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Characterization of entire functions of exponential type with respect to the Lie norm

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Introduction

We consider the space of entire functions on $\tilde{\mathbf{E}}=\mathbf{C}^{n+1}$ and denote it by $\mathcal{O}(\tilde{\mathbf{E}})$. Let $F(z)=\sum_{k=0}^{\infty}F_k(z)\in\mathcal{O}(\tilde{\mathbf{E}})$ be the homogeneous expansion of F into homogeneous polynomials F_k of degree k. For a norm $N(z)$ on $\tilde{\mathbf{E}}$ put

$$\mathrm{Exp}\,(\tilde{\mathbf{E}};(r,N))=\{F\in\mathcal{O}(\tilde{\mathbf{E}});\forall r'>r,\exists C\geq 0 \text{ s.t. } |F(z)|\leq C\exp(r'N(z))\}$$

and $\|F\|_{C(\tilde{B}_N[1])}=\sup\{|F(z)|;N(z)\leq 1\}$. Then we know that

$$F\in\mathrm{Exp}\,(\tilde{\mathbf{E}};(r,N))\Longleftrightarrow\limsup_{k\to\infty}(k!\|F_k\|_{C(\tilde{B}_N[1])})^{1/k}\leq r.$$

An entire function can also be expanded into the double series with $(k-2l)$-homogeneous harmonic polynomials $F_{k,k-2l}, k=0,1,\cdots,l=0,1,\cdots,[k/2]$;

$$F(z)=\sum_{k=0}^{\infty}F_k(z)=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^lF_{k,k-2l}(z),$$

where the convergence is uniform on compact sets in $\tilde{\mathbf{E}}$.

In this note, we consider the case that the norm $N(z)$ is the Lie norm $L(z)$ or the dual Lie norm $L^*(z)$. First, we formulate, in terms of the growth behavior of $F_{k,k-2l}$, the necessary and sufficient conditions for an entire function F to belong to $\mathrm{Exp}\,(\tilde{\mathbf{E}};(r,N))$. Here we will present the following results according to [1]:

For $F(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l F_{k,k-2l}(z)$, we have

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L)) \iff \limsup_{2k-2l\to\infty} \left(\frac{k!}{r^k} \|F_{k,k-2l}\|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1,$$

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)) \iff \limsup_{2k-2l\to\infty} \left(\frac{2^k l!(k-l)!}{r^k} \|F_{k,k-2l}\|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1,$$

where S_1 is the unit real sphere. (See Theorems 1.4 and 2.1.)

Second, we will study the spaces of entire eigenfunctions of exponential type of the Laplacian; $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L))$ and $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*))$. For these spaces we will prove the following relation which generalizes a theorem in [5]:
Theorem

$$\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)) = \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}}; \left(\frac{r^2 + |\lambda|^2}{2r}, L\right)\right), \quad |\lambda| \leq r.$$

(See Theorem 3.3.) From this relation we have

$$\mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)) \subsetneq \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L)) \subsetneq \mathrm{Exp}\,(\tilde{\mathbf{E}}; (2r, L^*)).$$

1 Lie norm

Let $N(z)$ be a norm on $\tilde{\mathbf{E}} = \mathbf{C}^{n+1}$. Its dual norm $N^*(z)$ is defined by

$$N^*(z) = \sup\{|z \cdot \zeta|; N(\zeta) \leq 1\}.$$

The open and the closed N-balls of radius r with center at 0 are defined by

$$\tilde{B}_N(r) = \{z \in \tilde{\mathbf{E}}; N(z) < r\},\ r > 0, \quad \tilde{B}_N[r] = \{z \in \tilde{\mathbf{E}}; N(z) \leq r\},\ r \geq 0.$$

Note that $\tilde{B}_N(\infty) = \tilde{\mathbf{E}}$. We denote by $\mathcal{O}(\tilde{B}_N(r))$ the space of holomorphic functions on $\tilde{B}_N(r)$. Put $\mathcal{O}(\tilde{B}_N[r]) = \liminf_{r'>r} \mathcal{O}(\tilde{B}_N(r'))$,

$$\begin{aligned}
\mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N)) &= \{F \in \mathcal{O}(\tilde{\mathbf{E}}); \forall r' > r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \exp(r' N(z))\}, \\
\mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, N]) &= \{F \in \mathcal{O}(\tilde{\mathbf{E}}); \exists r' < r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \exp(r' N(z))\}.
\end{aligned}$$

Note that for any norm N on $\tilde{\mathbf{E}}$ we have $\mathrm{Exp}\,(\tilde{\mathbf{E}}; (0, N)) = \mathrm{Exp}\,(\tilde{\mathbf{E}}; (0))$.

We denote by $\mathcal{P}^k(\tilde{\mathbf{E}})$ the space of homogeneous polynomials of degree k. Define the k-homogeneous component $f_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$ of $f \in \mathcal{O}(\{0\})$ by

$$f_k(z) = \frac{1}{2\pi i} \int_{|t|=\rho} \frac{f(tz)}{t^{k+1}} dt, \tag{1}$$

where ρ is sufficiently small. Then we know the following theorem (see, for example, [2]):

THEOREM **1.1** *Let $N(z)$ be a norm on $\tilde{\mathbf{E}}$ and $F_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$. Then we have*

$$F = \sum_{k=0}^{\infty} F_k(z) \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N)) \iff \limsup_{k\to\infty} (k! \|F_k\|_{C(\tilde{B}_N[1])})^{1/k} \leq r,$$

$$F = \sum_{k=0}^{\infty} F_k(z) \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, N]) \iff \limsup_{k\to\infty} (k! \|F_k\|_{C(\tilde{B}_N[1])})^{1/k} < r,$$

where $\|F\|_{C(\tilde{B}_N[1])} = \sup\{|F(z)|; N(z) \leq 1\}$.

We define the Lie norm $L(z)$ of $z \in \tilde{\mathbf{E}}$ by

$$L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}.$$

Then $L(z)$ is the cross norm of the Euclidean norm $\|x\|$; that is,

$$L(z) = \inf \left\{ \sum_{j=1}^{m} |\lambda_j| \|x_j\|; z = \sum_{j=1}^{m} \lambda_j x_j, \lambda_j \in \mathbf{C}, x_j \in \mathbf{R}^{n+1}, m \in \mathbf{Z}_+ \right\}.$$

Thus putting $\|f_k\|_{C(S_1)} = \sup\{|f_k(x)|; x \in S_1\}$, for $f_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$ we can see

$$\|f_k\|_{C(\tilde{B}_L[1])} = \|f_k\|_{C(S_1)}.$$

Therefore as a corollary of Theorem 1.1, we have

COROLLARY **1.2** *Let $F(z) = \sum_{k=0}^{\infty} F_k(z)$, $F_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$. Then we have*

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L)) \iff \limsup_{k\to\infty} (k! \|F_k\|_{C(S_1)})^{1/k} \leq r,$$

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, L]) \iff \limsup_{k\to\infty} (k! \|F_k\|_{C(S_1)})^{1/k} < r.$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$. The harmonic extension $\tilde{P}_{k,n}(z,w)$ of $P_{k,n}(z\cdot w)$ is given by

$$\tilde{P}_{k,n}(z,w) = (\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right).$$

Then $\tilde{P}_{k,n}(z,w)$ is a k-homogeneous harmonic polynomial in z and in w and satisfies $|\tilde{P}_{k,n}(z,w)| \le L(z)^k L(w)^k$. We denote by $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ the space of homogeneous harmonic polynomials of degree k. The dimension of $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ is known to be $(2k+n-1)(k+n-2)!/(k!(n-1)!) \equiv N(k,n)$.

When $N(z) = L(z)$, we omit the subscript; for example, we write $\tilde{B}(r)$ for $\tilde{B}_L(r)$. For a holomorphic function on $\tilde{B}(r)$ we know the following theorem:

THEOREM **1.3 ([3, Theorem 3.1])**
Let $f \in \mathcal{O}(\tilde{B}(r))$. Define the k-homogeneous component of f by (1) *and define the (k,j)-component of f by*

$$f_{k,j}(z) = N(j,n)\int_{S_1} f_k(\tau)\tilde{P}_{j,n}(z,\tau)d\dot{\tau}, \tag{2}$$

where $d\dot{\tau}$ is the normalized invariant measure on the unit real sphere S_1. Then $f_{k,j}$ is a j-homogeneous harmonic polynomial and we can expand f into the double series:

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(\sqrt{z^2})^{k-j}f_{k,j}(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z), \tag{3}$$

where the convergence is uniform on compact sets in $\tilde{B}(r)$ and we have

$$\limsup_{2k-2l\to\infty}(r^k\|f_{k,k-2l}\|_{C(S_1)})^{1/(2k-2l)} \le 1. \tag{4}$$

Conversely, if we are given a double sequence $\{f_{k,k-2l}\}$ of homogeneous harmonic polynomials $f_{k,k-2l}(z)$ satisfying (4)*, then the right-hand side of* (3) *converges to a holomorphic function f uniformly on compact sets in $\tilde{B}(r)$ and the $(k,k-2l)$-component of f is equal to the given $f_{k,k-2l}$.*

For an entire function of exponential type, [1] proved the following theorem: We can prove it by the property of the Lie norm. Here, we omit its proof.

THEOREM **1.4 ([1, Theorem 3.7])** *Let* $F(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l F_{k,k-2l}(z)$, $F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}})$, *be the expansion of* $F \in \mathcal{O}(\tilde{\mathbf{E}})$. *Then we have*

$$F \in \operatorname{Exp}(\tilde{\mathbf{E}}; (r, L)) \iff \limsup_{2k-2l\to\infty} \left(\frac{k!}{r^k}\|F_{k,k-2l}\|_{C(S_1)}\right)^{1/(2k-2l)} \leq 1.$$

2 Dual Lie norm

The dual Lie norm $L^*(z)$ is given by

$$L^*(z) = \sqrt{(\|z\|^2 + |z^2|)/2}.$$

Since $|\sqrt{z^2}| \leq L^*(z) \leq \|z\| \leq L(z) \leq 2L^*(z)$, we have

$$\operatorname{Exp}(\tilde{\mathbf{E}}; (r, L^*)) \subset \operatorname{Exp}(\tilde{\mathbf{E}}; (r, L)) \subset \operatorname{Exp}(\tilde{\mathbf{E}}; (2r, L^*)). \tag{5}$$

Similar to Theorem 1.4, for the dual Lie norm $L^*(z)$, we have the following theorem:

THEOREM **2.1 ([1, Theorem 5.2])** *Let* $F(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{[k/2]}(z^2)^l F_{k,k-2l}(z)$, $F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}})$, *be the expansion of* $F \in \mathcal{O}(\tilde{\mathbf{E}})$. *Then we have*

$$F \in \operatorname{Exp}(\tilde{\mathbf{E}}; (r, L^*)) \Leftrightarrow \limsup_{2k-2l\to\infty} \left(\frac{2^k l!(k-l)!}{r^k}\|F_{k,k-2l}\|_{C(S_1)}\right)^{\frac{1}{2k-2l}} \leq 1. \tag{6}$$

For a proof, we use the Cauchy-Hua transformation and the Fourier transformation. First we introduce the invariant measure on the Lie sphere.

2.1 Lie sphere

The Shilov boundary of $\tilde{B}[r]$ is the Lie sphere Σ_r:

$$\Sigma_r = \{re^{i\theta}\omega; 0 \leq \theta < 2\pi,\ \omega \in S_1\} = \{e^{i\theta}\omega; 0 \leq \theta < 2\pi,\ \omega \in S_r\}.$$

Note that $-xe^{i(\theta+\pi)} = xe^{i\theta}$ and $\Sigma_r = (\mathbf{R}/(2\pi\mathbf{Z}) \times S_r)/\sim$, where $\sim$ is the equivalence relation defined by $(\theta, x) \sim (\theta+\pi, -x)$, and that for $f \in \mathcal{O}(\tilde{B}[r])$ we have $\sup\{|f(z)|; z \in \tilde{B}[r]\} = \sup\{|f(z)|; z \in \Sigma_r\}$.

We define the invariant integral over Σ_r by

$$\int_{\Sigma_r} f(z)\dot{d}z = \frac{1}{2\pi}\int_0^{2\pi}\int_{S_1} f(re^{i\theta}\omega)\dot{d}\omega d\theta.$$

For $f, g \in \mathcal{O}(\tilde{B}[r])$, the integral $\int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z$ is well-defined. Since

$$\begin{aligned}(f,g)_{\Sigma_r} \equiv \int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z &= \sum_{k=0}^{\infty} r^{2k}\int_{S_1} f_k(\omega)\overline{g_k(\omega)}\dot{d}\omega \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]} r^{2k}\int_{S_1} f_{k,k-2l}(\omega)\overline{g_{k,k-2l}(\omega)}\dot{d}\omega,\end{aligned} \tag{7}$$

$(\ ,\ )_{\Sigma_r}$ is an inner product on $\mathcal{O}(\tilde{B}[r])$. If $f \in \mathcal{O}(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}(r))$, then for $s > 1$ sufficiently close to 1 the integral $\int_{\Sigma_r} f(z/s)\overline{g(sz)}\dot{d}z$ is well-defined and does not depend on s by (7). Thus for $f \in \mathcal{O}(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}(r))$ or for $g \in \mathcal{O}(\tilde{B}[r])$ and $f \in \mathcal{O}(\tilde{B}(r))$ we write

$$\int_{\Sigma_r} f(z/s)\overline{g(sz)}\dot{d}z = s\int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z.$$

Let $H^2(\tilde{B}(r))$ be the completion of $\mathcal{O}(\tilde{B}[r])$ with respect to the inner product $(\ ,\ )_{\Sigma_r}$, and put $\|f\|_{S_r}^2 = \int_{S_r}|f(\omega)|^2\dot{d}\omega$. Then by the definition,

$$\begin{aligned}H^2(\tilde{B}(r)) &= \Big\{f(z) = \sum_{k=0}^{\infty} f_k(z); \\ &\qquad f_k \in \mathcal{P}^k(\tilde{\mathbf{E}}),\ \sum_{k=0}^{\infty}\|f_k\|_{\Sigma_r}^2 = \sum_{k=0}^{\infty} r^{2k}\|f_k\|_{S_1}^2 < \infty\Big\} \\ &= \Big\{f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z); \\ &\qquad f_{k,k-2l} \in \mathcal{P}_\Delta^{k-2l}(\tilde{\mathbf{E}}),\ \sum_{k=0}^{\infty} r^{2k}\sum_{l=0}^{[k/2]}\|f_{k,k-2l}\|_{S_1}^2 < \infty\Big\}.\end{aligned} \tag{8}$$

Note that $H^2(\tilde{B}(r))|_{\Sigma_r} \subsetneq L^2(\Sigma_r)$, where $L^2(\Sigma_r)$ is the Hilbert space of square integrable functions on Σ_r.

Furthermore, we can see that $H^2(\tilde{B}(r))$ is isomorphic to the Hardy space:

$$H^2(\tilde{B}(r)) = \left\{ f \in \mathcal{O}(\tilde{B}(r)); \sup_{0<t<1} \int_{\Sigma_r} |f(tz)|^2 \dot{d}z < \infty \right\}.$$

Clearly, we have

$$\mathcal{O}(\tilde{B}[r]) \hookrightarrow H^2(\tilde{B}(r)) \hookrightarrow \mathcal{O}(\tilde{B}(r)). \tag{9}$$

2.2 Cauchy-Hua transformation

The Cauchy-Hua kernel $H_r(z, w)$ is defined by

$$H_r(z, w) = H_1(z/r, w/r), \quad H_1(z, w) = \frac{1}{(1 - 2z \cdot \overline{w} + z^2 \overline{w}^2)^{(n+1)/2}}.$$

Then $H_r(z, \overline{w})$ is holomorphic on $\{(z, w) \in \tilde{\mathbf{E}} \times \tilde{\mathbf{E}}; L(z)L(w) < r^2\}$. Note that $H_r(z, w) = \overline{H_r(w, z)}$ and $H_1(z, \overline{w})$ is expanded as follows;

$$\begin{aligned} H_1(z, \overline{w}) &= \sum_{k=0}^{\infty} \frac{N(k, n+2)(n+1)}{2k+n+1} \tilde{P}_{k,n+2}(z, w) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} N(k-2l, n)(z^2)^l (w^2)^l \tilde{P}_{k-2l,n}(z, w). \end{aligned}$$

For $f \in \mathcal{O}(\tilde{B}(r))$, we have the following integral representation:

$$f(z) = s \int_{\Sigma_r} H_r(z, w) f(w) \dot{d}w.$$

(See, for example, [4].)

We denote by X' the dual space of X; for example, $\mathcal{O}'(\tilde{B}_N(r))$ means the dual space of $\mathcal{O}(\tilde{B}_N(r))$.

Let $T \in \mathcal{O}'(\tilde{B}[r])$. If $w \in \tilde{B}(r)$, then the mapping $z \mapsto H_r(z, w)$ belongs to $\mathcal{O}(\tilde{B}[r])$. Thus we can define the Cauchy-Hua transform $\mathcal{C}T$ of T by

$$\mathcal{C}T(w) = \overline{\langle T_z, H_r(z, w) \rangle}, \quad w \in \tilde{B}(r).$$

We call the mapping $\mathcal{C} : T \mapsto \mathcal{C}T$ the Cauchy-Hua transformation.

THEOREM **2.2** *Let $r > 0$. The Cauchy-Hua transformation $\mathcal{C}$ establishes the following topological antilinear isomorphisms:*

$$\begin{aligned} \mathcal{C} &: \mathcal{O}'(\tilde{B}[r]) \xrightarrow{\sim} \mathcal{O}(\tilde{B}(r)), \\ \mathcal{C} &: \mathcal{O}'(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}(\tilde{B}[r]). \end{aligned}$$

Further, we have

$$\langle T, g\rangle = s\int_{\Sigma_r} g(w)\overline{\mathcal{C}T(w)}\dot{dw}$$

for $T \in \mathcal{O}'(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}[r])$ or for $T \in \mathcal{O}'(\tilde{B}(r))$ and $g \in \mathcal{O}(\tilde{B}(r))$, which gives the inverse of $\mathcal{C}$.

(For a proof see, for example, [4].)

2.3 Fourier transformation

The Fourier-Borel transform $\mathcal{F}T$ of $T \in \mathcal{O}'(\tilde{B}_N[r])$ is defined by

$$\mathcal{F}T(\zeta) = \langle T_z, \exp(z\cdot\zeta)\rangle.$$

We call the mapping $\mathcal{F} : T \mapsto \mathcal{F}T$ the Fourier-Borel transformation.

In [2], A.Martineau proved the following theorem:

THEOREM **2.3** *Let $N(z)$ be a norm on $\tilde{\mathbf{E}}$. The Fourier-Borel transformation $\mathcal{F}$ establishes the following topological linear isomorphisms:*

$$\begin{aligned} \mathcal{F}:\ & \mathcal{O}'(\tilde{B}_N[r]) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N^*)), \quad 0 \le r < \infty, \\ \mathcal{F}:\ & \mathcal{O}'(\tilde{B}_N(r)) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, N^*]), \quad 0 < r \le \infty. \end{aligned}$$

Composing the Fourier-Borel transformation $\mathcal{F}$ and the Cauchy-Hua transformation $\mathcal{C}$ on $\mathcal{O}'(\tilde{B}[r])$, we can consider the Fourier transformation $\mathcal{Q}$ on $\mathcal{O}(\tilde{B}(r))$ as $\mathcal{Q} = \mathcal{F}\circ\mathcal{C}^{-1}$. Then by Theorems 2.2 and 2.3, for $f \in \mathcal{O}(\tilde{B}(r))$ we have

$$\mathcal{Q}f(\zeta) = s\int_{\Sigma_r} \exp(z\cdot\zeta)\overline{f(z)}\dot{dz}.$$

By the definition of $\mathcal{Q}$, Theorems 2.2 and 2.3 imply the following corollary:

Corollary **2.4** *Let $r > 0$. The Fourier transformation $\mathcal{Q}$ establishes the following topological antilinear isomorphisms:*

$$\mathcal{Q}: \ \mathcal{O}(\tilde{B}(r)) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)),$$
$$\mathcal{Q}: \ \mathcal{O}(\tilde{B}[r]) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, L^*]).$$

By (9) and Corollary 2.4, we have

$$\mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, L^*]) \hookrightarrow \mathcal{Q}(H^2(\tilde{B}(r))) \hookrightarrow \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)).$$

By a simple calculation we can determine the image $\mathcal{Q}f$ of $f \in \mathcal{O}(\tilde{B}(r))$, concretely as follows:

Lemma **2.5** *Let $f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z) \in \mathcal{O}(\tilde{B}(r))$, $f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}})$. Then we have*

$$\mathcal{Q}f(\zeta) \ = \ \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k}\Gamma(\frac{n+1}{2})}{2^k l! \Gamma(k-l+\frac{n+1}{2})} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta),$$

where we write $\overline{f}(z) = \overline{f(\overline{z})}$.

By Lemma 2.5 and (8),

$$\mathcal{Q}(H^2(\tilde{B}(r))) = \Bigg\{ F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \in \mathcal{O}(\tilde{\mathbf{E}}); F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}}),$$
$$\sum_{k=0}^{\infty} (\frac{2}{r})^{2k} \sum_{l=0}^{[k/2]} \Big(l!\Gamma(k-l+\frac{n+1}{2}) \Big)^2 \|F_{k,k-2l}\|_{S_1}^2 < \infty \Bigg\}.$$

2.4 Proof of Theorem 2.1

Proof. Let $F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*))$. By Corollary 2.4, there exists $f \in \mathcal{O}(\tilde{B}(r))$ such that $F(\zeta) = \mathcal{Q}f(\zeta) \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*))$. By Lemma 2.5, for $f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z)$, $f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}})$, we have

$$F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k}\Gamma(\frac{n+1}{2})}{2^k l! \Gamma(k-l+\frac{n+1}{2})} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta).$$

Thus we have

$$F_{k,k-2l}(\zeta) = \frac{r^{2k}\Gamma(\frac{n+1}{2})}{2^k l!\Gamma(k-l+\frac{n+1}{2})}\overline{f_{k,k-2l}(\zeta)}.$$

Since $f \in \mathcal{O}(\tilde{B}(r))$, by Theorem 1.3, we have

$$\limsup_{2k-2l\to\infty}\left(r^k\|f_{k,k-2l}\|_{C(S_1)}\right)^{1/(2k-2l)} \leq 1.$$

Therefore

$$\limsup_{2k-2l\to\infty}\left(\frac{2^k l!\Gamma(k-l+\frac{n+1}{2})}{r^k\Gamma(\frac{n+1}{2})}\|F_{k,k-2l}\|_{C(S_1)}\right)^{1/(2k-2l)} \leq 1,$$

and it is equivalent to (6).

Conversely, assume that the sequence $\{F_{k,k-2l}\}$ of $(k-2l)$-homogeneous harmonic polynomials satisfies (6). Then for any $\delta > 0$ there exists $C \geq 0$ such that

$$\|F_{k,k-2l}\|_{C(S_1)} \leq C\frac{(1+\delta)^{2k-2l}r^k}{2^k l!(k-l)!}. \tag{10}$$

Put

$$f_{k,k-2l}(z) = \frac{2^k l!\Gamma(k-l+\frac{n+1}{2})}{r^{2k}\Gamma(\frac{n+1}{2})}\overline{F_{k,k-2l}(z)}. \tag{11}$$

Noting that $\lim_{p\to\infty}(\frac{\Gamma(p+q)}{\Gamma(p)})^{1/p} = 1$ for any constant $q \in \mathbf{R}$, by (10), we have

$$\limsup_{2k-2l\to\infty}\left(\frac{2^k l!\Gamma(k-l+\frac{n+1}{2})}{\Gamma(\frac{n+1}{2})r^k}\|F_{k,k-2l}\|_{C(S_1)}\right)^{1/(2k-2l)} \leq 1+\delta.$$

Since $\delta > 0$ is arbitrary we have $\limsup_{2k-2l\to\infty}\left(r^k\|f_{k,k-2l}\|_{C(S_1)}\right)^{1/(2k-2l)} \leq 1$. Therefore the function $f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z)$ belongs to $\mathcal{O}(\tilde{B}(r))$ by Theorem 1.3, and $\mathcal{Q}f(\zeta) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(\zeta^2)^l F_{k,k-2l}(\zeta)$ by Lemma 2.5 and (11). Further by Corollary 2.4, we have

$$F(\zeta) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(\zeta^2)^l F_{k,k-2l}(\zeta) \in \mathrm{Exp}\,(\tilde{\mathbf{E}};(r,L^*)).$$

q.e.d.

3 Entire eigenfunctions of the Laplacian

Let λ be a complex number. We denote the space of eigenfunctions of the Laplacian by $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r)); (\Delta_z - \lambda^2)f(z) = 0\}$, where Δ_z is the complex Laplacian: $\Delta_z = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \cdots + \frac{\partial^2}{\partial z_{n+1}^2}$.

LEMMA **3.1 ([6, Theorem 2.1])**
Let $f \in \mathcal{O}(\tilde{B}(r))$ and $f_{k,k-2l}$ be the $(k, k-2l)$-component of f defined by (2). Then we have

$$f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r)) \Longleftrightarrow f_{k,k-2l} = \frac{(\lambda/2)^{2l}\Gamma(k-2l+\frac{n+1}{2})}{\Gamma(l+1)\Gamma(k-l+\frac{n+1}{2})} f_{k-2l,k-2l}$$

for $l = 0, 1, 2, \cdots, [k/2]$ and $k = 0, 1, 2, \cdots$.

In case of the eigenfunctions of the Laplacian, by Lemma 3.1 the expansion of (3) reduces to

$$f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z) = \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2}) f_{k,k}(z),$$

where $\tilde{j}_k(t)$ is the entire Bessel function:

$$\tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t) = \Gamma(k+(n+1)/2)(t/2)^{-(k+\frac{n-1}{2})} J_{k+\frac{n-1}{2}}(t).$$

Then the (k, k)-component of $f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r))$ is given by

$$f_{k,k}(z) = N(k,n) \int_{S_1} \tilde{P}_{k,n}(z, \tau) f(\tau) \dot{d\tau}. \tag{12}$$

Let $N(z)$ be a norm on $\tilde{\mathbf{E}}$ and put

$$\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, N)) = \mathrm{Exp}(\tilde{\mathbf{E}}; (r, N)) \cap \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}).$$

We have the following theorem:

THEOREM **3.2 ([6, Theorem 2.1])** *Let $F \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$ and $F_{k,k}$ be the (k, k)-component of F defined by (12). Then we have*

$$F \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)) \Longleftrightarrow \limsup_{k\to\infty} \left(k! \|F_{k,k}\|_{C(S_1)}\right)^{1/k} \leq \frac{r}{2}.$$

We define the complex sphere $\tilde{S}_\lambda$ of complex radius λ with center at 0 by

$$\tilde{S}_\lambda = \{z \in \tilde{\mathbf{E}}; z^2 = \lambda^2\}.$$

If $z \in \tilde{S}_\lambda$, then

$$L^*(z) = \frac{1}{2}\left(L(z) + \frac{|\lambda|^2}{L(z)}\right). \tag{13}$$

Since $L(z) \geq L^*(z)$, (13) is equivalent to $L(z) = L^*(z) + \sqrt{L^*(z)^2 - |\lambda|^2}$. Putting $\tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}(r)$, for $|\lambda| < r$ we have

$$z \in \tilde{S}_\lambda(r) \Longleftrightarrow L^*(z) < \frac{r^2 + |\lambda|^2}{2r}, \quad z \in \tilde{S}_\lambda.$$

Therefore we have $\tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}_{L^*}(\frac{r^2+|\lambda|^2}{2r})$ and $\mathcal{O}'(\tilde{S}_\lambda(r)) = \mathcal{O}'\left(\tilde{S}_\lambda \cap \tilde{B}_{L^*}(\frac{r^2+|\lambda|^2}{2r})\right)$. Restrict the Fourier-Borel transformation on $\mathcal{O}'(\tilde{B}_N(r))$ to $\mathcal{O}'(\tilde{S}_\lambda \cap \tilde{B}_N(r))$ and apply Theorem 2.3. Then we have the following theorem:

THEOREM **3.3** *For $|\lambda| \leq r$, we have*

$$\operatorname{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)) = \operatorname{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}}; \left(\frac{r^2 + |\lambda|^2}{2r}, L\right)\right).$$

This generalizes a theorem in [5];

$$\operatorname{Exp}_{\Delta}(\tilde{\mathbf{E}}; (r, L^*)) = \operatorname{Exp}_{\Delta}\left(\tilde{\mathbf{E}}; (\frac{r}{2}, L)\right), \quad |\lambda| \leq r.$$

Moreover, if $|\lambda| = r$, then $\operatorname{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}}; (r, L^*)\right) = \operatorname{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}}; (r, L)\right)$. Therefore, more precisely, we can rewrite (5) as

$$\operatorname{Exp}(\tilde{\mathbf{E}}; (r, L^*)) \underset{\neq}{\subset} \operatorname{Exp}(\tilde{\mathbf{E}}; (r, L)) \underset{\neq}{\subset} \operatorname{Exp}(\tilde{\mathbf{E}}; (2r, L^*)).$$

From Theorems 3.2 and 3.3 we have the following corollary:

COROLLARY **3.4**
Let $F \in \operatorname{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L))$, $|\lambda| \leq r$. Define $F_{k,k}$ by (12). Then we have

$$\limsup_{k\to\infty} \left(k! \|F_{k,k}\|_{C(S_1)}\right)^{1/k} \leq \frac{r + \sqrt{r^2 - |\lambda|^2}}{2}.$$

Conversely, if we are given a sequence $\{F_{k,k}\}$ *of* k*-homogeneous harmonic polynomials* $F_{k,k}(z)$ *satisfying*

$$\limsup_{k\to\infty}\left(k!\|F_{k,k}\|_{C(S_1)}\right)^{1/k}\leq r,$$

then $\sum_{k=0}^{\infty}\tilde{j}_k(i\lambda\sqrt{z^2})F_{k,k}(z)$ *converges to* $F\in \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r+\frac{|\lambda|^2}{4r},L)\right)$ *and the* (k,k)*-component of* F *is equal to the given* $F_{k,k}$.

References

[1] Y.Iwahara, Entire functions of exponential type related to the Lie ball, Master's thesis, Sophia University, 1998. (in Japanese)

[2] A.Martineau, Sur les fonctionnelles analytiques et la transformation de Fourier-Borel, J. d'analyse Math. de Jérusalem 11, 1963, 1-164.

[3] M.Morimoto, Analytic functionals on the Lie sphere, Tokyo J.Math., 3(1980), 1–35

[4] M.Morimoto, Analytic Functionals on the Sphere, Translation of Mathematical Monographs, AMS, 1998.

[5] R.Wada, Holomorphic functions on the complex sphere, Tokyo J. Math., 11(1988), 205-218.

[6] R.Wada and M.Morimoto, A uniqueness set for the differential operator $\Delta_z+\lambda^2$, Tokyo J. Math., 10(1987), 93-105.