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SATO-KASHIWARA DETERMINANT AND LEVI CONDITIONS FOR SYSTEMS

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ABSTRACT. In a paper with A. D'AGNOLO [10] we have introduced a variant of the SATO-KASHIWARA determinant [33]. This determinant computes the Newton polygon of determined systems of linear partial differential operators with constant multiplicities, which gives a necessary and sufficient condition for $\mathcal{C}^\infty$ well-posedness.

We give here a different presentation of this result. We give also applications to the Cauchy problem in Gevrey classes that are not discussed in [10].

1. NEWTON POLYGON FOR SCALAR OPERATOR

1. Let h be a scalar operator of order M, with analytic coefficients and *characteristics of constant multiplicities*, that is

$$\sigma_M(h) = \prod_j H_j^{m_j}(x,\xi),$$

where $H_j(x,\xi)$ are homogeneous irreducibles polynomials such that $\prod_j H_j$ is strictly hyperbolic.

Let H be one of the H_j. DE PARIS [11, Prop. 1] proved that, given an operator H' with principal symbol H, there exist operators l'_r, $r = 1, \ldots, M$, of order $\leq M - r - \nu_r \deg(H)$, such that one can locally decompose h in the following manner:

$$h = \sum_{r=0}^{M} l'_r H'^{\nu_r}. \tag{1}$$

2. According to such decomposition, we construct the *Newton polygon* of h, with respect to the characteristic factor H.

Set

$$N_H^0(h) = \left\{ (\operatorname{ord}(l'_r H'^{\nu_r}), \operatorname{ord}(l'_r H'^{\nu_r}) - \nu_r) \,\middle|\, r = 1, \ldots, M \right\}.$$

Consider the family $\mathcal{N}$ of the half-planes π of $\mathbb{R}^2$ of the form

$$\pi = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, mx + ny + p \leq 0 \right\},$$

with $m, n, p \in \mathbb{Z}$ and $mn \geq 0$. The *geometric Newton polygon* is the intersection of half-planes π in $\mathcal{N}$ containing $N_H^0(h)$:

$$\mathrm{New}_H^0(h) = \bigcap_{\substack{\pi \in \mathcal{N} \\ N_H^0(h) \subset \pi}} \pi.$$

The boundary of $\mathrm{New}_H^0(h)$ has a finite number, say $e+2$, of edges with slopes $-\infty = m_0 < m_1 < \cdots < m_e < m_{e+1} = 0$. Denote $\partial' \mathrm{New}_H^0(h)$ the set of vertices of $\mathrm{New}_H^0(h)$.

The *full Newton polygon of H with respect to H* is the set of couples

$$\Big(\big(\mathrm{ord}(l'_r H'^{\nu_r}), \mathrm{ord}(l'_r H'^{\nu_r}) - \nu_r\big), \sigma(l'_r) H^{\nu_r}\Big),$$

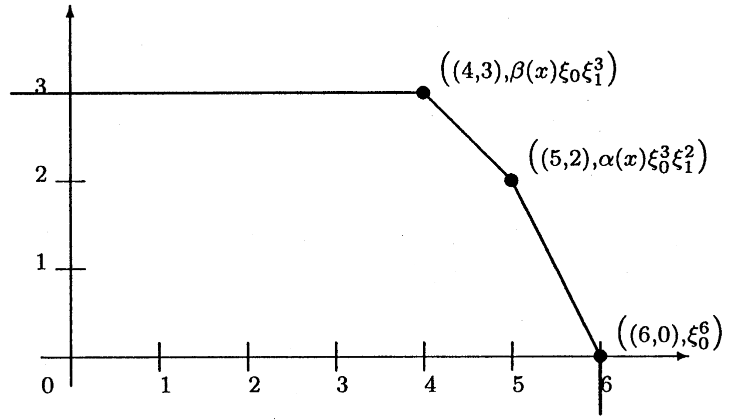
where $\big(\mathrm{ord}(l'_r H'^{\nu_r}), \mathrm{ord}(l'_r H'^{\nu_r}) - \nu_r\big)$ belongs to $\partial' \mathrm{New}_H^0(h)$. We denote it by $\mathrm{New}_H(h)$.

Example 1. Let $x = (x_0, x_1)$, and

$$h = D_0^6 + \alpha(x) D_0^3 D_1^2 + \beta(x) D_0 D_1^3 + \gamma(x) D_1^3 + \delta(x) D_0^2 D_1^2,$$

with α, β, γ, δ analytic functions in some open set $\Omega \subset \mathbb{R}^2$.

Assuming $\alpha \not\equiv 0$, and $\beta \not\equiv 0$, the Newton polygon of h is



3. The decomposition and the ν_j in (1) depends on the choice of H', only ν_0 is invariant (it is the multiplicity of H in the principal symbol of h). However, the Newton polygon does not depends on the choice of the operator H', of principal symbol H.

Let $\mathring{H}'$ be an operator of principal symbol H, it's easy to show by induction (cf. [39, Lemma II.1.7]) that for any $r \in \mathbb{N}$ there exist operators $C'_{r,j}$, $j = 0, \ldots, r$, of order $\leq j(\deg H - 1)$, such that

$$H'^r = \sum_{j=0}^{r} C'_{r,j} \mathring{H}'^{r-j}.$$

Given a decomposition of h with respect to H' as in (1), we can obtain a decomposition of h with respect to $\mathring{H}'$. Each term $l'_r H'^{\nu_r}$ is replaced by terms of the form $l'_r C'_{\nu_r,j} \mathring{H}'^{\nu_r - j}$, $j = 1, \dots, \nu_r$. Each of these terms will produce a point

$$\left(\operatorname{ord}(l'_r C'_{\nu_r,j} \mathring{H}'^{\nu_r-j}), \operatorname{ord}(l'_r C'_{\nu_r,j} \mathring{H}'^{\nu_r-j}) - (\nu_r - j)\right),$$

and it's easy to see that all of them are on the same horizontal line, on the left of the point

$$\left(\operatorname{ord}(l'_r \mathring{H}'^{\nu_r}), \operatorname{ord}(l'_r \mathring{H}'^{\nu_r}) - \nu_r\right),$$

so they will not change the Newton polygon. Note that also symbols belonging to an edge of $\mathrm{New}^0_H(h)$ with non zero slope are well defined. However we will not consider them here.

4. Using the Newton polygon we can state the known results for C^∞ and Gevrey well-posedness as follows:

Theorem (De Paris [11], Flaschka-Strang [14], Chazarain [8]). *In order the Cauchy problem for h to be C^∞ well posed is necessary and sufficient that* $\mathrm{New}_H(h)$ *is reduced to a quadrant, for any H.*

Theorem (Ivrii [17], De Paris-Wagschal [12], Komatsu [23]). *If the maximum slope of* $\mathrm{New}_H(h)$ *is p, then the Cauchy problem for h is γ^d well posed, for any $d < 1 + \dfrac{1}{p}$, for any H.*

If the Cauchy problem for h is γ^d well posed, then the maximum slope of $\mathrm{New}_H(h)$ *is smaller than $\dfrac{1}{d-1}$, for any H.*

The Cauchy problem for the operator in Example 1 is γ^d well posed, for any $d < \dfrac{3}{2}$. It is not well posed in γ^d, with $d > \dfrac{3}{2}$ if $\alpha \not\equiv 0$.

5. We give the definition of *upper* and *lower Gevrey order* of an operator, that we will use in the following.

Consider the ordering of $\mathbb{Z}^2$ for which

$$(i', j') \underset{(\cdot, s)}{\leq} (i, j) \qquad \Longleftrightarrow \qquad i' - i \leq (1 - s)(j' - j)$$

the inequality being strict if $j' > j$. The *upper Gevrey s-order* of h is the maximum of the couples (i, j) belonging to $\mathrm{New}_H^0(h)$, according to the order $\underset{(\cdot,s)}{\leq}$. The *upper Gevrey s-symbol* is the associated symbol in $\mathrm{New}_H(h)$, and we note it by $\sigma_H^{(\cdot,s)}(h)$.

Similary, we define the *lower Gevrey r-order* as the maximum of the couples (i, j) belonging to $\mathrm{New}_H^0(h)$, according to the order $\underset{(r,\cdot)}{\leq}$:

$$(i', j') \underset{(r,\cdot)}{\leq} (i, j) \qquad \Longleftrightarrow \qquad j' - j \leq (i' - i)/(1 - r),$$

the inequality being strict if $i' > i$. The *lower Gevrey r-symbol* is the associated symbol in $\mathrm{New}_H(h)$, and we note it by $\sigma_H^{(r,\cdot)}(h)$.

Necessary and sufficient condition for Gevrey and C^∞ well posedness can be stated as follows:

Theorem. *If $\sigma_H^{(\cdot,s)}(h) = \sigma_H^{(\cdot,1)}(h)$, then the Cauchy problem for h is well posed in γ^d, for all $1 \leq d < s$.*

If the Cauchy problem for h is well posed in γ^d, then $\sigma_H^{(r,\cdot)}(h) = \sigma_H^{(\cdot,1)}(h)$, for all $1 \leq r \leq d$.

In order the Cauchy problem for h to be well posed in C^∞, it's necessary and sufficient that $\sigma_H^{(\cdot,s)}(h) = \sigma_H^{(\cdot,1)}(h)$ for all s (or equivalently $\sigma_H^{(r,\cdot)}(h) = \sigma_H^{(\cdot,1)}(h)$ for all r).

6. We define the "sum" of two Newton polygons as follows: given N_1 and N_2 Newton polygons, let h_1 and h_2 be differential operators such that $N_1 = \mathrm{New}_H(h_1)$ and $N_2 = \mathrm{New}_H(h_2)$; then

$$N_1 + N_2 = \mathrm{New}_H(h_1 \circ h_2).$$

The sum does not depends on the choice of h_1 and h_2, it is commutative and *regular*, that is

$$N_1 + N_2 = N_1 + N_3 \qquad \Longrightarrow \qquad N_2 = N_3.$$

With this sum the set of Newton polygons becomes a commutative monoid, and the application

$$\{\text{differential operators}\} \qquad \mapsto \qquad \{\text{Newton polygons}\}$$

is a morphism from a (non commutative) ring into a (commutative) monoid. The problem is now to extend such morphism to matrices of differential operators.

2. Non commutative determinant

1. Many authors have studied the problem of extension of a morphism from a ring into a monoid to matrices with entries in the ring.

The most important example is the morphism "principal symbol" from a ring of differential operator to a monoid of symbols.

Let A a square matrix of differential operators of order $\leq M$, the "classical" principal part of A is defined by

$$\det \sigma_M(A_{IJ}),$$

where $\sigma_M(A_{IJ})$ is the homogeneous part of degree M of the symbol of A_{IJ}.

A more refined principal part can been defined as follows (cf. [27]): let r_i, s_i integers such that $\operatorname{ord}(A_{IJ}) \leq r_i - s_j$, then consider

$$\det \sigma_{r_i - s_j}(A_{IJ}). \tag{2}$$

If $\det \sigma_{r_i-s_j}(A_{IJ}) \neq 0$, one say that A is *normal* and one can use (2) as principal part of A.

Hoverer one can find invertible matrices such that $\det \sigma_{r_i-s_j}(A_{IJ}) \equiv 0$, then such definition is useless for matrices that are not normal. Moreover product of normal matrices is not necessarly normal.

2. Since in the constant coefficient case one can consider the principal part of determinant of the full symbols of the elements of A, as principal part of A, HUFFORD [15] defined the determinant of a general matrix as the principal part of the DIEUDONNÉ determinant. This principal part coincides with (2) if the matrix is normal.

However, since DIEUDONNÉ determinant is defined on fields, this principal part is a priori a meromorphic function. SATO-KASHIWARA [33] proved however that it is in fact holomorphic.

3. We now recall DIEUDONNÉ determinant. (See [13] and [6] for complete details).

Let K be a field, not necessarily commutative, and set $\mathsf{K}^* = \mathsf{K} \setminus \{0\}$ and $[\mathsf{K}^*, \mathsf{K}^*]$ the *commutator multiplicative subgroup* of K^*, that is the subgroup of K^* generated by the elements of the form $xyx^{-1}y^{-1}$, with $x, y \in \mathsf{K}^*$. Denote $\overline{\mathsf{K}} = (\mathsf{K}^*/[\mathsf{K}^*, \mathsf{K}^*]) \cup \{0\}$.

Let $\operatorname{Mat}_m(\mathsf{K})$ be the ring of $m \times m$ matrices with elements in K, Dieudonné [13] (see also [6]) proved that there exists a unique multiplicative morphism

$$\operatorname{Det}\colon \operatorname{Mat}_m(\mathsf{K}) \to \overline{\mathsf{K}},$$

satisfying the axioms:

1. $\operatorname{Det}(B) = \bar{c} \operatorname{Det}(A)$ if B is obtained from A by multiplying one row of A on the left by $c \in \mathsf{K}$ (where $\bar{c}$ denotes the image of c by the map $\mathsf{K} \to \overline{\mathsf{K}}$);
2. $\operatorname{Det}(B) = \operatorname{Det}(A)$ if B is obtained from A by adding one row to another;
3. the unit matrix has determinant $\bar{1}$.

Such a determinant satisfies natural properties as

1. $\mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B)$,
2. $\mathrm{Det}(A \oplus B) = \mathrm{Det}(AB)$,
3. an $m \times m$ matrix A is invertible as a left (resp. right) K-linear endomorphism of K^m if and only if $\mathrm{Det}(A) \neq 0$;
4. if K is commutative, then $\overline{\mathsf{K}} = \mathsf{K}$, and the DIEUDONNÉ determinant coincides with the usual determinant.

4. The DIEUDONNÉ determinant is computed with the usual *Gauss method.* Let $\mathrm{GL}_m(\mathsf{K})$ be the group of non-singular matrices, $\mathrm{SL}_m(\mathsf{K})$ the subgroup of unitary matrices (a matrix U is unitary if it is obtained from the unit matrix I_m by replacing the zero in the i-th row and j-th column ($i \neq j$) by some element of K). The usual Gauss method shows that given $A \in \mathrm{Mat}_m(\mathsf{K})$ there exist unitary matrices $U_1, \ldots, U_\ell$ such that $U_1 \cdots U_\ell A$ is a matrix obtained from the identity matrix by replacing the 1 in the m-th row and m-th column by some element in K.

5. Now, let R be a noncommutative ring having the *Ore property* [32]: given $a, b \in \mathsf{R}$ there exists $p, q \in \mathsf{R}$ such that $pa = qb$. The Ore property is the necessary and sufficient condition, in order that R admits a quotient field K.

Any morphism φ from R into a commutative monoid M can be extended as a morphism (that we still denote by φ) from K to KM, where KM is the quotient monoid. By the universal property of $\overline{\mathsf{K}}$, φ factorizes trough $\overline{\mathsf{K}}$, according to the following diagram:

$$
\begin{tikzcd}
\mathsf{R} \arrow[rr, "\varphi"] \arrow[d, hook] & & \mathsf{M} \arrow[d, hook] \\
\mathsf{K} \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & \mathsf{KM} \\
& \overline{\mathsf{K}} \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
\quad .
$$

In order to extend the morphism φ to $\mathrm{Mat}_n(\mathsf{R})$, one can consider the map $\imath \circ \mathrm{Det} \circ \overline{\varphi}$ where $\imath$ is the natural injection of $\mathrm{Mat}_n(\mathsf{R})$ in $\mathrm{Mat}_n(\mathsf{K})$ induced by the injection $\mathsf{R} \hookrightarrow \mathsf{K}$:

$$
\begin{tikzcd}
\mathrm{Mat}_n(\mathsf{R}) \arrow[d, hook, "\imath"'] \arrow[rrrd, bend left=10, "\det_\varphi"] & \mathsf{R} \arrow[rr, "\varphi"] \arrow[d, hook] & & \mathsf{M} \arrow[d, hook] \\
\mathrm{Mat}_n(\mathsf{K}) \arrow[drr, "\mathrm{Det}"'] & \mathsf{K} \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & \mathsf{KM} \\
& & \overline{\mathsf{K}} \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
$$

So we have the following

Theorem (Adjamagbo [4], Moussy [31]). *Let R be an Ore domain, M a commutative monoid, and $\varphi\colon \mathsf{R} \to \mathsf{M}$ such that $\varphi(a)$ is a regular*[1] *element of M for any $a \in \mathsf{R}$. Let KM be the quotient monoid $\mathsf{KM} = \varphi(\mathsf{R})^{-1}\mathsf{M}$.*

There exists a unique map

$$\det{}_{\varphi}\colon \mathrm{Mat}_n(\mathsf{R}) \to \mathsf{KM}$$

such that

1. $\det{}_{\varphi}(AB) = \det{}_{\varphi}(A)\det{}_{\varphi}(B)$*;*
2. $\det{}_{\varphi}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \dots & 0 & a \end{pmatrix} = \bar{a}$*, where $\bar{a}$ denotes the image of a by the map* $\mathsf{K} \to \overline{\mathsf{K}}$.

Note however that $\det{}_{\varphi}$ has values in the quotient monoid. One may ask when the extension is "regular", in the sense that $\det{}_{\varphi}(A) \in \imath(\mathsf{M})$ for any $A \in \mathrm{Mat}_n(\mathsf{R})$.

6. ADJAMAGBO gave a positive answer in the case the ring R is a *filtered ring*, M is the associated graded ring (which is of course assumed to be commutative and factorial) and φ the natural symbol map $\mathsf{R} \to \mathsf{GR}$ [2], giving so an algebraic version of SATO-KASHIWARA result [33]. He obtain also a result for geometric Newton polygons on Weyl algebras [3].

7. We return to our problem. Let $\mathcal{O}_\Omega$ be the ring of homomorphic functions on a open set Ω, and $\mathcal{D}_\Omega$ the ring of differential operators, with homomorphic coefficients on Ω. Using ADJAMAGBO results we can prove that we can extend $\sigma_H^{(\cdot,s)}$ and $\sigma_H^{(r,\cdot)}$ to matrices with entries in $\mathcal{D}_\Omega$, and also that

$$\mathrm{Mat}_n(\mathcal{D}_\Omega) \to \{ \text{ geometric Newton polygons } \},$$

is well defined. This can be enough for the applications, but it's not enough to prove that the map

$$\mathrm{Mat}_n(\mathcal{D}_\Omega) \to \{ \text{ full Newton polygons } \},$$

is well defined.

To prove this, we can use SATO-KASHIWARA original argument.

[1] An element m in a commutative monoid M is called *regular* if $mn = mp$ implies $n = p$.

8. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Mat}_n(\mathcal{E}_\Omega) \arrow[dd, hook, "\imath"'] \arrow[drr, dotted] & \mathcal{E}_\Omega \arrow[rr, "\sigma"] \arrow[dr, "\pi|_{\mathcal{E}_\Omega}"] \arrow[dd, hook] & & \mathcal{O}_\Omega \arrow[dd, hook] \\
 & & \overline{\mathcal{E}_\Omega} \arrow[dd, hook] & \\
\mathrm{Mat}_n(\mathsf{K}\mathcal{E}_\Omega) \arrow[drr, "\mathrm{Det}"'] & \mathsf{K}\mathcal{E}_\Omega \arrow[rr, "\sigma"] \arrow[dr, "\pi"] & & \mathcal{M}_\Omega \\
 & & \overline{\mathsf{K}\mathcal{E}_\Omega} \arrow[ur, "\overline{\sigma}"'] &
\end{tikzcd}$$

where σ is the principal symbol, and Det the DIEUDONNÉ determinant.

Set $\det_{\mathrm{SK}}(A) = \overline{\sigma}\big(\mathrm{Det}(A)\big)$. A priori one has $\det_{\mathrm{SK}}(A) = \dfrac{f}{g} \in \mathcal{M}_\Omega$. SATO-KASHIWARA proved that if $Z = \big\{(x,\xi) \mid g(x,\xi) = 0\big\}$, then there exists $U \subset Z$ with $\operatorname{codim} U \geq 2$ such that in the complement of U $\det_{\mathrm{SK}}(A)$ is holomorphic. Using then Hartog's Theorem they conclude that $\det_{\mathrm{SK}}(A)$ is holomorphic everywhere.

Now, considering $\mathrm{Det}(A)$, one has $\mathrm{Det}(A) = \overline{Q^{-1}P}$, for some P and Q in $\mathcal{E}_\Omega$. Repeating SATO-KASHIWARA proof with $Z = \big\{(x,\xi) \mid \sigma(Q)(x,\xi) = 0\big\}$, we can prove that there exists $U \subset Z$ with $\operatorname{codim} U \geq 2$ and such that, in the complement of U, $\mathrm{Det}(A)$ is the image by π of some $P \in \mathcal{E}_\Omega$, that is there exists $P \in \mathcal{E}_\Omega$ defined up to commutators that represent generically (out a set of codimension 2) DIEUDONNÉ determinant.

Using the trick of the dummy variable we can prove that if A is a matrix of differential operators, then $\mathrm{Det}(A)$ is generically defined in $\overline{\mathcal{D}}$, where $\overline{\mathcal{D}_\Omega}$ is the canonical image of $\mathcal{D}_\Omega$ in $\overline{\mathsf{K}\mathcal{E}_\Omega}$.

3. LEVI CONDITION FOR SYSTEMS

1. Using previous remark we obtain then

Theorem. *Let A a square matrix of differential operators of order $\leq M$, and assume that*

$$\det \sigma_M(A_{\mathrm{IJ}}) = \prod_j H_j^{m_j}(x,\xi), \tag{3}$$

where $H_j(x,\xi)$ are homogeneous irreducibles polynomials such that $\prod\limits_j H_j$ is strictly hyperbolic.

Then there exists a canonically define Newton polygon $\mathrm{New}_H(A)$ *along each irreducible factor* H*, having the following properties*

1. *the Cauchy problem for* A *is* C^∞ *well posed if and only if* $\mathrm{New}_H(A)$ *is reduced to a quadrant, for any* H*;*
2. *if the maximum slope of* $\mathrm{New}_H(A)$ *is* $\leq p$*, for any* H*, then the Cauchy problem for* A *is* γ^d *well posed, for any* $d < 1 + \dfrac{1}{p}$*;*
   if the Cauchy problem for h *is* γ^d *well posed, then the maximum slope of* $\mathrm{New}_H(A)$ *is smaller than* $\dfrac{1}{d-1}$*, for any* H*.*

The first part of this Theorem can be proved for more general matrices. Indeed we can replace (3) with

$$\det{}_{SK}(A) = \prod_j H_j^{m_j}(x,\xi).$$

We can prove then that $\mathrm{New}_H(A)$ is reduced to a quadrant if, and only if, the $\mathcal{D}_\Omega$-module associated to A has *regular singularities* in the sense of KASHIWARA-OSHIMA [22]. D'AGNOLO-TONIN [9] have prove that the Cauchy problem for such $\mathcal{D}_\Omega$-module is well posed in C^∞.

However as we are interested also in Gevrey well-posedness we will restrict to "classical" matrices and we will assume (3).

2. In order to prove our result, we recall that MATSUMOTO [28, Theorem 3.1] proved that any system with constant multiplicities can be microlocally reduced, out of an analytic set, to a direct sum of matrices of pseudo-differential operator having the following normal form

$$\tilde{A}_j = I\big(D_0 - \lambda_j(x;D')\big) + J_j|D'| + \tilde{b}_j(x;D'),$$

where

$$J_j = \begin{pmatrix} 0 & 1 & \dots & & 0 \\ \vdots & 0 & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad \tilde{b}_j = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \\ (\tilde{b}_j)_1^{\nu_j} & & \dots & (\tilde{b}_j)_{\nu_j}^{\nu_j} \end{pmatrix}.$$

Moreover, one can assume $\lambda_j \equiv 0$ and $(\tilde{b}_j)_{\nu_j}^{\nu_j} \equiv 0$.

3. To prove the Theorem, it's enough to prove the Theorem for systems in the normal form. We have

$$\begin{pmatrix} D_0 & D_1 & 0 & \dots & 0 \\ 0 & D_0 & D_1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & D_0 & D_1 \\ (\tilde{b}_j)_1^\nu & & \dots & (\tilde{b}_j)_{\nu-1}^\nu & D_0 \end{pmatrix} \begin{pmatrix} 1 & -D_1 & D_1^2 & \dots & (-1)^{\nu-1} D_1^{\nu-1} \\ 0 & D_0 & -D_0 D_1 & & \vdots \\ & & D_0^2 & \ddots & \\ & & & \ddots & -D_0^{\nu-2} D_1 \\ 0 & & & 0 & D_0^{\nu-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & & \dots & 0 \\ 0 & D_0 & 0 & & \vdots \\ & & \ddots & & 0 \\ & & & D_0^{\nu-1} & 0 \\ * & * & & * & W \end{pmatrix}$$

where

$$W = D_0^\nu + \sum_{\mathrm{J}=1}^{\nu} (-1)^{\nu-\mathrm{J}} (\tilde{b}_j)_{\mathrm{J}}^\nu D_0^{\mathrm{J}-1} D_1^{\nu-\mathrm{J}}$$

(we don't need to explicit the others terms on the last line). We have then

$$\operatorname{Det} A = \overline{W}.$$

4. Kajitani [19, Theorem 3], proved that the Cauchy problem for $\tilde{A}_j$ is C^∞-well-posed if and only if

$$\operatorname{ord}(\tilde{b}_j)_{\mathrm{J}}^{\nu_j} \leq -(\nu_j - \mathrm{J}), \tag{4}$$

for $j = 1, \dots, r$, $l = 1, \dots, l_j$, $\mathrm{J} = 1, \dots, \nu_j - 1$, and (4) is equivalent to say that the Newton polygon of W is reduced to a quadrant. This proves first statement of the Theorem.

5. Assume that the maximum slope of $\mathrm{New}_{\xi_0}(W)$ is p, we have

$$\frac{\nu - \mathrm{J} + \operatorname{ord}(\tilde{b}_j)_{\mathrm{J}}^\nu}{1 - \operatorname{ord}(\tilde{b}_j)_{\mathrm{J}}^\nu} \leq p$$

that is

$$\operatorname{ord}(\tilde{b}_j)_{\mathrm{J}}^\nu \leq -(m - \mathrm{J}) + \frac{1}{s}(m - \mathrm{J} + 1), \tag{5}$$

where $s = 1 + \dfrac{1}{p}$. Condition (5) is sufficient for γ^d well-posedness if $d < s$ (cf. [35]). On the other side if Cauchy problem for h is γ^d well-posed, then (5) is verified with $d \leq s$ (cf. [30]), and we obtain also the necessity.

6. By similar method we can prove the following

Theorem. *If* $\det^{(\cdot,s)}(A) = \det^{(\cdot,1)}(A)$, *then the Cauchy problem for h is well posed in γ^d, for all $1 \le d < s$.*

If the Cauchy problem for h is well posed in γ^d, then $\det^{(r,\cdot)}(A) = \det^{(\cdot,1)}(A)$, *for all $1 \le r \le d$.*

In order the Cauchy problem for h to be well posed in C^∞, it's necessary and sufficient that $\det^{(\cdot,s)}(A) = \det^{(\cdot,1)}(A)$ *for all s* ($\det^{(r,\cdot)}(A) = \det^{(\cdot,1)}(A)$ *for all r*).

This last result is very useful when we have to compute the determinant. Indeed if A is $(\cdot,s)$-normal, that is there exists $n_I, m_J \in \mathbb{Z}^2$ such that $\mathrm{ord}^{(\cdot,s)} A_{IJ} \le n_I - m_J$ and $\det \sigma^{(\cdot,s)}_{n_I - m_J} A_{IJ} \neq 0$, then

$$\det^{(\cdot,s)}(A) = \det \sigma^{(\cdot,s)}_{n_I - m_J} A_{IJ}.$$

4. Examples

Example 2 (cf. [19, 38]). Let $A = \begin{pmatrix} D_0^2 + \alpha D_1 & \beta D_1 \\ \gamma D_1 & D_0^2 + \delta D_1 \end{pmatrix}$, with α, β, γ, δ analytic functions of $x = (x_0, x_1)$, and $\gamma\beta \not\equiv 0$. If $s \le 2$, A is $(\cdot,s)$-normal, and

$$\det^{(\cdot,s)} A = \begin{cases} \xi_0^4 & \text{if } s < 2, \\ \xi_0^4 + (\alpha+\delta)\xi_0^2\xi_1 + (\alpha\delta - \beta\gamma)\xi_1^2 & \text{if } s = 2. \end{cases}$$

If $\alpha + \delta \not\equiv 0$ or $\alpha\delta - \beta\gamma \not\equiv 0$, then $\det^{(\cdot,s)} A \neq \det^{(\cdot,1)} A$, so Cauchy problem for A is not C^∞-well-posed. If $\alpha + \delta \equiv 0$ and $\alpha\delta - \beta\gamma \equiv 0$, A is not $(\cdot,s)$-normal, for $s > 2$. Let

$$P_1 = \gamma^3 D_1 \quad \text{and} \quad P_2 = \gamma^2 D_0^2 - 2\gamma\gamma_0' D_0 + \alpha\gamma^2 D_1 + \mu,$$

with $\mu = 2(\gamma_0')^2 - \gamma\gamma_{00}'' - \alpha\gamma\gamma_1' + \alpha_1'\gamma^2$, so that $P_1 \circ (D_0^2 + \alpha D_1) = P_2 \circ (\gamma D_1)$. We have

$$\begin{pmatrix} 1 & 0 \\ -P_1 & P_2 \end{pmatrix} \circ \begin{pmatrix} D_0^2 + \alpha D_1 & \beta D_1 \\ \gamma D_1 & D_0^2 + \delta D_1 \end{pmatrix} = \begin{pmatrix} D_0^2 + \alpha D_1 & \beta D_1 \\ 0 & W \end{pmatrix}$$

with

$$\begin{aligned} W = \gamma^2 D_0^4 - 2\gamma\gamma_0' D_0^3 + \mu D_0^2 - 2\gamma(\gamma_0'\delta - \gamma\delta_0') D_0 D_1 \\ - \left[\gamma(\gamma_0'\delta - \gamma\delta_0')_0' + 2\gamma_0'(\gamma\delta_0' - \gamma_0'\delta)\right] D_1. \end{aligned}$$

We have $\sigma_H^{(\cdot,s)}(D_0^2 + \alpha D_1) = \gamma^2 \sigma_H^{(\cdot,s)}(P_2)$, for every s. Then

$$\det^{(\cdot,s)} A = \frac{1}{\gamma^2}\sigma_H^{(\cdot,s)}(W) = \begin{cases} \xi_0^4 & \text{if } s < 3, \\ \xi_0^4 + [(\gamma_0'\delta - \gamma\delta_0')/\gamma]\xi_0^2\xi_1 & \text{if } s = 3. \end{cases}$$

Note that if $\gamma_0'\delta - \gamma\delta_0' \equiv 0$ then $W = \gamma^2 D_0^4 - 2\gamma\gamma_0' D_0^3 + \mu D_0^2$, so $\det^{(\cdot,s)} A = \xi_0^4$, for all s. Remark that the function $(\gamma_0'\delta - \gamma\delta_0')/\gamma$ is analytic.

Example 3. Consider the matrix

$$A = \begin{pmatrix} D_1^2 + D_0 D_1 - D_0 - D_1 & D_0^2 - D_1^2 + \alpha D_0 + (1+\alpha) D_1 + \beta \\ D_1^2 - D_0 + 1 & D_0 D_1 - D_1^2 + D_0 + \alpha D_1 + \gamma \end{pmatrix}.$$

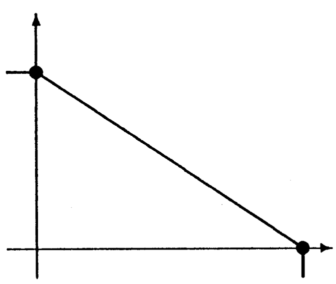
Since A_{11} and A_{21} are operators with constant coefficients, we have

$$\begin{pmatrix} 1 & 0 \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & W \end{pmatrix},$$

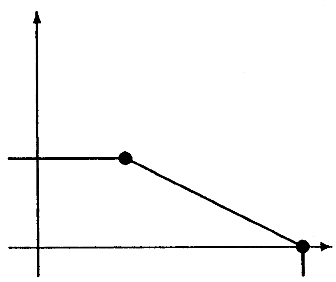
where $W = A_{11}A_{22} - A_{21}A_{12} = D_0^3 + \sum_{i+j \leq 2} W_{ij} D_0^i D_1^j$ and

$$\begin{cases} W_{02} = 1 - \alpha + \alpha_0 - \beta + \gamma \\ W_{11} = \gamma - \alpha_1 \\ W_{01} = -1 - \alpha - \alpha_1 + \alpha_{01} - 2\beta_1 - \gamma + \gamma_0 + 2\gamma_1 \end{cases}$$

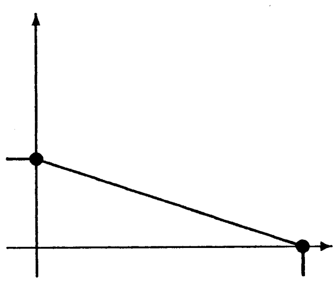
(we don't need to explicit the terms $W_{i,0}$, since they will never contribute to Newton polygon). Note that A_{11} is not invertible as an operator acting in C^∞, but Gauss algorithm is performed in the quotient field of $\mathcal{E}_\Omega$, where it is invertible, so we can write $\det^{(\cdot,s)} A = \sigma_H^{(\cdot,s)} W$, for all s. The Newton polygon of A is then



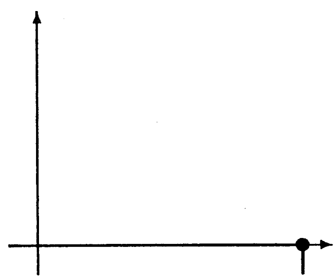
$W_{0,2} \not\equiv 0$



$W_{0,2} \equiv 0,\ W_{1,1} \not\equiv 0$



$W_{0,2} = W_{1,1} \equiv 0,\ W_{0,1} \not\equiv 0$



$W_{0,2} = W_{1,1} = W_{0,1} \equiv 0$

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