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SMALL STABLE STATIONARY SOLUTIONS IN MORREY SPACES OF THE SEMILINEAR HEAT EQUATION

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Abstract. In this paper, we consider in Morrey spaces the Cauchy problem of the semilinear heat equation with an external force. Under appropriate assumptions on the external force, we proved the unique existence of small solution to the corresponding stationary problem. Moreover, if the initial data is close enough to the stationary solution, we verified the time-global solvability of the Cauchy problem, which leads to the stability of the small stationary solution.

In this talk, we consider the Cauchy problem of the following semilinear heat equation with an external force $f(x)$ in $\mathbf{R}^n$ for $n \geq 3$:

$$\frac{\partial v}{\partial t}(t,x) = \Delta v(t,x) + v(t,x)^{\nu} + f(x) \quad \text{in } (0,\infty)\times\mathbf{R}^n, \tag{0.1}$$

$$v(0,x) = a(x) \quad \text{on } \mathbf{R}^n, \tag{0.2}$$

where $\nu \geq 3$, $\nu \in \mathbf{Z}$.

Furthermore, the corresponding stationary problem of the above equation is as follows:

$$-\Delta w(x) = w(x)^{\nu} + f(x) \quad \text{on } \mathbf{R}^n. \tag{0.3}$$

For the equation (0.1) without external forces, *i.e.* $f(x)\equiv 0$, Fujita [3] first showed that the Cauchy problem admits a time-global strong solution with $\nu > 1+\frac{2}{n}$, provided that $\|a(x)\|_{C^2(\mathbf{R}^n)}$ is sufficiently small. At the same time he also showed that the condition $\nu > 1 + \frac{2}{n}$ is necessary for the existence of a time-global solution for nonnegative nontrivial initial data (also see Haraux and Weissler [4], Hayakawa [5], Kobayashi, Sirao and Tanaka [6]). Furthermore, Weissler [11] proved the global existence of the solution when $\|a(x)\|_{L^p(\mathbf{R}^n)}$ with $p = n(\nu-1)/2 > 1$ is sufficiently small.

On the other hand, there have been some researches on the Cauchy problem with measures as initial data. Brezis and Friedman [2] proved that a time-local solution exists with initial data $\delta(x)$ if and only if $\nu < 1 + \frac{2}{n}$. Baras and Pierre [1] studied various capacities of the Radon measures as initial data. Niwa [9] obtained a sufficient condition for the local well-posedness and the global well-posedness of the Cauchy problem with initial data in the measure spaces of the Morrey type. Kozono and Yamazaki [7] obtained time-local and time-global solutions as initial data in the Besov-type Morrey spaces. Under appropriate assumptions on the external forces, they [8] also proved the stability of small staionary solutions in Morrey spaces for the Navier-Stokes equations. Recently Wu [12] concluded the well-posedness of the Cauchy problem with initial data in the homogeneous Lebesgue spaces.

Inspired by the researches mentioned above, we are interested in studying the Cauchy problem (0.1)-(0.2) in the Morrey spaces. First, under appropriate assumptions on $f(x)$ in Morrey spaces, we show the unique existence of a small solution of the corresponding stationary problem (0.3) by applying the Banach inverse mapping theorem. Next, we verify the time-global unique solvability of the Cauchy problem (0.1)-(0.2) by the semigroup methods when the initial data $a(x)$ is close enough to the stationary solution. Then we get the stability of the small stationary solution in the same Morrey spaces.

Our results are the following theorems.

Theorem 0.1. *Suppose that $\nu \geq 3$, $\nu \in \mathbf{Z}$, $\nu < q_0 \leq p_0 = \frac{n(\nu-1)}{2}$. Then we can find a positive number δ_0 and a continuous, strictly monotone-increasing function $\omega(\delta)$ on $[0, \delta_0]$ with $\omega(0) = 0$ such that:*

(1) *For every $f(x) \in \mathcal{D}'$, there exists at most one solution $w(x)$ of (0.3) in $\{w(x) \in \mathcal{M}_{p_0,q_0} | \|w(x)\|_{\mathcal{M}_{p_0,q_0}} < \omega(\delta_0)\}$.*

(2) *For every $f(x) \in \mathcal{M}^{-2}_{p_0,q_0}$ with $\|f(x)\|_{\mathcal{M}^{-2}_{p_0,q_0}} = \delta < \delta_0$, there exists a solution $w(x) \in \mathcal{M}_{p_0,q_0}$ of (0.3) with $\|w(x)\|_{\mathcal{M}_{p_0,q_0}} \leq \omega(\delta)$.*

Example 0.1. It is shown in [7] that for every p and q such that $1 \leq q < p < \infty$, the Lorentz space $L^{p,\infty}(\mathbf{R}^n)$ is contained in $\mathcal{M}_{p,q}$. In our case, if $q_0 < p_0$, put

$f(x) = c_0|x|^{-\frac{2\nu}{\nu-1}}$, then $f(x) \in L^{\frac{p_0}{\nu},\infty}(\mathbf{R}^n) \subset \mathcal{M}_{\frac{p_0}{\nu},\frac{q_0}{\nu}} \subset \mathcal{M}^{-2}_{p_0,q_0}$. It follows that we can take this function as the external force $f(x)$ in Theorem 0.1 provided that the constant $|c_0|$ is sufficiently small.

Theorem 0.2. *Let ν, p_0, q_0 be the same as in Theorem 0.1, $p_0 < p < \frac{\nu}{2}p_0$, $q_0 < q \leq \frac{pq_0}{p_0}$ and $\frac{n}{2p} < \sigma_0 < \frac{n}{p}$. Then we can find positive numbers $\delta_1 (\leq \delta_0)$, ε_0 and M_0 satisfying the following:*

For every $f(x) \in \mathcal{M}^{-2}_{p_0,q_0}$ with $\|f(x)\|_{\mathcal{M}^{-2}_{p_0,q_0}} < \delta_1$, take the solution $w(x)$ of (0.3) in Theorem 0.1, and take $a(x) \in \mathcal{M}^{\frac{n}{p}-\frac{2}{\nu-1}}_{p,q}$ with $\|a(x) - w(x)\|_{\mathcal{M}^{\frac{n}{p}-\frac{2}{\nu-1}}_{p,q}} = \varepsilon < \varepsilon_0$, there exists a time-global solution $v(t,x)$ of (0.1)-(0.2) such that:

$$\sup_{0<t\leq T'} t^{\frac{1}{\nu-1}-\frac{n}{4p}}\|v(t,\cdot) - w\|_{\mathcal{M}^{\frac{n}{2p}}_{p,q}} < \infty, \text{ for every } 0 < T' \leq \infty, \tag{0.4}$$

$$\limsup_{t\to 0+} t^{\frac{1}{\nu-1}-\frac{n}{4p}}\|v(t,\cdot) - w\|_{\mathcal{M}^{\frac{n}{2p}}_{p,q}} < M_0. \tag{0.5}$$

Moreover, the initial condition (0.2) holds in the following sense: For every s satisfying $-\frac{n(\nu-2)}{2p} \leq s \leq \frac{n}{p} - \frac{2}{\nu-1}$ and every $T' > 0$, we have

$$\sup_{0<t\leq T'} t^{\frac{s}{2}+\frac{1}{\nu-1}-\frac{n}{2p}}\|v(t,\cdot) - a\|_{\mathcal{M}^{s}_{p,q}} < \infty. \tag{0.6}$$

Remark 0.1. In general, $v(t,\cdot) - a \longrightarrow 0$ in $\mathcal{M}_{p_0,q_0}$ as $t \to 0+$ does not hold. But it is true for the special case $a(x) \in \mathcal{M}_{p_0,q_0}$ since we can obtain a unique time-global solution such that $v(t,\cdot) - w \in \mathcal{M}^{\frac{n}{2p_0}}_{p_0,q_0}$. We omit the details because the modifications are straightforward.

Theorem 0.3. *Under the same assumptions and notations as in Theorem 0.2, for every $0 < T \leq \infty$, any solution of (0.1)-(0.2) on $(0,T)\times\mathbf{R}^n$ satisfying (0.4) for every $T' \in (0,T)$, (0.5) and $v(t,\cdot) - a \longrightarrow 0$ in $\mathcal{M}^{-\frac{n(\nu-2)}{2p}}_{p,q}$ coincides with the restriction on $(0,T) \times \mathbf{R}^n$ of the time-global solution in Theorem 0.2.*

Remark 0.2. This result shows the uniqueness of the time-global solution in Theorem 0.2. Moreover, for ν such that $1+2/n < \nu < (n+2)/(n-2)$, Haraux and Weissler [9] constructed a nontrivial solution $v(t,x)$ of (0.1)-(0.2) with $f(x) \equiv a(x) \equiv 0$ of the self-similar form $v(t,x) = t^{-1/(\nu-1)}\varphi(x/\sqrt{t})$. Then we have $\|v(t,\cdot)\|_{\mathcal{M}^{s}_{p,q}} = Ct^{-1/(\nu-1)+n/2p-s/2}$ for every p, q and s. It follows that, even in the case $a(x) \equiv f(x) \equiv 0$, the condition (0.5) is necessary for the uniqueness in Theorem 0.3.

Theorem 0.4. *Under the same assumptions and notations as in Theorem 0.2, for every σ satisfying $\frac{n}{p} - \frac{2}{\nu-1} \le \sigma \le \sigma_0$, there exists a continuous, strictly monotone-increasing function $\psi_\sigma(\varepsilon)$ on $[0, \varepsilon_0]$ with $\psi_\sigma(0) = 0$ such that:*

$$\sup_{t>0} t^{\frac{\sigma}{2}+\frac{1}{\nu-1}-\frac{n}{2p}} \|v(t,\cdot) - w\|_{\mathcal{M}^\sigma_{p,q}} \le \psi_\sigma(\varepsilon), \quad \textit{for every} \quad \varepsilon < \varepsilon_0. \tag{0.7}$$

Example 0.2. Let $w(x)$ be a small stationary solution with $f(x) = c_0|x|^{-\frac{2\nu}{\nu-1}}$ as shown in the previous Example 0.1. If $q < p$, we can take $a(x) = w(x) + c_1|x|^{-\frac{2}{\nu-1}}$ in Theorem 0.2 provided that the constant $|c_1|$ is sufficiently small, since $c_1|x|^{-\frac{2}{\nu-1}} \in \mathcal{M}_{p_0,q_0} \subset \mathcal{M}^{\frac{n}{p}-\frac{2}{\nu-1}}_{p,q}$.

Remark 0.3. The estimate (0.7) with $\sigma = \frac{n}{p} - \frac{2}{\nu-1}$ in Theorem 0.4 together with the fact that $\lim_{\varepsilon\to 0+} \psi_\sigma(\varepsilon) = 0$ asserts the Lyapunov stability of the stationary solution in the topology of $\mathcal{M}^{\frac{n}{p}-\frac{2}{\nu-1}}_{p,q}$. Other estimates in (0.7) give the asymptotic stability in different topologies of $\mathcal{M}^\sigma_{p,q}$.

Remark 0.4. The main result of Kozono and Yamazaki [7] can be regarded as the stability of the stationary solution 0 of the equations (0.1)-(0.2) with the external force $f(x) \equiv 0$. Hence our results can be regarded as a generalization of theirs to the case with more general stationary solutions. Furthermore, Wu [12] consider the Cauchy problem with initial data in the homogeneous Lebesgue space $\dot{L}_{s,p}(\mathbf{R}^n)$ which is slightly smaller than $\mathcal{M}^s_{p,q}$ (by noting that $\mathcal{M}_{p,q} \supset \mathcal{M}_{p,p} \equiv L^p$). Hence we have extended his results to the general case when $\nu \ge 3$, $\nu \in \mathbf{Z}$.

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