数理解析研究所講究録
1158 巻 2000 年 1-16

Cauchy-Kowalewski theorem with a large parameter and an application to microlocal analysis

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1 Introduction

We consider a class of differential equations with a parameter and show the existence of solutions of infra-exponential type by using the Cauchy-Kowalewski theorem with a parameter.

As application, we study solvability of some class of differential equations in the sheaf of 2-analytic functions, that is, microfunctions with holomorphic parameters. We also give a characterization of solutions of a class of differential equations. In particular we treat *transversally elliptic operators* and other related operators, which are difficult to study in the former theory of second microlocal analysis.

Let V and Σ be the following regular involutive and Lagrangian submanifolds of $\sqrt{-1}T^*\mathbb{R}^n$ respectively:

$$V=\left\{(x,\sqrt{-1}\xi\cdot dx)\in\sqrt{-1}\dot{T}^*\mathbb{R}^n;\xi_1=\cdots=\xi_{n-1}=0\right\},$$
$$\Sigma=\left\{(x,\sqrt{-1}\xi\cdot dx)\in\sqrt{-1}\dot{T}^*\mathbb{R}^n;\xi_1=\cdots=\xi_{n-1}=x_n=0\right\},$$

where $\sqrt{-1}\dot{T}^*\mathbb{R}^n=\sqrt{-1}T^*\mathbb{R}^n\setminus\mathbb{R}^n$. So, we set $x=(x',x_n)$ with $x'=(x_1,\ldots,x_{n-1})$ and $\xi=(\xi',\xi_n)$ with $\xi'=(\xi_1,\ldots,\xi_{n-1})$. Let P be a differential operator with analytic coefficients defined near a point $0\in\mathbb{R}^n$. Assume P is transversally elliptic in a neighborhood of $p_\circ=(0,\sqrt{-1}\,dx_n)\in\Sigma$, that is, P satisfies the property:

$$|\sigma(P)(x,\sqrt{-1}\xi/|\xi|)|\sim(|x_n|+|\xi'|/|\xi|)^l$$

for some non-negative integer l in a neighborhood of $p_\circ$. Here $\sigma(P)$ denotes the principal symbol of P. Grigis-Schapira-Sjöstrand [3] has given a theorem on the propagation of analytic singularities for this operator P along the bicharacteristic leaf of V passing through $p_\circ$.

On the other hand, assume P satisfies the property:

$$|\sigma(P)(x,\sqrt{-1}\xi/|\xi|)| \sim (|x_n|^k + |\xi'|/|\xi|)^l$$

for some non-negative integers k and l in a neighborhood of $p_\circ \in \Sigma$. We have proved in [2] unique solvability in the sheaf $\tilde{\mathcal{C}}^2_V$ of *small 2-microfunctions* for this operator P. This result was obtained by using our elementary construction of $\tilde{\mathcal{C}}^2_V$ and the estimate of the support of solution complexes with coefficients in $\tilde{\mathcal{C}}^2_V$. In this case, the structure of solutions of $Pu = f$ in the sheaf $\mathcal{C}_M$ of Sato microfunctions is reduced to that in the sheaf $\mathcal{A}^2_V$ of 2-analytic functions. Therefore our result implies the above theorem due to Grigis-Schapira-Sjöstrand [3] because any section of $\mathcal{A}^2_V$ has the property of the uniqueness of analytic continuation along the bicharacteristic leaves of V. Moreover, we showed solvability in $\mathcal{A}^2_V$ and in $\mathcal{C}_M$ of a class of differential equations of the first order with Lagrangian characteristics. Refer to [2].

In connection with these operators, we introduce a new class of differential operators with analytic coefficients defined near $0 \in \mathbb{R}^n$:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha|\le m} a_\alpha(x) D_{x'}^{\alpha'} (x_n D_{x_n})^{\alpha_n},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, and $D_j = D_{x_j} = \partial/\partial x_j$ for $\alpha = (\alpha', \alpha_n) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. One makes the hypothesis $a_{(0,\ldots,0,m)}(0) \neq 0$. Then we will get result of a characterization of the kernel of the operator $P\colon \mathcal{A}^2_V \to \mathcal{A}^2_V$ at $p_\circ = (0, \sqrt{-1}\,dx_n) \in \Sigma$. Furthermore, one makes the hypothesis $a_{(m,0,\ldots,0)}(0) \neq 0$. We will prove solvability of $Pu = f$ in $\mathcal{A}^2_V$ at $p_\circ \in \Sigma$ on some suitable condition of $f \in \mathcal{A}^2_V$.

2 The Cauchy-Kowalewski theorem with a large parameter

We consider the following differential operator:

$$P(z, \lambda, D_z) = \sum_{j=0}^{m} \lambda^j P_j(z, D_z),$$

where $\lambda \in (1,\infty)$ is a parameter and the P_j's are holomorphic differential operators of order $m-j$ defined on a neighborhood D of $0 \in \mathbb{C}^n$:

$$P_j(z, D_z) = \sum_{|\alpha| \le m-j} a^j_\alpha(z) D^\alpha_z.$$

One makes the hypothesis:

$$\begin{cases} a^0_{(m,0,\ldots,0)}(0) \neq 0, \\ a^m_{(0,0,\ldots,0)}(0) \neq 0. \end{cases} \tag{2.1}$$

Consider the following differential equation:

$$P(z,\lambda,D_z)U(z,\lambda) = F(z,\lambda), \tag{2.2}$$

where $F(z,\lambda)$ is a holomorphic function on D for each fixed λ. Assume $F(z,\lambda)$ possesses infra-exponential growth order as $\lambda \to \infty$, that is, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that we have:

$$|F(z,\lambda)| \le C_\varepsilon e^{\varepsilon\lambda}.$$

Theorem 2.1. *There exist a neighborhood D' of $0 \in \mathbb{C}^n$ and a solution $U(z,\lambda)$ of (2.2) which is holomorphic in D' for each fixed λ and possesses infra-exponential growth order.*

We give a sketch of the proof of this theorem. First, we may assume that $a^m_{(0,\ldots,0)}(z) \equiv 1$ from the hypothesis (2.1). By the differential equation (2.2) one has formally:

$$U(z,\lambda) = \lambda^{-m} \sum_{k=0}^{\infty} \left(- \sum_{j=0}^{m-1} \lambda^{j-m} P_j(z, D_z) \right)^k F(z,\lambda),$$

but in general this infinite sum is divergent. So we introduce another modified function instead of U. Let A be a sufficiently large constant. We set $W_0 = (1,\infty)$ and $W_k = \{\lambda \in (1,\infty); \lambda > Ak\}$ for $k \in \mathbb{N} \setminus \{0\}$. Then we define:

$$U_1(z,\lambda) = \lambda^{-m} \sum_{k=0}^{\infty} \chi_{W_k}(\lambda) \left(- \sum_{j=0}^{m-1} \lambda^{j-m} P_j(z, D_z) \right)^k F(z,\lambda) \tag{2.3}$$

for $(z,\lambda) \in D \times (1,\infty)$, where χ_{W_k} is the characteristic function of W_k:

$$\chi_{W_k}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in W_k \\ 0 & \text{if } \lambda \notin W_k. \end{cases}$$

We find immediately that the function U_1 is well-defined, since the sum in (2.3) is locally finite on $D \times (1, \infty)$. The function $U_1(z, \lambda)$ is holomorphic in D for each fixed $\lambda \in (1, \infty)$. We can claim that U_1 is the function of infra-exponential type as $\lambda \to \infty$ in a sufficiently small neighborhood D_1 of $0 \in \mathbb{C}^n$ by Cauchy's inequalities.

Lemma 2.2. *For any positive ε there exists a positive constant M_ε such that one has*

$$|U_1(z, \lambda)| \leq M_\varepsilon e^{\varepsilon\lambda} \quad \textit{for} \quad z \in D_1, \quad \lambda \in (1, \infty).$$

Needless to say, the function U_1 is not a solution of (2.2), but it gives sufficient approximation of solutions of (2.2) in the following sense. By its construction, we have on $D_1 \times (1, \infty)$:

$$P(z, \lambda, D_z)U_1(z, \lambda) = F(z, \lambda) - F_0(z, \lambda),$$

where we set:

$$F_0(z, \lambda) = \sum_{k=1}^{\infty} \chi_{W_{k-1} \setminus W_k}(\lambda) \left(-\sum_{j=0}^{m-1} \lambda^{j-m} P_j(z, D_z) \right)^k F(z, \lambda). \tag{2.4}$$

Here the error $F_0(z, \lambda)$ is also a holomorphic function in D_1 for each fixed $\lambda \in (1, \infty)$, since the sum in (2.4) is locally finite on $D_1 \times (1, \infty)$. Then one can claim that F_0 is exponentially decreasing as $\lambda \to \infty$.

Lemma 2.3. *There exist positive constants δ_1 and M_1 such that one has*

$$|F_0(z, \lambda)| \leq M_1 e^{-\delta_1 \lambda} \quad \textit{for} \quad z \in D_1, \quad \lambda \in (1, \infty).$$

Next, we show the existence of exponentially decreasing solutions of the differential equation

$$P(z, \lambda, D_z)U_0(z, \lambda) = F_0(z, \lambda)$$

by using the classical Cauchy-Kowalewski theorem with a parameter. One sets $U = U_0 + U_1$. At that time we find immediately that U is a solution of infra-exponential type of (2.2).

Set:

$$\Omega_r = \{z \in \mathbb{C}^n; |z_1| + |z_2| + \cdots + |z_n| < r\},$$
$$\Omega_{L,r} = \{z \in \mathbb{C}^n; L|z_1| + |z_2| + \cdots + |z_n| < r\}$$

for $r > 0$, $L \geq 1$. For a sufficiently small $r > 0$, the differential operator $P(z, \lambda, D_z)$ is defined on a neighborhood of $\bar{\Omega}_{2r}$ and $F_0(z, \lambda) \in \mathcal{O}(\Omega_r)$ for each fixed $\lambda \in (1, \infty)$.

By the condition (2.1), we may assume from the beginning that

$$a^0_{(m,0,\dots,0)}(z) \equiv 1.$$

Consider the Cauchy problem:

$$\begin{cases} P(z, \lambda, D_z)U_0 = F_0 \\ D^j_{z_1} U_0 = 0 \quad \text{when } z_1 = 0,\ j < m. \end{cases} \tag{2.5}$$

Proposition 2.4. *There exists $L \geq 1$ such that the Cauchy problem* (2.5) *has a unique solution $U_0 \in \mathcal{O}(\Omega_{L,r})$ for any λ. Moreover, there exist $r_1 > 0$, $M_0 > 0$, $\delta_2 > 0$ such that for any $z \in \Omega_{L,r_1}$, any λ, we have*

$$|U_0(z, \lambda)| \leq M_0 e^{-\delta_2 \lambda}. \tag{2.6}$$

Proof. We make use of the majorant series of the solution $U_0(z, \lambda)$ for each fixed λ. The estimation (2.6) follows from Lemma 2.3. □

Remark 2.5. In the situation of Proposition 2.4, we can choose the constant L independent of the parameter λ because the principal symbol of $P(z, \lambda, D_z)$ does not depend on λ. So the domain $\Omega_{L,r}$ in which the solution is holomorphic does not shrink as $\lambda \longrightarrow \infty$.

From Lemma 2.2 and Proposition 2.4, we find that the solution $U(z, \lambda) = U_0(z, \lambda) + U_1(z, \lambda)$ possesses infra-exponential growth order as $\lambda \longrightarrow \infty$ in a neighborhood of $0 \in \mathbb{C}^n$. This completes the proof of Theorem 2.1.

Remark 2.6. Our construction of the solution U has made it possible to get the results in Theorem 2.1. If we consider (2.2) and the Cauchy boundary conditions directly, we cannot find a solution of infra-exponential type. For this reason one has reduced F to F_0 which is exponentially decreasing, and considered the Cauchy problem (2.5).

3 Solvability in the sheaf of microfunctions with holomorphic parameters

As application, we give the theorem of solvability and a characterization of solutions of some class of differential equations in the sheaf of 2-analytic functions, that is, microfunctions with holomorphic parameters. In the introduction, we have seen the propagation of analytic singularities for each operator along the bicharacteristic leaves of the regular involutive submanifold. However, they are not sufficient to get results of solvability for these operators.

First, let M be an open neighborhood of 0 in $\mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$, X a complex neighborhood of M in $\mathbb{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$, and $Y = \{z \in X; z_n = 0\}$. Let V and Σ be the following regular involutive and Lagrangian submanifolds of $\dot{T}^*_M X$ respectively:

$$\begin{aligned} V &= \left\{(x, \sqrt{-1}\xi \cdot dx) \in \dot{T}^*_M X; \xi_1 = \cdots = \xi_{n-1} = 0\right\}, \\ \Sigma &= \left\{(x, \sqrt{-1}\xi \cdot dx) \in \dot{T}^*_M X; \xi_1 = \cdots = \xi_{n-1} = x_n = 0\right\}, \end{aligned}$$

where $\dot{T}^*_M X = T^*_M X \setminus M$. So, we put $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$, $z = (z', z_n)$ with $z' = (z_1, \ldots, z_{n-1})$, $\xi = (\xi', \xi_n)$ with $\xi' = (\xi_1, \ldots, \xi_{n-1})$, and $\alpha = (\alpha', \alpha_n)$ with $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$.

Let $p_\circ = (0, \sqrt{-1}\, dx_n) \in \Sigma$. In connection with the transversally elliptic operator, we consider the following differential operator of order m with analytic coefficients defined on M:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha| \leq m} a_\alpha(x) D_{x'}^{\alpha'} (x_n D_{x_n})^{\alpha_n}. \tag{3.1}$$

The operator P is not partially elliptic along V provided that $a_{(0,\ldots,0,m)}(0) \neq 0$. Hence one cannot apply Bony-Schapira's theory to this operator. See Bony-Schapira [1].

Now recall that $\mathcal{A}^2_V = \mathcal{C}_{\tilde{V}}|_V = H^1(\mu_N(\mathcal{O}_X))|_V$ and that

$$\begin{aligned} \mathcal{A}^2_{V,p_\circ} &\simeq H^1_Z(\mathcal{O}_X)_0 \\ &\simeq \varinjlim_{r>0} \mathcal{O}(D_r^{n-1} \times U_r)/\mathcal{O}(D_r^n). \end{aligned} \tag{3.2}$$

Here we have set the closed subset $Z \subset X$, the open polydisc $D_r^k \subset \mathbb{C}^k$ and the open subset $U_r \subset \mathbb{C}$ respectively by:

$$\begin{aligned} Z &= \{z \in X; \operatorname{Im} z_n \leq 0\}, \\ D_r^k &= \{z \in \mathbb{C}^k; |z_j| < r, j = 1, \ldots, k\}, \\ U_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im} z_n > 0\} \end{aligned}$$

for $k \leq n$ and $r > 0$. Then any germ $f(x) \in \mathcal{A}^2_{V,p_\circ}$ is obtained as boundary value of a holomorphic function:

$$f(x) = b_{D_r^{n-1} \times U_r}(F(z)), \tag{3.3}$$

where $F(z) \in \mathcal{O}(D_r^{n-1} \times U_r)$ for some $r > 0$.

Recall, moreover, the sheaf $\mathcal{C}_{Y|X}^{\mathbb{R}}$ of microfunctions on Y defined by: $\mathcal{C}_{Y|X}^{\mathbb{R}} = H^1(\mu_Y(\mathcal{O}_X))$. The stalk of $\mathcal{C}_{Y|X}^{\mathbb{R}}$ at $p_\circ \in \Sigma$ is also written:

$$\begin{aligned}\mathcal{C}_{Y|X,p_\circ}^{\mathbb{R}} &\simeq \varinjlim_{r>0} H^1_{Z_r}(\mathcal{O}_X)_0 \\ &\simeq \varinjlim_{r>0} \mathcal{O}(D_r^{n-1} \times V_r)/\mathcal{O}(D_r^n).\end{aligned}$$

Here we have set the closed subset $Z_r \subset X$ and the open subset $V_r \subset \mathbb{C}$ respectively by:

$$\begin{aligned}Z_r &= \{z \in X; \operatorname{Im} z_n \leq -r|\operatorname{Re} z_n|\}, \\ V_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im} z_n > -r|\operatorname{Re} z_n|\}.\end{aligned}$$

It is clear by the definitions that there exists the exact sequence on Σ concerning these sheaves:

$$0 \longrightarrow \mathcal{C}_{Y|X}^{\mathbb{R}}|_\Sigma \longrightarrow \mathcal{A}_V^2|_\Sigma.$$

Now one makes the hypothesis:

$$a_{(0,\ldots,0,m)}(0) \neq 0. \tag{3.4}$$

Then we can obtain the following result on a characterization of the kernel of the operator $P\colon \mathcal{A}_V^2 \longrightarrow \mathcal{A}_V^2$ at $p_\circ$.

Theorem 3.1. *Assume* (3.4) *for the differential operator* (3.1). *If* $Pu = 0$ *for* $u \in \mathcal{A}_{V,p_\circ}^2$, *then* $u \in \mathcal{C}_{Y|X,p_\circ}^{\mathbb{R}}$.

Proof. One introduces the new local coordinates

$$w = \log z_n, \quad -\frac{1}{2}\pi < \arg z_n < \frac{3}{2}\pi.$$

We continue analytically a defining function of a 2-analytic function by means of the Cauchy-Kowalewski theorem. □

Now one makes the hypothesis:

$$\begin{cases} a_{(m,0,\ldots,0)}(0) \neq 0, \\ a_{(0,\ldots,0,m)}(0) \neq 0. \end{cases} \tag{3.5}$$

One can obtain the following theorem on the solvability for the operator $P\colon \mathcal{A}_V^2 \longrightarrow \mathcal{A}_V^2$ at $p_\circ$.

Theorem 3.2. *Assume* (3.5) *for the differential operator* (3.1). *We assume, furthermore, a germ $f \in \mathcal{A}^2_{V,p_\circ}$ represented by* (3.3) *satisfies the following growth condition. There exist positive constants $p < 1$, C such that*

$$|F(z)| \leq C|\operatorname{Im} z_n|^{-p}, \quad z \in D_r^{n-1} \times U_r.$$

Then we can find a solution $u \in \mathcal{A}^2_{V,p_\circ}$ of $Pu = f$.

In order to show the existence of solutions in Theorem 3.2, we will make the following steps. First, we turn holomorphic functions into the form of integral representation which is easy to deal with by means of Fourier transformation. Secondly, regarding the variable of integration as a parameter, we solve the differential equation with the parameter. Then we get a real solution by superposing a solution with respect to the parameter. At this time, we have to find a infra-exponential solution with the parameter. For that purpose, we make use of the theorem in the preceding section.

Set $U_r^\infty = \mathbb{P}^1 \setminus \{z_n \in \mathbb{C}; |z_n| \leq r, \operatorname{Im} z_n \leq 0\}$. Note that the open set $(D_r^{n-1} \times U_r^\infty) \cup D_r^n \subset \mathbb{C}^{n-1} \times \mathbb{P}^1$ is a Stein manifold. Therefore one can find functions $F_\infty \in \mathcal{O}(D_r^{n-1} \times U_r^\infty)$ and $F_0 \in \mathcal{O}(D_r^n)$ such that $F = F_\infty - F_0$ in $D_r^{n-1} \times U_r$ and $F_\infty(z', \infty) \equiv 0$ by the solvability of the first Cousin problem. Then one obtains:

$$f(x) = b_{D_r^{n-1} \times U_r}(F(z)) = b_{D_r^{n-1} \times U_r}(F_\infty(z))$$

by the isomorphisms in (3.2). In this way it is enough to consider $F_\infty \in \mathcal{O}(D_r^{n-1} \times U_r^\infty)$ which satisfies $F_\infty(z', \infty) \equiv 0$ instead of F.

Next, choose the system of local coordinates $(z', w) = (z_1, \ldots, z_{n-1}, w)$ with

$$w = \log z_n, \quad 0 < \arg z_n < \pi,$$

and set $w = u + \sqrt{-1}v$. We choose a C^∞-function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \leq \psi(v) \leq 1$ for $v \in \mathbb{R}$, $\psi(v) = 0$ for $v \leq \delta_\circ$ and $\psi(v) = 1$ for $v \geq \pi - \delta_\circ$, where $\delta_\circ > 0$ is a small constant. Using this function, we define:

$$G(z', w) = \frac{\partial}{\partial \bar{w}}(F_\infty(z', e^w)\psi(v)) = \frac{i}{2} F_\infty(z', e^w)\psi'(v)$$

for $0 < v < \pi$. We can consider $G(z', w)$ as a C^∞-function on $D_r^{n-1} \times \mathbb{C}$ by setting $G(z', w) \equiv 0$ for $\operatorname{Im} w \in \mathbb{R} \setminus (0, \pi)$.

Lemma 3.3. *Let q be a positive constant with $0 < p < q < 1$, and set $\varphi(u) = -2qu$. Then $G(z', w) \in L^2(D_r^{n-1} \times \mathbb{C}, \varphi)$ by shrinking D_r^{n-1}, that is, one has*

$$\int_{D_r^{n-1} \times \mathbb{C}} |G(z', w)|^2 e^{-\varphi(u)}\, dV(z', w) < \infty,$$

where the symbol $dV(z', w)$ is the standard Euclidean volume form on $\mathbb{C}^n$.

By existence theorems for the $\bar{\partial}$ operator due to Hörmander [4] and Lemma 3.3, there is a solution $H(z', w) \in L^2(D_r^{n-1} \times \mathbb{C}, \mathrm{loc})$ of the equation $\bar{\partial} H = G\, d\bar{w}$ such that

$$\int_{D_r^{n-1}\times\mathbb{C}} |H|^2 e^{-\varphi} \left(1 + |(z', w)|^2\right)^{-2} dV \leq \int_{D_r^{n-1}\times\mathbb{C}} |G|^2 e^{-\varphi}\, dV.$$

In fact, $H \in L^2(D_r^{n-1} \times \mathbb{C}, \Phi)$, where $\Phi(z', w) := \varphi(u) + 2\log(1 + |(z', w)|^2)$.

Set:

$$\begin{aligned} V &= \{w \in \mathbb{C}; 0 < \operatorname{Im} w < \pi\}, \\ V_+ &= \{w \in \mathbb{C}; \operatorname{Im} w > 0\}, \\ V_- &= \{w \in \mathbb{C}; \operatorname{Im} w < \pi\}, \end{aligned}$$

and

$$\begin{aligned} F_+(z', w) &= F_\infty(z', e^w)(1 - \psi(v)) + H(z', w), \\ F_-(z', w) &= F_\infty(z', e^w)\psi(v) - H(z', w). \end{aligned}$$

We find immediately that $F_\pm \in \mathcal{O}(D_r^{n-1} \times V_\pm)$ and that

$$F_\infty(z', e^w) = F_+(z', w) + F_-(z', w) \quad \text{for} \quad (z', w) \in D_r^{n-1} \times V.$$

We study values of the holomorphic functions $F_\pm$ as $|w| \to \infty$. Let

$$\begin{aligned} V_{+\delta} &= \{w \in \mathbb{C}; \operatorname{Im} w > \delta\}, \\ V_{-\delta} &= \{w \in \mathbb{C}; \operatorname{Im} w < \pi - \delta\} \end{aligned}$$

for $\delta > 0$, and set $r_1 = r/2 > 0$.

Proposition 3.4. *For any small positive δ there exists a positive constant C_δ such that one has*

$$|F_\pm(z', w)| \leq C_\delta \left|w^2 e^{-qw}\right| \tag{3.6}$$

for $(z', w) \in D_{r_1}^{n-1} \times V_{\pm\delta}$ with $|w| \gg 1$.

Proof. First, one has

$$\begin{aligned} F_+|_{D_r^{n-1}\times V_{+\delta/2}} &\in L^2(D_r^{n-1} \times V_{+\delta/2}, \Phi), \\ F_-|_{D_r^{n-1}\times V_{-\delta/2}} &\in L^2(D_r^{n-1} \times V_{-\delta/2}, \Phi). \end{aligned}$$

Choose any point $(z'_\circ, w_\circ) \in D_{r_1}^{n-1} \times V_{\pm\delta}$. Since $F_\pm$ is holomorphic on $D_r^{n-1} \times V_\pm$, we have

$$F_\pm(z'_\circ, w_\circ) = \frac{1}{\operatorname{vol}(B((z'_\circ, w_\circ), \delta/2))} \int_{B((z'_\circ, w_\circ),\delta/2)} F_\pm(z', w)\, dV(z', w),$$

where $B((z'_\circ, w_\circ), \delta/2) = \{(z', w); |(z', w) - (z'_\circ, w_\circ)| < \delta/2\} \subset\subset D_r^{n-1} \times V_{\pm\delta/2}$. Then we have the inequalities:

$$\begin{aligned}
&|F_\pm(z'_\circ, w_\circ)| \\
&\quad \leq \frac{1}{\operatorname{vol}(B((z'_\circ, w_\circ), \delta/2))} \left(\int_{B((z'_\circ, w_\circ),\delta/2)} |F_\pm|^2 e^{-\Phi}\, dV \right)^{1/2} \\
&\qquad\qquad\qquad\qquad \times \left(\int_{B((z'_\circ, w_\circ),\delta/2)} e^{\Phi}\, dV \right)^{1/2} \\
&\quad \leq \frac{1}{\operatorname{vol}(B((z'_\circ, w_\circ), \delta/2))^{1/2}} \left\| F_\pm|_{D_r^{n-1} \times V_{\pm\delta/2}} \right\|_\Phi \sup_{B((z'_\circ, w_\circ),\delta/2)} e^{\Phi/2}.
\end{aligned}$$

From these inequalities and the fact that $e^{\Phi/2} = e^{-qu}(1 + |(z', w)|^2)$, we can get the required inequality (3.6). $\square$

Now, we define the following holomorphic functions on $D_r^{n-1} \times V_+$:

$$\begin{aligned}
\tilde{F}_+(z', w) &= e^{qw} F_+(z', w), \\
\tilde{F}_-(z', w) &= e^{-qw} F_-(z', \pi i - w).
\end{aligned}$$

Corollary 3.5. *For any small positive δ there exists a positive constant C_δ such that one has*

$$\left| \tilde{F}_\pm(z', w) \right| \leq C_\delta \left| w^2 \right| \tag{3.7}$$

for $(z', w) \in D_{r_1}^{n-1} \times V_{+\delta}$ with $|w| \gg 1$.

This corollary shows that the holomorphic function $\tilde{F}_\pm$ is slowly increasing with respect to w. Therefore, the boundary value

$$b_{D_r^{n-1} \times V_+} \left(\tilde{F}_\pm(z', w) \right)$$

represents a slowly increasing Fourier hyperfunction with respect to the variable w. Refer to Kaneko [5] for the notion of Fourier hyperfunctions. We introduce the Fourier transformation of $b_{D_r^{n-1} \times V_+}(\tilde{F}_\pm)$ with respect to w in the following way. First we decompose $\tilde{F}_\pm$ by using $\chi_1(w) = e^w/(1+e^w)$, $\chi_2(w) = 1/(1+e^w)$ into the form of:

$$\tilde{F}_\pm(z', w) = \chi_1(w)\tilde{F}_\pm(z', w) + \chi_2(w)\tilde{F}_\pm(z', w).$$

Then we define

$$G_{\pm j}(z', \zeta) = \int_{\operatorname{Im} w = v} e^{-iw\zeta}\chi_j(w)\tilde{F}_\pm(z', w)\, du \tag{3.8}$$

for an arbitrary fixed v with $0 < v < \pi$ and $j = 1, 2$. Set $\zeta = \xi + \sqrt{-1}\eta$ and define the open subsets:

$$\begin{aligned} W &= \{\zeta \in \mathbb{C}; -1 < \operatorname{Im}\zeta < 1\} \setminus [0, +\infty), \\ W_+ &= \{\zeta \in \mathbb{C}; 0 < \operatorname{Im}\zeta < 1\}, \\ W_- &= \{\zeta \in \mathbb{C}; -1 < \operatorname{Im}\zeta < 0\}. \end{aligned}$$

The integral transform (3.8) is independent of the choice of $\operatorname{Im} w = v$ in the path of integration as long as $0 < v < \pi$. We have, moreover, $G_{\pm 1} \in \mathcal{O}(D_{r_1}^{n-1} \times W_-)$, $G_{\pm 2} \in \mathcal{O}(D_{r_1}^{n-1} \times W_+)$.

Then we can introduce the Fourier transformation:

$$\begin{aligned} \mathcal{F}b_{D_r^{n-1}\times V_+}\left(\tilde{F}_+\right) &= b_{D_{r_1}^{n-1}\times W_-}(G_{+1}) + b_{D_{r_1}^{n-1}\times W_+}(G_{+2}), \\ \mathcal{F}b_{D_r^{n-1}\times V_+}\left(\tilde{F}_-\right) &= b_{D_{r_1}^{n-1}\times W_-}(G_{-1}) + b_{D_{r_1}^{n-1}\times W_+}(G_{-2}). \end{aligned}$$

Using the holomorphic functions $G_{\pm j}$, one defines

$$G_\pm(z', \zeta) = \begin{cases} G_{\pm 2}(z', \zeta) & \text{on } D_{r_1}^{n-1} \times W_+ \\ -G_{\pm 1}(z', \zeta) & \text{on } D_{r_1}^{n-1} \times W_-. \end{cases}$$

Note that the function $\tilde{F}_\pm$ is holomorphic not only on $D_r^{n-1} \times V$ but also on $D_r^{n-1} \times V_+$, and that $\tilde{F}_\pm$ satisfies the growth condition (3.7) in Corollary 3.5. In this special situation we can claim the following proposition on the holomorphic function $G_\pm$.

Proposition 3.6. *The holomorphic function $G_\pm(z', \zeta)$ can be extended to a function in $\mathcal{O}(D_{r_1}^{n-1} \times W)$.*

Proof. We make use of the following function:

$$H_\pm(z', \zeta) = \begin{cases} H_{\pm 2}(z', \zeta) & \text{on } D_{r_1}^{n-1} \times W_+ \\ -H_{\pm 1}(z', \zeta) & \text{on } D_{r_1}^{n-1} \times W_-, \end{cases}$$

where one sets

$$H_{\pm j}(z',\zeta) = \int_{\operatorname{Im} w = v} e^{-iw\zeta}\chi_j(w)\tilde{F}_\pm(z',w)(w+i)^{-4}\,du$$

for an arbitrary fixed v with $0 < v < \pi$. We find immediately that $H_{\pm 1} \in \mathcal{O}(D_{r_1}^{n-1} \times W_-)$, $H_{\pm 2} \in \mathcal{O}(D_{r_1}^{n-1} \times W_+)$ and that $H_{\pm j}$ are independent of the choice of v as long as $0 < v < \pi$. Note that there is the relation between $G_\pm$ and $H_\pm$ on $D_{r_1}^{n-1} \times (W_+ \cup W_-)$:

$$(D_\zeta + 1)^4 H_\pm(z',\zeta) = G_\pm(z',\zeta). \tag{3.9}$$

The holomorphic functions $H_{\pm j}$ have finite boundary values at any point of $D_{r_1}^{n-1} \times \mathbb{R}$:

$$H_{\pm 1}(z',\xi) := \lim_{W_- \ni \zeta \to \xi} H_{\pm 1}(z',\zeta),$$
$$H_{\pm 2}(z',\xi) := \lim_{W_+ \ni \zeta \to \xi} H_{\pm 2}(z',\zeta).$$

For each fixed $\xi \in \mathbb{R}$, $H_{\pm j}(z',\xi)$ is holomorphic on $D_{r_1}^{n-1}$. Furthermore, the functions $-H_{\pm 1}$ and $H_{\pm 2}$ have the same boundary values at $(z',\xi) \in D_{r_1}^{n-1} \times \mathbb{R}$ with $\xi < 0$. We have indeed on $D_{r_1}^{n-1} \times \mathbb{R}$

$$H_{\pm 1}(z',\xi) + H_{\pm 2}(z',\xi) = e^{v\xi} \int_{\operatorname{Im} w = v} e^{-iu\xi}\tilde{F}_\pm(z',w)(w+i)^{-4}\,du. \tag{3.10}$$

Since the function $\tilde{F}_\pm(z',w)(w+i)^{-4}$ in (3.10) is holomorphic not only on $D_r^{n-1} \times V$ but also on $D_r^{n-1} \times V_+$, we can choose an arbitrary v in the path of integration as long as $v > 0$. By using the estimation in Corollary 3.5, there exists a positive constant C such that we have the inequalities for $v \gg 1$:

$$\begin{aligned}
&\left| e^{v\xi} \int_{\operatorname{Im} w = v} e^{-iu\xi}\tilde{F}_\pm(z',w)(w+i)^{-4}\,du \right| \\
&\quad \le e^{v\xi} \int_{-\infty}^{\infty} \left| \tilde{F}_\pm(z',u+iv)(u+iv+i)^{-4} \right| du \\
&\quad \le e^{v\xi} \int_{-\infty}^{\infty} C|u+iv|^2 |u+iv+i|^{-4}\,du \\
&\quad \le Ce^{v\xi} \int_{-\infty}^{\infty} \frac{1}{1+u^2}\,du.
\end{aligned}$$

Therefore one obtains

$$H_{\pm 1}(z',\xi) + H_{\pm 2}(z',\xi) = 0 \quad \text{for} \quad \xi < 0,$$

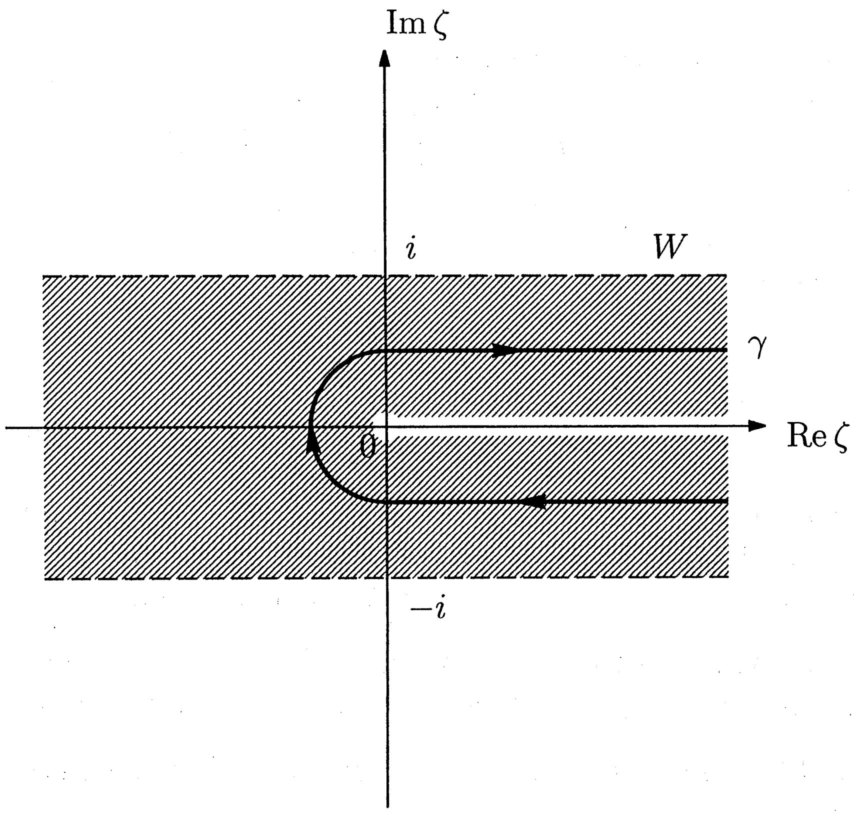

Figure 1: The path γ

since v is arbitrary as long as $v > 0$ in the preceding inequalities. Thus the function $H_{\pm}$ is holomorphic on $D^{n-1}_{r_1} \times (W_+ \cup W_-)$ and extended to a continuous function on $D^{n-1}_{r_1} \times W$. Then we find that $H_{\pm}$ is holomorphic on $D^{n-1}_{r_1} \times W$. We find, furthermore, that $G_{\pm}$ can be extended to a holomorphic function on the domain $D^{n-1}_{r_1} \times W$ through the relation (3.9). $\square$

Note values of the holomorphic function $G_{\pm}$ as $\operatorname{Re}\zeta \to \pm\infty$. We find easily that $G_{\pm}$ possesses infra-exponential growth order as $\operatorname{Re}\zeta \to \infty$ and decreases exponentially as $\operatorname{Re}\zeta \to -\infty$. We have constructed the holomorphic function $G_{\pm}$ from $\tilde{F}_{\pm}$ by using the Fourier transformation. Now we shall restore $G_{\pm}$ to the original holomorphic function $\tilde{F}_{\pm}$ by using Fourier's inversion formula.

Proposition 3.7. *For $(z', w) \in D^{n-1}_{r_1} \times V$, one has*

$$\tilde{F}_{\pm}(z', w) = \frac{1}{2\pi} \int_{\gamma} e^{iw\zeta} G_{\pm}(z', \zeta)\, d\zeta,$$

where γ is the infinite path in Figure 1.

Proof. By the inverse transformation, we have

$$\chi_j(w)\tilde{F}_\pm(z',w) = \frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_j} e^{iw\zeta} G_{\pm j}(z',\zeta)\,d\xi \tag{3.11}$$

for any point $(z',w) \in D_{r_1}^{n-1} \times V$.

Therefore by (3.11):

$$\begin{aligned}\tilde{F}_\pm(z',w) &= \chi_1(w)\tilde{F}_\pm(z',w) + \chi_2(w)\tilde{F}_\pm(z',w)\\ &= -\frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_1} e^{iw\zeta} G_\pm(z',\zeta)\,d\xi + \frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_2} e^{iw\zeta} G_\pm(z',\zeta)\,d\xi.\end{aligned}$$

We can deform the path of integration into γ, since the integration as $\operatorname{Re}\zeta \to -\infty$ can be neglected by the exponential decay of the integrand. Then we can get the required integral representation of the holomorphic function $\tilde{F}_\pm$. □

By Proposition 3.7 and the definition of $\tilde{F}_\pm$, we reach a conclusion of the following representation through the variable z_n in the first situation.

Corollary 3.8. *One has* $F_\infty(z) = F_+(z', \log z_n) + F_-(z', \log z_n)$ *with*

$$\begin{cases} F_+(z',\log z_n) = \dfrac{1}{2\pi}\displaystyle\int_\gamma (z_n)^{i\zeta-q} G_+(z',\zeta)\,d\zeta, \\ F_-(z',\log z_n) = \dfrac{1}{2\pi}\displaystyle\int_\gamma e^{(-\zeta+iq)\pi}(z_n)^{-i\zeta-q} G_-(z',\zeta)\,d\zeta \end{cases} \tag{3.12}$$

for $z' \in D_{r_1}^{n-1}$, $0 < \arg z_n < \pi$.

Now, regarding the variable of integration as a parameter, we construct a solution with infra-exponential growth order. By Corollary 3.8, our differential equation is turned into:

$$\begin{cases} Pu_+(x) = b_{D_{r_1}^{n-1}\times U_{r_1}}(F_+(z',\log z_n)), \\ Pu_-(x) = b_{D_{r_1}^{n-1}\times U_{r_1}}(F_-(z',\log z_n)) \end{cases}$$

at $p_\circ = (0, \sqrt{-1}\,dx_n) \in \varSigma$ with the integral representation (3.12). It suffices to consider the differential equation on the complex domain $D_{r_1}^{n-1} \times U_{r_1}$

$$P(z, D_{z'}, z_n D_{z_n}) U_\pm(z) = F_\pm(z', \log z_n).$$

In order to study the existence of $U_\pm(z)$, we consider the differential equations with a parameter ζ on $D_{r_1}^{n-1} \times U_{r_1}$:

$$\begin{aligned} P(z, D_{z'}, z_n D_{z_n}) U_+(z,\zeta) &= (z_n)^{i\zeta-q} G_+(z',\zeta),\\ P(z, D_{z'}, z_n D_{z_n}) U_-(z,\zeta) &= e^{(-\zeta+iq)\pi}(z_n)^{-i\zeta-q} G_-(z',\zeta). \end{aligned}$$

Here the parameter ζ ranges through the path γ.

Note that these equations are equivalent to:

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)\tilde{U}_\pm(z,\zeta) = G_\pm(z',\zeta),$$

where we set

$$\tilde{U}_+(z,\zeta) = (z_n)^{-i\zeta+q}U_+(z,\zeta),$$
$$\tilde{U}_-(z,\zeta) = e^{(\zeta-iq)\pi}(z_n)^{i\zeta+q}U_-(z,\zeta).$$

Note, moreover, that the differential operator $P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)$ is written as:

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) = \sum_{j=0}^{m} (\pm\zeta)^j P_j(z, D_z),$$

where the P_j's are holomorphic differential operators of order $m - j$ defined on D_r^n

$$P_j(z, D_z) = \sum_{|\alpha|\leq m-j} a_\alpha^j(z) D_z^\alpha.$$

By Theorem 2.1, we can find a solution $\tilde{U}_\pm(z,\zeta)$ of infra-exponential type with respect to ζ on the path γ. Therefore one has a solution

$$U_\pm(z) = \frac{1}{2\pi}\int_\gamma U_\pm(z,\zeta)\, d\zeta.$$

This completes the proof of Theorem 3.2.

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