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Characterization of entire functions of exponential type with respect to the Lie norm

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Introduction

We consider the space of entire functions on $\tilde{\mathbb{E}} = \mathbb{C}^{n+1}$ and denote it by $\mathcal{O}(\tilde{\mathbb{E}})$. Let $F(z) = \sum_{k=0}^{\infty} F_k(z) \in \mathcal{O}(\tilde{\mathbb{E}})$ be the homogeneous expansion of $F$ into homogeneous polynomials $F_k$ of degree $k$. For a norm $N(z)$ on $\tilde{\mathbb{E}}$ put

$$\text{Exp}(\tilde{\mathbb{E}}; (r, N)) = \{F \in \mathcal{O}(\tilde{\mathbb{E}}); \forall r' > r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \exp(r' N(z))\}$$

and $\|F\|_{C(B_N[1])} = \sup\{|F(z)|; N(z) \leq 1\}$. Then we know that

$$F \in \text{Exp}(\tilde{\mathbb{E}}; (r, N)) \iff \lim_{k \to \infty} \sup_{|z| \leq 1} (k!)^{1/k} \|F_k\|_{C(B_N[1])}^{1/k} \leq r.$$ 

An entire function can also be expanded into the double series with $(k-2l)$-homogeneous harmonic polynomials $F_{k,k-2l}$, $k = 0, 1, \cdots, l = 0, 1, \cdots, [k/2]$;

$$F(z) = \sum_{k=0}^{\infty} F_k(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l F_{k,k-2l}(z),$$

where the convergence is uniform on compact sets in $\tilde{\mathbb{E}}$.

In this note, we consider the case that the norm $N(z)$ is the Lie norm $L(z)$ or the dual Lie norm $L^*(z)$. First, we formulate, in terms of the growth behavior of $F_{k,k-2l}$, the necessary and sufficient conditions for an entire function $F$ to belong to $\text{Exp}(\tilde{\mathbb{E}}; (r, N))$. Here we will present the following results according to [1]:
For $F(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{k/2} (z^2)^l F_{k,k-2l}(z)$, we have

\[ F \in \text{Exp}(\tilde{E}; (r, L)) \iff \limsup_{2k-2l \to \infty} \left( \frac{k!}{r^k} \|F_{k,k-2l}\|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1, \]

\[ F \in \text{Exp}(\tilde{E}; (r, L^*)) \iff \limsup_{2k-2l \to \infty} \left( \frac{2^k l! (k-l)!}{r^k} \|F_{k,k-2l}\|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1, \]

where $S_1$ is the unit real sphere. (See Theorems 1.4 and 2.1.)

Second, we will study the spaces of entire eigenfunctions of exponential type of the Laplacian; $\text{Exp}_{\Delta-\lambda^2}(\tilde{E}; (r, L))$ and $\text{Exp}_{\Delta-\lambda^2}(\tilde{E}; (r, L^*))$. For these spaces we will prove the following relation which generalizes a theorem in [5]:

Theorem

\[ \text{Exp}_{\Delta-\lambda^2}(\tilde{E}; (r, L^*)) = \text{Exp}_{\Delta-\lambda^2}(\tilde{E}; \left( \frac{r^2 + |\lambda|^2}{2r}, L \right)), \quad |\lambda| \leq r. \]

(See Theorem 3.3.) From this relation we have

\[ \text{Exp}(\tilde{E}; (r, L^*)) \subset \text{Exp}(\tilde{E}; (r, L)) \subset \text{Exp}(\tilde{E}; (2r, L^*)). \]

1 Lie norm

Let $N(z)$ be a norm on $\tilde{E} = \mathbb{C}^{n+1}$. Its dual norm $N^*(z)$ is defined by

\[ N^*(z) = \sup\{|z \cdot \zeta|; N(\zeta) \leq 1\}. \]

The open and the closed $N$-balls of radius $r$ with center at 0 are defined by

\[ \tilde{B}_N(r) = \{z \in \tilde{E}; N(z) < r\}, \quad r > 0, \quad \tilde{B}_N[r] = \{z \in \tilde{E}; N(z) \leq r\}, \quad r \geq 0. \]

Note that $\tilde{B}_N(\infty) = \tilde{E}$. We denote by $O(\tilde{B}_N(r))$ the space of holomorphic functions on $\tilde{B}_N(r)$. Put $O(\tilde{B}_N[r]) = \lim\limits_{r' > r} \text{ind} O(\tilde{B}_N(r'))$, $O(\tilde{B}_N(\infty)) = \tilde{E}$.

\[ \text{Exp}(\tilde{E}; (r, N)) = \{F \in O(\tilde{E}); \forall r' > r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \exp(r'N(z))\}, \]

\[ \text{Exp}(\tilde{E}; [r, N]) = \{F \in O(\tilde{E}); \exists r' < r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \exp(r'N(z))\}. \]
Note that for any norm $N$ on $\tilde{E}$ we have $\text{Exp}(\tilde{E}; (0, N)) = \text{Exp}(\tilde{E}; (0))$.

We denote by $\mathcal{P}^{k}(\tilde{E})$ the space of homogeneous polynomials of degree $k$. Define the $k$-homogeneous component $f_{k} \in \mathcal{P}^{k}(\tilde{E})$ of $f \in \mathcal{O}((0))$ by
\[
 f_{k}(z) = \frac{1}{2\pi i} \int_{|t|=\rho} \frac{f(tz)}{t^{k+1}} dt,
\]
where $\rho$ is sufficiently small. Then we know the following theorem (see, for example, [2]):

**Theorem 1.1** Let $N(z)$ be a norm on $\tilde{E}$ and $F_{k} \in \mathcal{P}^{k}(\tilde{E})$. Then we have
\[
 F = \sum_{k=0}^{\infty} F_k(z) \in \text{Exp}(\tilde{E}; (r, N)) \iff \limsup_{k \to \infty} (k!||F_k||_{C(\tilde{B}_{N}[1])})^{1/k} \leq r,
\]
\[
 F = \sum_{k=0}^{\infty} F_k(z) \in \text{Exp}(\tilde{E}; [r, N]) \iff \limsup_{k \to \infty} (k!||F_k||_{C(\tilde{B}_{N}[1])})^{1/k} < r,
\]
where $||F||_{C(\tilde{B}_{N}[1])} = \sup\{|F(z)|; N(z) \leq 1\}$.

We define the Lie norm $L(z)$ of $z \in \tilde{E}$ by
\[
 L(z) = \sqrt{||z||^2 + \sqrt{||z||^4 - |z|^2}}.
\]
Then $L(z)$ is the cross norm of the Euclidean norm $||x||$; that is,
\[
 L(z) = \inf \left\{ \sum_{j=1}^{m} |\lambda_j||x_j||; z = \sum_{j=1}^{m} \lambda_j x_j, \lambda_j \in \mathbb{C}, x_j \in \mathbb{R}^{n+1}, m \in \mathbb{Z}_+ \right\}.
\]
Thus putting $||f_k||_{C(S_1)} = \sup\{|f_k(x)|; x \in S_1\}$, for $f_k \in \mathcal{P}^{k}(\tilde{E})$ we can see
\[
 ||f_k||_{C(\tilde{B}_{L}[1])} = ||f_k||_{C(S_1)}.
\]
Therefore as a corollary of Theorem 1.1, we have

**Corollary 1.2** Let $F(z) = \sum_{k=0}^{\infty} F_k(z)$, $F_k \in \mathcal{P}^{k}(\tilde{E})$. Then we have
\[
 F \in \text{Exp}(\tilde{E}; (r, L)) \iff \limsup_{k \to \infty} (k!||F_k||_{C(S_1)})^{1/k} \leq r,
\]
\[
 F \in \text{Exp}(\tilde{E}; [r, L]) \iff \limsup_{k \to \infty} (k!||F_k||_{C(S_1)})^{1/k} < r.
\]
Let $P_{k,n}(t)$ be the Legendre polynomial of degree $k$ and of dimension $n+1$. The harmonic extension $\tilde{P}_{k,n}(z, w)$ of $P_{k,n}(z \cdot w)$ is given by

$$\tilde{P}_{k,n}(z, w) = (\sqrt{z^2})^k (\sqrt{w^2})^k P_{k,n} \left( \frac{z}{\sqrt{z^2}} \cdot \frac{w}{\sqrt{w^2}} \right).$$

Then $\tilde{P}_{k,n}(z, w)$ is a $k$-homogeneous harmonic polynomial in $z$ and in $w$ and satisfies $|\tilde{P}_{k,n}(z, w)| \leq L(z)^k L(w)^k$. We denote by $\mathcal{P}_k^k(\mathcal{E})$ the space of homogeneous harmonic polynomials of degree $k$. The dimension of $\mathcal{P}_k^k(\mathcal{E})$ is known to be $(2k + n - 1)(k + n - 2)!/(k!(n-1)!) \equiv N(k, n)$.

When $N(z) = L(z)$, we omit the subscript; for example, we write $\tilde{B}(r)$ for $\tilde{B}_L(r)$. For a holomorphic function on $\tilde{B}(r)$ we know the following theorem:

**Theorem 1.3 ([3, Theorem 3.1])**

Let $f \in \mathcal{O}(\tilde{B}(r))$. Define the $k$-homogeneous component of $f$ by (1) and define the $(k, j)$-component of $f$ by

$$f_{k,j}(z) = N(j, n) \int_{S_1} f_k(\tau) \tilde{P}_{j,n}(z, \tau) d\tau,$$

where $d\tau$ is the normalized invariant measure on the unit real sphere $S_1$. Then $f_{k,j}$ is a $j$-homogeneous harmonic polynomial and we can expand $f$ into the double series:

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (\sqrt{z^2})^{k-j} f_{k,j}(z) = \sum_{k=0}^{[k/2]} \sum_{l=0}^{k} (z^2)^{l} f_{k,k-2l}(z),$$

where the convergence is uniform on compact sets in $\tilde{B}(r)$ and we have

$$\lim_{2k-2l \to \infty} \sup_{z \in \overline{B}(r)} |r^k ||f_{k,k-2l}|_{C(S_1)}^{1/(2k-2l)} \leq 1.$$  \hfill (4)

Conversely, if we are given a double sequence $\{f_{k,k-2l}\}$ of homogeneous harmonic polynomials $f_{k,k-2l}(z)$ satisfying (4), then the right-hand side of (3) converges to a holomorphic function $f$ uniformly on compact sets in $\tilde{B}(r)$ and the $(k, k-2l)$-component of $f$ is equal to the given $f_{k,k-2l}$.
For an entire function of exponential type, [1] proved the following theorem: We can prove it by the property of the Lie norm. Here, we omit its proof.

**Theorem 1.4 ([1, Theorem 3.7])** Let \( F(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^{l} F_{k,k-2l}(z), F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{E}) \), be the expansion of \( F \in \mathcal{O}(\tilde{E}) \). Then we have

\[
F \in \text{Exp}(\tilde{E}; (r, L)) \iff \lim_{2k-2l \to \infty} \sup_{l \to \infty} \left( \frac{k!}{r^k} \| F_{k,k-2l} \|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1.
\]

## 2 Dual Lie norm

The dual Lie norm \( L^*(z) \) is given by

\[
L^*(z) = \sqrt{(\|z\|^2 + |z|^2)/2}.
\]

Since \(|\sqrt{z^2}| \leq L^*(z) \leq \|z\| \leq L(z) \leq 2L^*(z)\), we have

\[
\text{Exp}(\tilde{E}; (r, L^*)) \subset \text{Exp}(\tilde{E}; (r, L)) \subset \text{Exp}(\tilde{E}; (2r, L^*)).
\]

(5)

Similar to Theorem 1.4, for the dual Lie norm \( L^*(z) \), we have the following theorem:

**Theorem 2.1 ([1, Theorem 5.2])** Let \( F(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{[k/2]} (z^2)^{j} F_{k,k-2l}(z), F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{E}) \), be the expansion of \( F \in \mathcal{O}(\tilde{E}) \). Then we have

\[
F \in \text{Exp}(\tilde{E}; (r, L^*)) \iff \lim_{2k-2l \to \infty} \sup_{l \to \infty} \left( \frac{2^k l! (k-l)!}{r^k} \| F_{k,k-2l} \|_{C(S_1)} \right)^{1/(2k-2l)} \leq 1.
\]

(6)

For a proof, we use the Cauchy-Hua transformation and the Fourier transformation. First we introduce the invariant measure on the Lie sphere.

### 2.1 Lie sphere

The Shilov boundary of \( \tilde{B}[r] \) is the Lie sphere \( \Sigma_r \):

\[
\Sigma_r = \{ re^{i\theta} \omega; 0 \leq \theta < 2\pi, \omega \in S_1 \} = \{ e^{i\theta} \omega; 0 \leq \theta < 2\pi, \omega \in S_r \}.
\]
Note that \(-xe^{i(\theta+\pi)} = xe^{i\theta}\) and \(\Sigma_r = (\mathbb{R}/(2\pi\mathbb{Z}) \times S_r)/\sim\), where \(\sim\) is the equivalence relation defined by \((\theta, x) \sim (\theta + \pi, -x)\), and that for \(f \in \mathcal{O}(\tilde{B}[r])\) we have \(\sup\{|f(z)|; z \in \tilde{B}[r]\} = \sup\{|f(z)|; z \in \Sigma_r\}\

We define the invariant integral over \(\Sigma_r\) by
\[
\int_{\Sigma_r} f(z) \dot{dz} = \frac{1}{2\pi} \int_0^{2\pi} \int_{S_1} f(re^{i\theta}) \dot{d\omega} d\theta.
\]
For \(f, g \in \mathcal{O}(\tilde{B}[r])\), the integral \(\int_{\Sigma_r} f(z)\overline{g(z)} \dot{dz}\) is well-defined. Since
\[
(f, g)_{\Sigma_r} \equiv \int_{\Sigma_r} f(z)\overline{g(z)} \dot{dz} = \sum_{k=0}^{\infty} r^{2k} \int_{S_1} f_k(\omega)\overline{g_k(\omega)} \dot{d\omega}
\]
(\(\cdot, \cdot\)) is an inner product on \(\mathcal{O}(\tilde{B}[r])\). If \(f \in \mathcal{O}(\tilde{B}[r])\) and \(g \in \mathcal{O}(\tilde{B}(r))\), then for \(s > 1\) sufficiently close to 1 the integral \(\int_{\Sigma_r} f(z/s)\overline{g(sz)} \dot{dz}\) is well-defined and does not depend on \(s\) by (7). Thus for \(f \in \mathcal{O}(\tilde{B}[r])\) and \(g \in \mathcal{O}(\tilde{B}(r))\) or for \(g \in \mathcal{O}(\tilde{B}[r])\) and \(f \in \mathcal{O}(\tilde{B}(r))\) we write
\[
\int_{\Sigma_r} f(z/s)\overline{g(sz)} \dot{dz} = s \int_{\Sigma_r} f(z)\overline{g(z)} \dot{dz}.
\]

Let \(H^2(\tilde{B}(r))\) be the completion of \(\mathcal{O}(\tilde{B}[r])\) with respect to the inner product \((\cdot, \cdot)_{\Sigma_r}\), and put \(\|f\|^2_{\Sigma_r} = \int_{S_1} |f(\omega)|^2 \dot{d\omega}\). Then by the definition,
\[
H^2(\tilde{B}(r)) = \left\{f(z) = \sum_{k=0}^{\infty} f_k(z); \right. \\
\left. f_k \in \mathcal{P}^k(\tilde{E}), \sum_{k=0}^{\infty} \|f_k\|^2_{\Sigma_r} = \sum_{k=0}^{\infty} r^{2k}\|f_k\|^2_{S_1} < \infty\right\}
\]
(8)
\[
= \left\{f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z); \right. \\
\left. f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{E}), \sum_{k=0}^{\infty} r^{2k} \sum_{l=0}^{[k/2]} \|f_{k,k-2l}\|^2_{S_1} < \infty\right\}.
\]

Note that \(H^2(\tilde{B}(r))|_{\Sigma_r} \subsetneq L^2(\Sigma_r)\), where \(L^2(\Sigma_r)\) is the Hilbert space of square integrable functions on \(\Sigma_r\).
Furthermore, we can see that $H^2(\tilde{B}(r))$ is isomorphic to the Hardy space:

$$H^2(\tilde{B}(r)) = \left\{ f \in \mathcal{O}(\tilde{B}(r)); \sup_{0 < t < 1} \int_{\Sigma_r} |f(tz)|^2 \, dz < \infty \right\}.$$  

Clearly, we have

$$\mathcal{O}(\tilde{B}[r]) \hookrightarrow H^2(\tilde{B}(r)) \hookrightarrow \mathcal{O}(\tilde{B}(r)).$$  \hfill (9)

### 2.2 Cauchy-Hua transformation

The Cauchy-Hua kernel $H_r(z, w)$ is defined by

$$H_r(z, w) = H_1(z/r, w/r), \quad H_1(z, w) = \frac{1}{(1 - 2z \cdot \overline{w} + z\overline{w}^2)^{(n+1)/2}}.$$  

Then $H_r(z, \overline{w})$ is holomorphic on $\{(z, w) \in \tilde{E} \times \tilde{E}; L(z)L(w) < r^2\}$. Note that $H_r(z, w) = \overline{H_r(w, z)}$ and $H_1(z, \overline{w})$ is expanded as follows;

$$H_1(z, \overline{w}) = \sum_{k=0}^{\infty} \frac{N(k, n+2)(n+1)}{2k+n+1} \tilde{P}_{k,n+2}(z, w)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\lfloor k/2 \rfloor} N(k-2l, n)(z^2)^l(w^2)^l \tilde{P}_{k-2l,n}(z, w).$$

For $f \in \mathcal{O}(\tilde{B}(r))$, we have the following integral representation:

$$f(z) = s \int_{\Sigma_r} H_r(z, w)f(w) \, dw.$$  

(See, for example, [4].)

We denote by $X'$ the dual space of $X$; for example, $\mathcal{O}'(\tilde{B}_N(r))$ means the dual space of $\mathcal{O}(\tilde{B}_N(r))$.

Let $T \in \mathcal{O}'(\tilde{B}[r])$. If $w \in \tilde{B}(r)$, then the mapping $z \mapsto H_r(z, w)$ belongs to $\mathcal{O}(\tilde{B}[r])$. Thus we can define the Cauchy-Hua transform $CT$ of $T$ by

$$CT(w) = \langle T_z, H_r(z, w) \rangle, \quad w \in \tilde{B}(r).$$

We call the mapping $C : T \mapsto CT$ the Cauchy-Hua transformation.
Theorem 2.2 Let $r > 0$. The Cauchy-Hua transformation $C$ establishes the following topological antilinear isomorphisms:

\[ C : O'(\tilde{B}[r]) \xrightarrow{\sim} O(\tilde{B}(r)), \]
\[ C : O'(\tilde{B}(r)) \xrightarrow{\sim} O(\tilde{B}[r]). \]

Further, we have

\[ \langle T, g \rangle = s \int_{\Sigma} g(w) \overline{C\tau}(\Gamma w) dw \]

for $T \in O'(\tilde{B}[r])$ and $g \in O(\tilde{B}(r))$ or for $T \in O'(\tilde{B}(r))$ and $g \in O(\tilde{B}(r))$, which gives the inverse of $C$.

(For a proof see, for example, [4].)

2.3 Fourier transformation

The Fourier-Borel transform $\mathcal{F}T$ of $T \in O'(\tilde{B}_N[r])$ is defined by

\[ \mathcal{F}T(\zeta) = \langle T, \exp(Z \cdot \zeta) \rangle. \]

We call the mapping $\mathcal{F} : T \mapsto \mathcal{F}T$ the Fourier-Borel transformation.

In [2], A.Martineau proved the following theorem:

Theorem 2.3 Let $N(z)$ be a norm on $\tilde{E}$. The Fourier-Borel transformation $\mathcal{F}$ establishes the following topological linear isomorphisms:

\[ \mathcal{F} : O'(\tilde{B}_N[r]) \xrightarrow{\sim} \text{Exp}(\tilde{E};(r,N^*)), \quad 0 \leq r < \infty, \]
\[ \mathcal{F} : O'(\tilde{B}_N(r)) \xrightarrow{\sim} \text{Exp}(\tilde{E};[r,N^*]), \quad 0 < r \leq \infty. \]

Composing the Fourier-Borel transformation $\mathcal{F}$ and the Cauchy-Hua transformation $C$ on $O'(\tilde{B}[r])$, we can consider the Fourier transformation $Q$ on $O(\tilde{B}(r))$ as $Q = \mathcal{F} \circ C^{-1}$. Then by Theorems 2.2 and 2.3, for $f \in O(\tilde{B}(r))$ we have

\[ Qf(\zeta) = s \int_{\Sigma_r} \exp(z \cdot \zeta)f(z)dz. \]

By the definition of $Q$, Theorems 2.2 and 2.3 imply the following corollary:
COROLLARY 2.4 Let $r > 0$. The Fourier transformation $\mathcal{Q}$ establishes the following topological antilinear isomorphisms:

$$\mathcal{Q} : \mathcal{O}(\tilde{B}(r)) \rightarrow \text{Exp}(\tilde{\mathcal{E}}; (r, L^*)),$$
$$\mathcal{Q} : \mathcal{O}(\tilde{B}[r]) \rightarrow \text{Exp}(\tilde{\mathcal{E}}; [r, L^*)].$$

By (9) and Corollary 2.4, we have

$$\text{Exp}(\tilde{\mathcal{E}}; [r, L^*)] \rightarrow \mathcal{Q}(\mathcal{H}^2(\tilde{B}(r))) \rightarrow \text{Exp}(\tilde{\mathcal{E}}; (r, L^*)).$$

By a simple calculation we can determine the image $\mathcal{Q}f$ of $f \in \mathcal{O}(\tilde{B}(r))$, concretely as follows:

**LEMMA 2.5** Let $f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{[k/2]} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z) \in \mathcal{O}(\tilde{B}(r))$, $f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathcal{E}}).$ Then we have

$$\mathcal{Q}f(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k} \Gamma(n+1)}{2^k l! \Gamma(k - l + \frac{n+1}{2})} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta),$$

where we write $\overline{f}(z) = \overline{f(\overline{z})}$.

By Lemma 2.5 and (8),

$$\mathcal{Q}(\mathcal{H}^2(\tilde{B}(r))) = \left\{ F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \in \mathcal{O}(\tilde{\mathcal{E}}); F_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathcal{E}}), \right.$$

$$\left. \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \left( l! \Gamma(k - l + \frac{n+1}{2}) \right)^2 ||F_{k,k-2l}||^2_{S^1} < \infty \right\}.$$  

### 2.4 Proof of Theorem 2.1

**Proof.** Let $F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \in \text{Exp}(\tilde{\mathcal{E}}; (r, L^*)).$ By Corollary 2.4, there exists $f \in \mathcal{O}(\tilde{B}(r))$ such that $F(\zeta) = \mathcal{Q}f(\zeta) \in \text{Exp}(\tilde{\mathcal{E}}; (r, L^*)).$

By Lemma 2.5, for $f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z)$, $f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathcal{E}})$, we have

$$F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k} \Gamma(n+1)}{2^k l! \Gamma(k - l + \frac{n+1}{2})} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta).$$

149
Thus we have

\[ F_{k,k-2l}(\zeta) = \frac{r^{2k}\Gamma(\frac{n+1}{2})}{2^{k}l!\Gamma(k-l+n+1/2)}f_{k,k-2l}(\zeta). \]

Since \( f \in \mathcal{O}(\tilde{B}(r)) \), by Theorem 1.3, we have

\[ \limsup_{2k-2l \to \infty} (r^k\|f_{k,k-2l}\|_{C(S)}{1/(2k-2l)} \leq 1. \]

Therefore

\[ \limsup_{2k-2l \to \infty} \left( \frac{2^k l! \Gamma(k - l + n + 1)}{r^k \Gamma(n+1/2)} \|F_{k,k-2l}\|_{C(S)} \right)^{1/(2k-2l)} \leq 1, \]

and it is equivalent to (6).

Conversely, assume that the sequence \( \{F_{k,k-2l}\} \) of \((k-2l)\)-homogeneous harmonic polynomials satisfies (6). Then for any \( \delta > 0 \) there exists \( C \geq 0 \) such that

\[ \|F_{k,k-2l}\|_{C(S)} \leq C \frac{(1+\delta)^{2k-2l}r^{k}}{2^k l!(k-l)!}. \]  

(10)

Put

\[ f_{k,k-2l}(z) = \frac{2^k l! \Gamma(k - l + n + 1)}{r^k \Gamma(n+1/2)}F_{k,k-2l}(z). \]  

(11)

Noting that \( \lim_{p \to \infty} (\frac{\Gamma(p+q)}{\Gamma(p)})^{1/p} = 1 \) for any constant \( q \in \mathbb{R} \), by (10), we have

\[ \limsup_{2k-2l \to \infty} \left( \frac{2^k l! \Gamma(k - l + n + 1)}{\Gamma(n+1/2)r^k} \|F_{k,k-2l}\|_{C(S)} \right)^{1/(2k-2l)} \leq 1 + \delta. \]

Since \( \delta > 0 \) is arbitrary we have \( \limsup_{2k-2l \to \infty} (r^k\|f_{k,k-2l}\|_{C(S)}{1/(2k-2l)} \leq 1. \)

Therefore the function \( f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z) \) belongs to \( \mathcal{O}(\tilde{B}(r)) \) by Theorem 1.3, and \( \mathcal{Q}f(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \) by Lemma 2.5 and (11). Further by Corollary 2.4, we have

\[ F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (c^2)^l F_{k,k-2l}(\zeta) \in \text{Exp} \left( \mathbb{E}; (r, L^*) \right). \]

q.e.d.
3 Entire eigenfunctions of the Laplacian

Let $\lambda$ be a complex number. We denote the space of eigenfunctions of the Laplacian by $\mathcal{O}_{\Delta - \lambda^2}(\tilde{B}(r)) = \{ f \in \mathcal{O}(\tilde{B}(r)); (\Delta_z - \lambda^2)f(z) = 0 \}$, where $\Delta_z$ is the complex Laplacian: $\Delta_z = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z_n^2} + \cdots + \frac{\partial^2}{\partial z_n^{2}}$.

**Lemma 3.1 ([6, Theorem 2.1])**

Let $f \in \mathcal{O}(\tilde{B}(r))$ and $f_{k,k-2l}$ be the $(k, k-2l)$-component of $f$ defined by (2). Then we have

$$f \in \mathcal{O}_{\Delta - \lambda^2}(\tilde{B}(r)) \iff f_{k,k-2l} = \frac{(\lambda/2)^{2l}\Gamma(k - 2l + \frac{n+1}{2})}{\Gamma(l + 1)\Gamma(k - l + \frac{n+1}{2})}f_{k-2l,k-2l}$$

for $l = 0, 1, 2, \cdots, [k/2]$ and $k = 0, 1, 2, \cdots$.

In case of the eigenfunctions of the Laplacian, by Lemma 3.1 the expansion of (3) reduces to

$$f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^lf_{k,k-2l}(z) = \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2})f_{k,k}(z),$$

where $\tilde{j}_k(t)$ is the entire Bessel function:

$$\tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t) = \Gamma(k + (n + 1)/2)(t/2)^{-(k+n-1)/2}J_{k+(n-1)/2}(t).$$

Then the $(k, k)$-component of $f \in \mathcal{O}_{\Delta - \lambda^2}(\tilde{B}(r))$ is given by

$$f_{k,k}(z) = N(k,n) \int_{S_1} \tilde{P}_{k,n}(z, \tau) f(\tau) d\tau.$$

(12)

Let $N(z)$ be a norm on $\tilde{E}$ and put

$$\text{Exp}_{\Delta - \lambda^2}(\tilde{E}; (r, N)) = \text{Exp}(\tilde{E}; (r, N)) \cap \mathcal{O}_{\Delta - \lambda^2}(\tilde{E}).$$

We have the following theorem:

**Theorem 3.2 ([6, Theorem 2.1])** Let $F \in \mathcal{O}_{\Delta - \lambda^2}(\tilde{E})$ and $F_{k,k}$ be the $(k, k)$-component of $F$ defined by (12). Then we have

$$F \in \text{Exp}_{\Delta - \lambda^2}(\tilde{E}; (r, L^*)) \iff \limsup_{k \to \infty} \left( k!\|F_{k,k}\|_{C(S_1)} \right)^{1/k} \leq \frac{r}{2}.$$
We define the complex sphere $\tilde{S}_{\lambda}$ of complex radius $\lambda$ with center at 0 by

$$\tilde{S}_{\lambda} = \{ z \in \tilde{E}; z^2 = \lambda^2 \}.$$ 

If $z \in \tilde{S}_{\lambda}$, then

$$L^*(z) = \frac{1}{2} \left( L(z) + \frac{|\lambda|^2}{L(z)} \right). \quad (13)$$

Since $L(z) \geq L^*(z)$, (13) is equivalent to $L(z) = L^*(z) + \sqrt{L^*(z)^2 - |\lambda|^2}$. Putting $\tilde{S}_{\lambda}(r) = \tilde{S}_{\lambda} \cap \tilde{B}(r)$, for $|\lambda| < r$ we have

$$z \in \tilde{S}_{\lambda}(r) \iff L^*(z) < \frac{r^2 + |\lambda|^2}{2r}, \quad z \in \tilde{S}_{\lambda}.$$ 

Therefore we have $\tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}_L \left( \frac{r^2 + |\lambda|^2}{2r} \right)$ and $\mathcal{O}'(\tilde{S}_\lambda(r)) = \mathcal{O}' \left( \tilde{S}_\lambda \cap \tilde{B}_N \left( \frac{r^2 + |\lambda|^2}{2r} \right) \right)$. Restrict the Fourier-Borel transformation on $\mathcal{O}'(\tilde{B}_N(r))$ to $\mathcal{O}'(\tilde{S}_\lambda \cap \tilde{B}_N(r))$ and apply Theorem 2.3. Then we have the following theorem:

**Theorem 3.3** For $|\lambda| \leq r$, we have

$$\text{Exp}_{\Delta-\lambda^2}(\tilde{E}; (r, L^*)) = \text{Exp}_{\Delta-\lambda^2} \left( \tilde{E}; \left( \frac{r^2 + |\lambda|^2}{2r}, L \right) \right).$$

This generalizes a theorem in [5];

$$\text{Exp}_{\Delta}(\tilde{E}; (r, L^*)) = \text{Exp}_{\Delta} \left( \tilde{E}; \left( \frac{r}{2}, L \right) \right), \quad |\lambda| \leq r.$$ 

Moreover, if $|\lambda| = r$, then $\text{Exp}_{\Delta-\lambda^2} \left( \tilde{E}; (r, L^*) \right) = \text{Exp}_{\Delta-\lambda^2} \left( \tilde{E}; (r, L) \right)$. Therefore, more precisely, we can rewrite (5) as

$$\text{Exp} \left( \tilde{E}; (r, L^*) \right) \subset \text{Exp} \left( \tilde{E}; (r, L) \right) \subset \text{Exp} \left( \tilde{E}; (2r, L^*) \right).$$

From Theorems 3.2 and 3.3 we have the following corollary:

**Corollary 3.4**

Let $F \in \text{Exp}_{\Delta-\lambda^2}(\tilde{E}; (r, L))$, $|\lambda| \leq r$. Define $F_{k,k}$ by (12). Then we have

$$\limsup_{k \to \infty} \left( k! \| F_{k,k} \|_{C(S_1)} \right)^{1/k} \leq \frac{r + \sqrt{r^2 - |\lambda|^2}}{2}.$$
Conversely, if we are given a sequence \( \{F_{k,k}\} \) of \( k \)-homogeneous harmonic polynomials \( F_{k,k}(z) \) satisfying

\[
\lim_{k \to \infty} \sup_{\infty} (k! \|F_{k,k}\|_{C(S_1)})^{1/k} \leq r,
\]

then \( \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda \sqrt{z^2})F_{k,k}(z) \) converges to \( F \in \text{Exp}_{\Delta-\lambda^2} \left( \tilde{E}; (r + \frac{\lambda^2}{4r}, L) \right) \) and the \((k,k)\)-component of \( F \) is equal to the given \( F_{k,k} \).

References


