construct the holonomic system of linear partial differential equations for the algebraic local cohomology class. By making use of this holonomic system, we derive algorithms for computing point residues.

2 Notation and a former result

Let \( X = \mathbb{C}^n \) and fix a coordinate system \( z = (z_1, \ldots, z_n) \) of \( X \). We denote by \( \mathcal{O}_X \) the sheaf of holomorphic functions on \( X \). Denote by \( \mathcal{I} \) the zero dimensional ideal in \( \mathcal{O}_X \) generated by holomorphic functions \( f_1, \ldots, f_n \) of \( z \).

Put \( Y = \{ z \in X \mid f_1 = \cdots = f_n = 0 \} \). The algebraic local cohomology group \( H^n_{\{Y\}}(\mathcal{O}_X) \) which satisfies \( H^n_{\{Y\}}(\mathcal{O}_X) = \lim \text{ind}_k \text{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X) \), has a structure of a left \( \mathcal{D}_X \)-module, where \( \mathcal{D}_X \) is the sheaf of linear partial differential operators on

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X. Let $\begin{bmatrix} h & \cdot & \cdot \\ f_1 \cdot f_n \end{bmatrix}$ be a class in $\text{Ext}^{n}_{\mathcal{O}}(\mathcal{O}/\mathcal{O}, \mathcal{O})$ for $h \in \mathcal{O}$. Denote by $\eta$ the algebraic local cohomology class $\begin{bmatrix} h \\ f_1 \cdot f_n \end{bmatrix}$ defined by the image of $\begin{bmatrix} h \\ f_1 \cdot f_n \end{bmatrix}$ by the canonical mapping

$$\text{Ext}^{n}_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \mathcal{O}) \to \mathcal{H}_{\mathcal{I}}^{n}(\mathcal{O}).$$

Denote by $\text{Ann}$ the ideal in $\mathcal{D}$ consisting of annihilators of $\eta$. Then we have $\mathcal{H}_{\mathcal{I}}^{n}(\mathcal{O}) \cong \mathcal{D}/\text{Ann}$. For the Weyl algebra, it is possible to compute a Gröbner basis of $\text{Ann}$ by using the computer algebra system Kan ([8], [9], [14]).

We have the canonical pairing

$$\text{Res}_{\alpha}: \Omega_{X} \times \mathcal{H}_{\mathcal{I}}^{n}(\mathcal{O}) \to \mathbb{C}$$

$$\psi(dz, \eta) \mapsto \text{Res}_{\alpha}(\psi, \eta)$$

defined by the point residue $\text{Res}_{\alpha}((\psi dz/f_1 \cdot f_n)$ of a meromorphic differential form $(\psi dz/f_1 \cdot f_n)$ at $\alpha \in Y$.

The sheaf of holomorphic differential forms $\Omega_{X}$ is naturally endowed with a structure of a right $\mathcal{D}$-module by setting $(\phi(z) dz) R = ((R^*\phi(z)) dz)$ for a differential operator $R \in \mathcal{D}$, where $R^*$ stands for the formal adjoint operator of $R$. Then we have, for any $R \in \text{Ann}$,

$$\text{Res}_{\alpha}((R^*\phi(z)) dz, \eta) = \text{Res}_{\alpha}(\phi(z)dz, R\eta) = 0, \quad \alpha \in Y.$$

**Theorem 2.1** ([10], [11]) \textit{Put $\mathcal{K} = \left\{ \phi(z) dz \in \Omega_{X} \mid \text{Res}_{\alpha}(\phi(z) dz, \eta) = 0, \forall \alpha \in Y \right\}$. Then we have $\mathcal{K} = \left\{ (R^*\psi(z)) dz \mid R \in \text{Ann}, \psi(z) dz \in \Omega_{X} \right\}$.}

### 3 Construction of the holonomic system in the shape basis case

Let us consider the system

\begin{equation}
\begin{aligned}
f_1 &= g_1(z_1), \\
f_2 &= z_2 - g_2(z_1), \\
\vdots \\
f_n &= z_n - g_n(z_1),
\end{aligned}
\end{equation}

where $g_i(z_1) \in \mathbb{Q}[z_1]$. Denote by $Y$ the set of common zeros of the system (S), i.e., $Y = \{z = (z_1, \ldots, z_n) \in X \mid f_1 = \ldots = f_n = 0\}$. Put $\eta = [h/f_1 \cdot f_n] \in \mathcal{H}_{\mathcal{I}}^{n}(\mathcal{O})$ for $h \in \mathcal{O}$ with $h(\alpha) \neq 0, \alpha \in Y$. Since $\eta$ depends on the moduli class of $h$ in $\mathcal{O}/\mathcal{I}$, the numerator $h$ of the cohomology class $\eta$ can be expressed as an univariate function of the variable $z_1$.

Let $P, F_1, \ldots, F_n$ be differential operators defined by following forms:

\begin{equation}
\begin{aligned}
P &= \text{sf}(g_1) \partial_1 + \sum_{i=2}^{n} \text{sf}(g_1)g'_i(z_1) \partial_i + \frac{g'_i(z_1)}{\text{gcd}(g_1(z_1), g'_i(z_1))} - h'(z_1) \text{sf}(g_1), \\
F_1 &= g_1(z_1), \\
F_2 &= z_2 - g_2(z_1), \\
\vdots \\
F_n &= z_n - g_n(z_1),
\end{aligned}
\end{equation}

where $\text{sf}(g_1)$ is the square free part $g_1(z_1)/\text{gcd}(g_1(z_1), g'_i(z_1))$ of $g_1(z_1)$, $g'_i(z_1) := \partial g_1/\partial z_1$, and $\partial_i := \partial/\partial z_i, i = 1, \ldots, n$. Then we have the next theorem.

**Theorem 3.1** Let $\text{Ann}$ be the left ideal in $\mathcal{D}$ consisting of annihilators of $\eta$. Then $\text{Ann}$ is generated by $P$ and $F_i, i = 1, \ldots, n$ in (A).
Proof. Recall the isomorphism
\[ \mathcal{H}_{[Y]}^{n}(\mathcal{O}_X) \cong \frac{\mathcal{O}_X[*Z_1 \cup \cdots \cup Z_n]}{\sum_{i=1}^{n} \mathcal{O}_X[*Z_i \cup \cdots \cup Z_n]}, \]  
where \( Z_i = \{ z \in X \mid f_i(z) = 0 \} \) and \( \mathcal{O}_X[*Z] \) stands for a sheaf of meromorphic functions with poles at \( Z \). By this isomorphism, we can readily see that operators in (A) annihilate \( \eta \). Let \( g_1 = \prod_{i=1}^{\nu} (z_1 - \alpha_{1,i})^{m_i} \) be the factorization of \( g_1 \) over \( \mathbb{C} \). Then we have \( \eta_1 \in \mathcal{H}_{[\alpha_1]}^{n}(\mathcal{O}_X) \) such that \( \eta = \eta_1 + \cdots + \eta_\nu \), where \( \alpha_i = (\alpha_{1,i}, g_2(\alpha_{1,i}), \ldots, g_n(\alpha_{1,i})) \in Y, i = 1, \ldots, \nu \). Let \( U_k \) be a sufficiently small neighborhood of a point \( \alpha_k \in Y \) and assume that \( U_k \cap Y = \{ \alpha_k \} \). Let us find the annihilators of \( \eta \) on \( U_k \). Denote by \( g_{i,k} \) the modulo class of \( g_i \) in \( \mathcal{O}_X/\langle (z_1 - \alpha_{1,k})^{m_i} \rangle \).

Put \( f_i,k(z_1) = z_1 - g_{i,k}(z_1) \). If we set \( h_k = h/\prod_{i \neq k}(z_1 - \alpha_{1,i})^m \), we have
\[ \eta_k = \left[ \frac{h_k}{(z_1 - \alpha_{1,k})^m f_{2,k} \cdots f_{n,k}} \right]. \]

Then we have
\[ P_k = (z_1 - \alpha_{1,k}) \partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g_{i,k} \partial_i + m_k - \frac{h_k}{h}(z_1 - \alpha_{1,k}), \]  
(3.2)
and
\[ F_{1,k} = (z_1 - \alpha_{1,k})^m, \]  
(3.3)
and
\[ F_{i,k} = z_i - g_{i,k}(z_1), \quad i = 2, \ldots, n \]  
(3.4)
as annihilators of \( \eta \) on \( U_k \). Note that the annihilator \( P_k \) can be rewritten as
\[ P_k = (z_1 - \alpha_{1,k}) \partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g_{i,k} \partial_i + \sum_{i=1}^{\nu} m_i \frac{z_1 - \alpha_{1,i}}{z_1 - \alpha_{1,t}} - \frac{h'}{h}(z_1 - \alpha_{1,k}), \]  
(3.5)
We set \( \text{Ann}_k = \{ R \in \mathcal{D}_X \mid R\eta_k = 0 \} \). Since \( \langle P_k, F_{1,k}, \ldots, F_{n,k} \rangle \subset \text{Ann}_k \), we have a surjective morphism \( \mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k} \rangle \to \mathcal{D}_X/\text{Ann}_k \to 0 \). Recall that \( \mathcal{D}_X/\text{Ann}_k \) is a simple holonomic system, the multiplicity of \( \mathcal{D}_X/\text{Ann}_k \) is equal to 1. We can see that the multiplicity of \( \mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k} \rangle \) is also equal to 1. Thus \( \mathcal{D}_X/\text{Ann}_k = \mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k} \rangle \) and finally we have \( \langle P_k, F_{1,k}, \ldots, F_{n,k} \rangle = \text{Ann}_k \). On the other hand, the localization of \( P \) and \( F_i, i = 1, \ldots, n \) to \( U_k \) have the following forms:
\[ P|_{\alpha_k} = \frac{1}{(z_1 - \alpha_{1,1}) \cdots (z_1 - \alpha_{1,k-1})(z_1 - \alpha_{1,k+1}) \cdots (z_1 - \alpha_{1,n})} \]  
(3.6)
\[ F_{1}|_{\alpha_k} = (z_1 - \alpha_{1,k})^m, \]  
(3.7)
\[ F_{i}|_{\alpha_k} = z_i - g_{i,k}(z_1), \quad i = 2, \ldots, n. \]  
(3.8)
According to the formulas from (3.3) to (3.8), we have $P|_{U_k} = P_k$, $F_i|_{U_k} = F_{i,k}$. Then we have $\text{Ann}|_{U_k} = \langle P|_{U_k}, F_1|_{U_k}, \ldots, F_n|_{U_k} \rangle$. If we denote by $\text{Ann}|_{U_k}$ the restriction of the ideal $\text{Ann}$ to $U_k$, we have $\text{Ann}|_{U_k} = \text{Ann}|_{U_k}$. Thus, we obtain that $\langle P|_{U_k}, F_1|_{U_k}, \ldots, F_n|_{U_k} \rangle$. Consequently, $\text{Ann} = \langle P, F_1, \ldots, F_n \rangle$. □

3.1 Properties of $P^*$

The following relations between operators $P$ and $F_i$, $i=1, \ldots, n$ hold:

**Corollary 3.1**

$$[P^*, F_i^*] = \begin{cases} -\frac{g_1'(z_1)}{\gcd(g_1(z_1), g_1'(z_1))}F_1, & i = 1, \\ 0, & i = 2, 3, \ldots, n. \end{cases}$$

**Proof.** Since $g_1$ is a univariate polynomial of $z_1$, we have

$$[P^*, F_1^*] = -s f(g_1) \cdot g_1' = -\frac{g_1'}{\gcd(g_1, g_1')}F_1.$$

For $i = 2, 3, \ldots, n$, we have

$$[P^*, F_i^*] = -s f(g_1)g_i' + s f(g_1)g_i' = 0.

□

This corollary implies that, if $\varphi \in \mathcal{I}$, then $P^*\varphi \in \mathcal{I}$ holds. Thus, we have the next proposition.

**Proposition 3.1** $P^*$ acts on the sheaf $\mathcal{O}_X/\mathcal{I}$, i.e.,

$$P^*: \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I}.$$ 

Let $\tilde{\mathcal{I}}$ be the ideal generated by $\gcd(g_1(z_1), g_1'(z_1))$, $z_2 - g_2(z_1)$, $\ldots$, $z_n - g_n(z_1)$ in $\mathcal{O}_X$. Then $P^*$ has the following property:

**Theorem 3.2** A necessary and sufficient condition for $P^*\varphi(z) \in \mathcal{I}$ is $\varphi(z) \in \tilde{\mathcal{I}}$.

**Proof.** We prove first that the condition is sufficient. Since $F_j^* = F_j = f_j$, we have $P^*(\chi f_i) = (P^*\chi)f_i$ for any $\chi \in \mathcal{O}_X$ by Corollary 3.1. Since the operator $P^*$ can be written in the form

$$P^* = -\frac{g_1(z_1)}{h(z_1)}\partial_1 - \frac{h(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} - \sum_{i=2}^{n} \frac{g_i(z_1)}{\gcd(g_1(z_1), g_i'(z_1))}g_i'(z_1)\partial_i,$$  

we have

$$P^*(\gcd(g_1, g_1')\varphi) = -\frac{g_1}{h}\partial_1(\frac{h}{\gcd(g_1, g_1')}\gcd(g_1, g_1')\varphi) - \sum_{i=2}^{n} \frac{g_i}{\gcd(g_1, g_i')}g_i'(\gcd(g_1, g_i')\varphi)$$

$$= -\left(\frac{1}{h}\partial_1 h\varphi + \sum_{i=2}^{n} g_i g_i'(\varphi)f_i \right).$$
These formulas imply the sufficiency. In order to prove the necessity, we set

$$\varphi(z) = \varphi_1(z) \gcd(g_1(z), g'_1(z)) + \varphi_2(z) f_2(z) + \cdots + \varphi_n(z) f_n(z) + \varphi_0(z),$$

where $\varphi_0, \varphi_1, \ldots, \varphi_n \in \mathcal{O}_X$ and $\varphi_0$ is a univariate polynomial of $z_1$ with $\deg \varphi_0(z_1) < \deg \gcd(g_1(z_1), g'_1(z_1))$. Since $P^* \varphi \in \mathcal{I}$ by Corollary 3.1, there is an univariate polynomial $\psi(z_1)$ of $z_1$ such that $P^* \varphi_0(z_1) = \psi(z_1) f_1$. On the other hand, we have

$$P^* \varphi_0 = -\frac{g_1}{h} \frac{\partial_1}{\gcd(g_1, g'_1)} \varphi_0.$$

Thus we have

$$-\frac{g_1}{h} \frac{\partial_1}{\gcd(g_1, g'_1)} \varphi_0 = \psi f_1,$$

$$\frac{h}{\gcd(g_1, g'_1)} \varphi_0 = -\int z_1 \frac{h(t)}{g_1(t)} \psi(t) f_1(t) dt,$$

$$\varphi_0 = \left( -\frac{1}{h} \int z_1 \frac{h(t)}{g_1(t)} \psi(t) f_1(t) dt \right) \gcd(g_1, g'_1).$$

Since $\varphi_0 \not\in \tilde{\mathcal{I}}$, we have $\varphi_0 = 0$. This completes the proof. $\square$

From the exact sequence $0 \to \tilde{\mathcal{I}} \to \mathcal{O}_X \to \mathcal{O}_X \to 0$, we have that $\dim \Gamma(X, \tilde{\mathcal{I}}/\mathcal{I}) = \dim \Gamma(X, \mathcal{O}_X/\mathcal{I}) - \dim \Gamma(X, \tilde{\mathcal{I}}/\mathcal{I}) = \nu$. Put $d = \deg g_1(z_1)$. Then, we have the following corollary:

**Corollary 3.2**

(i) $\dim \Gamma(X, \text{Im}(P^* : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I})) = d - \nu$.

(ii) $\dim \Gamma(X, \text{Ker}(P^* : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I})) = \nu$.

Let $v_j(z_1)$ be the image of $z_1^j$ by $P^*$ in $\Gamma(X, \mathcal{O}_X/\mathcal{I})$ for $j = 0, \ldots, d - \nu - 1$. Put $\mathcal{K} = \{v(z) \in \mathcal{O}_X \mid \text{Res}_\alpha(v(z)dz, \eta) = 0, \alpha \in Y\}$.

**Corollary 3.3**

$$\Gamma(X, \mathcal{K}/\mathcal{I}) \cong \text{Span}\{v_0(z_1), \ldots, v_{d-\nu-1}(z_1)\}.$$

That is, any $v(z_1)$ which satisfies $\text{Res}_\alpha(v(z_1)dz, \eta) = 0$ for $\alpha \in Y$ and $\deg v(z_1) \leq d - 1$ can be expressed as a linear combination of $v_0(z_1), \ldots, v_{d-\nu-1}(z_1)$.

### 3.2 Localization

Let $g_1(z_1) = g_{1,1}^{i_1}(z_1) \cdots g_{1,N}^{i_N}(z_1)$ be the factorization of $g_1(z_1)$ over $\mathbb{Q}$. Let $g_{i,k}(z_1)$ be the remainder of division of $g_i(z_1)$ by $g_{i,k}^{i_N}(z_1)$. Put $f_{1,k}(z) = g_{1,k}(z_1)$ and $f_{i,k}(z) = z_i - g_{i,k}(z_1)$ for $k = 1, \ldots, N$ and $i = 2, \ldots, n$. Denote by $I_k$ the ideal in $\mathbb{Q}[z]$ generated by $f_{1,k}(z), \ldots, f_{n,k}(z)$. Let $F_{i,k}$ be the differential operator of order zero defined by $F_{i,k} = f_{i,k}$. From Corollary 3.1, we have the following formulas:

**Corollary 3.4**

$$[P^*, F_{i,k}^*] = \begin{cases} -((\prod_{j \neq i} g_{1,j})g_{1,k}^{i_N})g_{i,k}^{i_N}, & i = 1, \\ 0, & i = 2, 3, \ldots, n. \end{cases}$$

These formulas imply the next result.

**Lemma 3.1** $P^*$ acts on the vector space $\mathcal{O}_X/I_k$, i.e.,

$$P^* : \mathcal{O}_X/I_k \to \mathcal{O}_X/I_k.$$
Thus we can localize results in Section 3.1 to $\mathcal{I}_k$. Put $\nu_k = \deg g_{1,k}(z_1)$ and $d_k = \nu_k \mu_k$. Then we have the following:

**Corollary 3.5**

(i) $\dim \Gamma(X, \text{Im}(P^*: \mathcal{O}_X/\mathcal{I}_k \to \mathcal{O}_X/\mathcal{I}_k)) = d_k - \nu_k$.

(ii) $\dim \Gamma(X, \text{Ker}(P^*: \mathcal{O}_X/\mathcal{I}_k \to \mathcal{O}_X/\mathcal{I}_k)) = \nu_k$.

Let $v_{k,j}(z_1)$ be the image of $z_1^j$ by $P^*$ in $\Gamma(X, \mathcal{O}_X/\mathcal{I}_k)$ for $j = 0, \ldots, d_k - \nu_k - 1$. Denote by $Y_k$ the set of common zeros of $f_{1,k}, \ldots, f_{n,k}$. Put $\mathcal{K}_k = \{v(z) \in \mathcal{O}_X \mid \text{Res}_\alpha \langle v(Z) dZ, \eta_k \rangle = 0, \alpha \in Y_k\}$.

**Corollary 3.6**

$$\Gamma(X, \mathcal{K}_k/\mathcal{I}_k) \cong \text{Span}\{v_{k,0}(z_1), \ldots, v_{k,d_k-\nu_k-1}(z_1)\}.$$ That is, any $v(z_1)$ which satisfies $\text{Res}_\alpha \langle v(z_1) dz, \eta_k \rangle = 0$ and $\deg v(z_1) \leq d_k - 1$ can be expressed as a linear combination of $v_{k,0}(z_1), \ldots, v_{k,d_k-\nu_k-1}(z_1)$.

### 4 Algorithm

We describe algorithms for computing point residues. Let $f_1(z), \ldots, f_n(z)$ be polynomials in $\mathbb{Q}[z_1, \ldots, z_n]$ of the form (S) and $dz = dz_1 \wedge \cdots \wedge dz_n$. Let us consider a meromorphic differential form $\theta(z)dz/f_1(z) \cdots f_n(z)$ with a polynomial $\theta(z) \in \mathbb{Q}[z]$. Denote by $\theta$ the remainder of $\theta$ by $I$. Now we introduce three vector spaces

$$U = \{u(z_1) \in \mathbb{Q}[z_1] \mid \deg u(z_1) \leq d-1\},$$

$$V = \{v(z_1) \in \mathbb{Q}[z_1] \mid \deg v(z_1) \leq d-1, \text{Res}_\alpha \langle v(z_1) dz, \frac{1}{f_1 \cdots f_n} \rangle = 0, \alpha \in Y\},$$

and

$$W = \{w(z_1) \in \mathbb{Q}[z_1] \mid \deg w(z_1) \leq d-1, \frac{w(z_1)}{f_1 \cdots f_n} \text{ has at most simple poles}\}.$$  

The dimensions of these vector spaces are $\dim U = d$, $\dim V = d-\nu$ and $\dim W = \nu$, respectively. Let $P$ be the annihilator of the cohomology class $[1/f_1 \cdots f_n]$ defined in (A), i.e.,

$$P = \text{sf}(g_1) \partial_1 + \sum_{i=2}^{n} \text{sf}(g_i) g'_i(z_1) \partial_i + \frac{g'_1(z_1)}{\gcd(g_1(z_1), g'_1(z_1))}.$$  

Denote by $v_j(z_1)$ the remainder of $P^* z_1^j$ by $g_i(z_1)$, $j = 1, \ldots, d-\nu-1$. Let Jac be Jacobian of $f_1, \ldots, f_n$. In this case, Jac = $g'_1(z_1)$. Let $w_j(z_1)$ be the remainder of Jac $\cdot z_1^j$ by $g_1(z_1)$ for $j = 0, \ldots, \nu-1$.

**Proposition 4.1**

(i) $U = V \oplus W$

(ii) $V = \text{Span}\{v_0(z_1), \ldots, v_{d-\nu-1}(z_1)\}$

(iii) $W = \text{Span}\{w_0(z_1), \ldots, w_{\nu-1}(z_1)\}$
For computing the residues, we write 
\[ \theta(z_1) = \sum_{j=0}^{d-\nu-1} a_j v_j(z_1) + \sum_{\ell=0}^{\nu-1} b_{\ell} w_\ell(z_1). \]

Then we have 
\[ \text{Res}_{\alpha \in Y} \left( \frac{\theta(z_1)}{f_1 \ldots f_n} dz \right) = \text{Res}_{\alpha \in Y} \left( \frac{\sum_{\ell=0}^{\nu-1} b_{\ell} w_\ell \sum_{\ell=0}^{\nu-1} b_{\ell} z_1^\ell}{f_1 \ldots f_n} dz \right). \]

Since Jac \( \sum_{\ell=0}^{\nu-1} b_{\ell} z_1^\ell dz/f_1 \ldots f_n \) is a meromorphic \( n \)-form with only simple poles, we can proceed as follows:

Let \( g_1(z_1) = g_1^{(1)}(z_1) \ldots g_1^{(N)}(z_1) \) be the factorization of \( g_1(z_1) \) over \( \mathbb{Q} \). Denote by \( g_{j,k} \) the remainder of \( g_j \) by \( g_{1,k}^{\mu} \) and \( \sigma_k \) the remainder of \( \sum_{\ell=0}^{\nu-1} b_{\ell} z_1^\ell \) by \( g_{1,k} \). Let \( J_k \) be the ideal of \( \mathbb{Q}[z,t] \) generated by \( g_{1,k}, z_2-g_2(z_1), \ldots, z_n-g_n(z_1) \).

We obtain a univariate polynomial \( g_k(t) \) of \( t \) as the generator of \( J_k \cap \mathbb{Q}[t] \). Then \( g_k(t) = 0 \) is the equation for residues of \( \theta dz/f_1 \ldots f_n \) at \( Y_k \).

Algorithm 1 (point residues for shape basis case)

Input \( g_1(z_1), z_2-g_2(z_1), \ldots, z_n-g_n(z_1) \) : the shape basis, \( \theta(z) \in \mathbb{Q}[z] \)
\[ \theta(z_1) \leftarrow \text{the remainder of } \theta(z) \text{ by } (g_1(z_1), z_2-g_2(z_1), \ldots, z_n-g_n(z_1)) \]
\[ \text{sf} \leftarrow \frac{g_1}{\gcd(g_1, g_1')} \]
\[ \nu \leftarrow \deg \text{sf} \]
\[ d \leftarrow \deg g_1 \]
for \( j \) from 0 to \( d-\nu-1 \)
\[ v_j \leftarrow \text{the remainder of } -\frac{g_1}{\gcd(g_1, g_1')} \cdot \frac{1}{z_1^j} \text{ by } f_1 \]
for \( \ell \) from 0 to \( \nu-1 \)
\[ w_{\ell} \leftarrow \text{the remainder of } g_1^{(\ell)} \cdot \frac{z_1^\ell}{\gcd(g_1, g_1')} \text{ by } f_1 \]
\[ \vartheta \leftarrow -\sum_{j=0}^{d-\nu-1} a_j v_j - \sum_{\ell=0}^{\nu-1} b_{\ell} w_{\ell} \]
\[ (a_0, \ldots, a_{d-\nu-1}, b_0, \ldots, b_{\nu-1}) \leftarrow \text{the coefficients s.t. } \vartheta = 0 \]
\[ g_1^{(1)} \ldots g_1^{(N)} \leftarrow \text{the squarefree factorization of } g_1 \]
for \( k \) from 1 to \( N \)
for \( i \) from 2 to \( n \)
\[ g_{i,k} \leftarrow \text{the remainder of } g_i \text{ by } g_{1,k}^{\mu} \]
\[ \sigma_k \leftarrow \text{the remainder of } \sum_{\ell=0}^{\nu-1} b_{\ell} z_1^\ell \text{ by } g_{1,k} \]
\[ J_k \leftarrow \langle g_{1,k}, z_2-g_2(z_1), \ldots, z_n-g_n(z_1), \mu_k \sigma_k - t \rangle \]
\[ G_k \leftarrow \text{Gröbner basis of } J_k \text{ w.r.t. the lexicographical order } z \succ t \]

Output \( \{ G_1, \ldots, G_N \} \)

Example 1 Put \( z = (x, y) \). Let us consider \( f_1 = x^4(2x^2-1)^3 \), \( f_2 = y - (x^3 + 1) \) and \( \theta = 35xy^3-x^2y+y-1 \). The annihilator \( P \) of the cohomology class \([1/f_1 f_2]\) is \( P = (2x^3-x)\partial_x + (6x^5-3x^3)\partial_y + 20x^2 - 4 \). Then we have \( V = \text{Span}\{v_0, \ldots, v_8\} \), where \( v_0 = 14x^2 - 3 \), \( v_1 = 12x^3 - 2x \), \( v_2 = 10x^4 - x^2 \), \( v_3 = 8x^5 \), \( v_4 = 6x^6 + x^4 \), \( v_5 = 4x^7 + 2x^5 \), \( v_6 = 2x^8 + 3x^6 \) and \( W = \text{Span}\{w_0, w_1, w_2\} \), where \( w_0 = 80x^9 - 96x^7 + 36x^5 - 4x^3 \), \( w_1 = 24x^8 - 24x^6 + 6x^4 \), \( w_2 = 24x^9 - 24x^7 + 6x^5 \).

The remainder \( \theta \) of \( \theta \) by \( \langle f_1, f_2 \rangle \) is \( \theta = (105/2)x^8 + 105x^7 - (105/4)x^6 - x^5 + (875/8)x^4 + x^3 - x^2 + 35x \). It can be written as \( \theta = -(35/2)v_1 + v_2 + (2425/16)v_3 + \).
\[(355/64)v_{4} - (739/4)v_{5} - (1965/32)v_{6} - (211/4)w_{0} + (935/128)w_{1} + (1055/6)w_{2}\]

Put \(\sigma = -(211/4) + (935/128)x + (1055/6)x^{2}\). For \(I_{1} = \langle x^{4}, y - (x^{3} + 1) \rangle\), we have \(\sigma_{1} = -211/4\). Thus, \(\text{Res}_{(0,1)}(\theta dz/f_{1}f_{2}) = 4(-211/4) = -211\). For \(I_{2} = \langle (2x^{2} - 1)^{3}, y - (x^{3} + 1) \rangle\), we have \(\sigma_{2} = (935/128)x + (211/6)x\).

\[\text{Thu8}, \quad \text{Re} \left[ t^{0,1} \right](\theta dZ/f_{1}f_{2}) = 4(-211/4) = -211\]

For \(I_{2} = \langle (2x^{2} - 1)^{3}, y - (x^{3} + 1) \rangle\), we have \(\sigma_{2} = (935/128)x + (211/6)x\).

\[G_{2} = \langle -32768t^{2} + 6914048t - 35684807, -2805x + 128t - 13504, -2805y + 64t - 3947 \rangle\]

We have \(\rho_{2}(t) = -32768t^{2} + 6914048t - 35684807 = 0\) which \(t = \text{Res}_{(V(J_{2}))}(\theta dZ/f_{1}f_{2})\).

4.1 Localization

By using Corollary 3.6, we get an algorithm for computing the point residues of \(\theta dz/f_{1} \ldots f_{n}\). Let \(U_{k}, V_{k}\) and \(W_{k}\) be vector spaces given by

\[U_{k} := \{ u(z_{1}) \in \mathbb{Q}[z_{1}] \mid \deg u(z_{1}) \leq d_{k} - 1 \}, \]
\[V_{k} := \{ v(z_{1}) \in \mathbb{Q}[z_{1}] \mid \deg v(z_{1}) \leq d_{k} - 1, \text{Res}_{\alpha}(v(z_{1})/f_{1,k} \cdots f_{n,k} dz) = 0, \alpha \in Y_{k} \}, \]

and

\[W_{k} = \{ w(z_{1}) \in \mathbb{Q}[z_{1}] \mid \deg w(z_{1}) \leq d_{k} - 1, \frac{w(z_{1})}{f_{1,k} \cdots f_{n,k}} \text{ has at most simple poles} \}. \]

The dimensions of these spaces are \(\dim U_{k} = d_{k}\), \(\dim V_{k} = d_{k} - \nu_{k}\) and \(\dim W_{k} = \nu_{k}\), respectively.

Denote by \(v_{k,j}(z_{1})\) the remainder of \(P^{*}z_{1}^{j}\) by \(f_{1,k}\), \(j = 0, \ldots, d_{k} - \nu_{k} - 1\). Let \(w_{k,j}(z_{1})\) the remainder of \(\text{Jac} \cdot z_{1}^{\ell}\) by \(f_{1,k}(z_{1})\) for \(\ell = 0, \ldots, \nu_{k} - 1\). Then we have the next proposition.

Proposition 4.2

(i) \(U_{k} = V_{k} \oplus W_{k}\)

(ii) \(V_{k} = \text{Span}\{v_{k,0}(z_{1}), \ldots, v_{k,d_{k}-\nu_{k}-1}(z_{1})\}\)

(iii) \(W_{k} = \text{Span}\{w_{k,0}(z_{1}), \ldots, w_{k,\nu_{k}-1}(z_{1})\}\)

Let \(\underline{\theta}_{k}(z_{1})\) be the remainder of \(\theta(z_{1})\) by \(f_{1,k}(z_{1})\), where \(\theta(z_{1})\) is the remainder of \(\theta(z)\) by \(I\). we can write \(\underline{\theta}_{k}(z_{1})\) into

\[\underline{\theta}_{k}(z_{1}) = \sum_{j=0}^{d_{k}-\nu_{k}-1} a_{k,j}v_{k,j}(z_{1}) + \sum_{\ell=0}^{\nu_{k}-1} b_{k,\ell}w_{k,\ell}(z_{1}) \]

and we have

\[\text{Res}_{\alpha \in Y_{k}} \left( \frac{\theta}{f_{1} \cdots f_{n}} dz \right) = \text{Res}_{\alpha \in Y_{k}} \left( \frac{\text{Jac}_{k} \sum_{\ell=0}^{\nu_{k}-1} b_{k,\ell}z_{1}^{\ell}}{f_{1,k} \cdots f_{n,k}} dz \right) . \]

Thus we have that the residue of \(\theta dz/f_{1} \ldots f_{n}\) at \(\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in Y_{k}\) is equal to \(\mu_{k}(\sum_{\ell=0}^{\nu_{k}-1} b_{k,\ell}z_{1}^{\ell}/\prod_{j \neq k} f_{j,k}(\alpha_{1}))\). In other words, for computing residues, we can proceed as follows:

Let \(J_{k}\) be the ideal of \(\mathbb{Q}[z, t]\) generated by \(f_{1,k}, f_{2,k}, \ldots, f_{n,k}\) and \(\mu_{k}(\sum_{\ell=0}^{\nu_{k}-1} b_{k,\ell}z_{1}^{\ell} - t\prod_{j \neq k} f_{j,k})\). We obtain an univariate polynomial \(\varrho_{k}(t)\) of \(t\) as the generator of \(J_{k} \cap \mathbb{Q}[t]\). Then \(\varrho_{k}(t) = 0\) is the equation for residues of \(\theta dz/f_{1} \ldots f_{n}\) at \(Y_{k}\).

Algorithm 2 (localized version)
Input $g_1(z_1), z_2-g_2(z_1), \ldots, z_n-g_n(z_1):$ the shape basis, $\theta(z) \in \mathbb{Q}[z]$
$\theta(z_1) \leftarrow$ the remainder of $\theta(z)$ by $(g_1(z_1), z_2-g_2(z_1), \ldots, z_n-g_n(z_1))$
$g_{11}^1(z_1) \cdots g_{1N}^1(z_1) \leftarrow$ the squarefree factorization of $g_1(z_1)$

for $k$ from 1 to $N$

$f_{1,k} \leftarrow g_{1k}^{\mu_k}$
$v_k \leftarrow \deg g_{1,k}$
$d_k \leftarrow \mu_k - v_k$
$\vartheta_k \leftarrow$ the remainder of $\theta$ by $f_{1,k}$

for $i$ from 2 to $n$

$g_{i,k} \leftarrow$ the remainder of $g_i$ by $f_{1,k}$
$f_{i,k} \leftarrow z_i - g_{i,k}$

for $j$ from 0 to $d_k - v_k - 1$

$v_{k,j} \leftarrow$ the remainder of $-\frac{g_1}{\gcd(g_1,g_1')} j z_1^{j-1} + \frac{g_1}{\gcd(g_1,g_1')} \frac{\gcd(g_1,g_1')}{\gcd(g_1,g_1')} z_1$ by $f_{1,k}$

for $\ell$ from 0 to $v_k - 1$

$w_{k,\ell} \leftarrow$ the remainder of $f_{1,k}' z^{\ell}$ by $f_{1,k}$
$\vartheta_k \leftarrow \vartheta_k - \sum_{j=0}^{v_k-1} a_{k,j} v_k^{j} - \sum_{\ell=0}^{v_k-1} b_{k,\ell} w_k^{\ell}$
$(a_{k,0}, \ldots, a_{k,d_k-v_k-1}, b_{k,0}, \ldots, b_{k,v_k-1}) \leftarrow$ the coefficients s.t. $\vartheta_k = 0$

$J_k \leftarrow (g_1, f_1, f_2, \cdots, f_{n,k}, \gamma_k \sum_{i=0}^{v_k-1} b_{i} z_1, -t \prod_{j \neq k} f_{1,k})$

$G_k \leftarrow$ Gröbner basis of $J_k$ w.r.t. the lexicographic order $z \succ t$

Output $\{G_1, \ldots, G_N\}$

Example 2 Let us consider the same $f_1$ and $f_2$ with Example 1. For $I_1 = \langle (x^4, y - (x^3+1)) \rangle$, $V_1 = \text{Span}\{4x^2-3, 12x^3-2x, -x^2\}$ and $W_1 = \text{Span}\{4x\}$. The remainder $\theta_1$ of $\theta$ by $x^4$ is $x^3 - x^2 + 35x$. It can be written as $\theta_1 = (114/4) v_{1,0} - (35/2) v_{1,1} + v_{1,2}$. Thus we have $G_1 = \langle t + 111, z, y - 1 \rangle$. In the same way, we have $\theta_2 = (211/24) v_{2,0} + (935/12) v_{2,1} + (375/26) v_{2,2} + (459/16) v_{2,3} + (1343/128) v_{2,4} - (513/16) v_{2,5}$. Thus we have $G_2 = \langle -32768 t^2 + 6914048 t - 356848007, -2805 x + 128 t - 13504, -2805 y + 64 t - 3947 \rangle$ for $I_2 = \langle (2x^2 - 1)^2, y - (x^3 + 1) \rangle$.

Example 3 Put $z = (x, y)$. Let us consider $f_1 = (x^2 + 1)^3 (2x^2 - 1)^9$ and $f_2 = y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2)$. The annihilator $P$ of the cohomology class $[1/f_1 f_2]$ given in $A$ is $P = \langle 2x^2 + 1 \rangle \partial_x + (36x^9 + 42x^7 - 12x^6 + 2x^5 - 10x^4 - 8x^3 + 4x^2 - 4x - 2) \partial_y + 8x^3 - 10x$. Let us compute the residue of $\theta dz/f_1 f_2$, where $\theta = 35x^3 y^5 - 2y^4 + 2xy - 1$. Along the algorithm 2, for $I_1 = \langle (2x^2 - 1)^9, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2) \rangle$, we have that

$$\text{Res}_{[V(I_1)]} \left( \frac{\theta}{f_1 f_2} dz \right) = \text{Res}_{[V(I_1)]} \left( \frac{\sigma_1}{f_{1,1} f_{1,2}} dz \right),$$

where

$$\sigma_1 = \left( -747718501 \quad 126787493190876461 \right) \begin{pmatrix} 5036466357 & 38024047776549504 \end{pmatrix} \begin{pmatrix} 2x^2 + 1 \end{pmatrix} \text{Jac}_1,$$

$\text{Jac}_1 = 9216 x^{17} - 36864 x^{15} + 64512 x^{13} - 64512 x^{11} + 40320 x^9 - 16128 x^7 + 4032 x^5 - 576 x^3 + 36 x$. Thus we have

$$G_1 = \langle (41773420433866686919639367682 - 16124476470496104851379700224 - 1778835134830001896609499526073, 1826154596005141 x + 457019805007872 t - 88204634267648, 14609236768041128 y - 10968475320188928 t - 39094030445746101) \rangle.$$

On the other hand, for $I_2 = \langle (x^2 + 1)^13, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2) \rangle$, we have that $\text{Res}_{[V(I_2)]}(\theta dz/f_1 f_2)$ satisfies
\[
\begin{align*}
855196504859980341686142500864t^2 + 3302292412508080227365641626058752t
+ 19261763972461410497003918883305 &= 0, \\
126787493190876461x + 29249267520503808t + 56450856593129472 &= 0, \\
y &= 0.
\end{align*}
\]

We can apply our algorithms for any 0-dimensional ideal which has the shape basis even though the given generators are not the shape basis.

**Example 4** Let \( I \) be the ideal in \( \mathbb{Q}[x, y] \) generated by \((x^2 + y^2)^2 + 3x^2y - y^3, x^2 + y^2 - 1\). Then \( I \) has the shape basis
\[
\{16x^6 - 24x^4 + 9x^2, y - (4x^4 - 5x^2 + 1)\}
\]
with respect to the lexicographical order \( y \succ x \). By the transformation law of the residue ([1]), we have
\[
\text{Res}_{\alpha \in Y} \left( \frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right) = \text{Res}_{\alpha \in Y} \left( \frac{h\Delta}{f_1f_2} \right)
\]
for some \( h \in \mathbb{Q}[x, y] \), where \( Y \) is the set of common zeros of \((x^2 + y^2)^2 + 3x^2y - y^3\) and \( x^2 + y^2 - 1 \) and \( \Delta = -4x^2 + 1 \).

Let us compute residues
\[
\text{Res}_{\alpha \in Y} \left( \frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right)
\]
for \( h = 34x^5y + 2x^3y^4 - 3x^2 + 42 \). Put \( \theta = h\Delta = -136yx^7 + (8y^4 + 34y)x^5 + 12x^4 + 2y^4x^3 - 171x^2 + 42 \).

The annihilator \( P \) of the algebraic local cohomology class \([1/f_1f_2]\) given in (A) is
\[
P = x(4x^2 - 3)\partial_x + (-8x^4 + 6x^2)\partial_y + 24x^2 - 6.
\]

Put \( I_1 = \langle x^2, y - 1 \rangle \) and \( I_2 = \langle 16x^4 - 24x^2 + 9, y - x^2 + 5/4 \rangle \). Then we have \( \langle f_1, f_2 \rangle = I_1 \cap I_2 \).

For \( I_1 \), we have \( v_{1,0} = -3, w_{1,0} = 2x \) and \( \theta_1 = -14v_{1,0} \). Thus the residue
\[
\text{Res}_{(0,0)} \left( \frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right)
\]
is equal to zero. On the other hand, for \( I_2 \), we have \( v_{2,0} = 12x^2 - 3, v_{2,1} = 8x^3, \)
\[w_{2,0} = 64x^3 - 48x, w_{2,1} = 48x^2 - 36 \text{ and} \]
\[
\theta_2 = \frac{213}{512}w_{2,0} + \frac{1}{8}w_{2,1} - \frac{53}{4}v_{2,0} + \frac{101}{32}v_{2,1}.
\]
Thus we have \( J_2 = \langle 2(213/512 + (1/8)x) - t, 4x^2 - 3, 2y + 1 \rangle \) and
\[
G_2 = (-12288t^2 + 27264t - 14099, -64x + 192t - 213, 2y + 1).
\]

**References**


