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<th>Singular Solutions of Nonlinear Differential Equations: an Application of Fuchsian Differential Equations (Microlocal Analysis and PDE in the Complex Domain)</th>
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1 Introduction

We consider nonlinear partial differential equations of Kovalevskian type

$$\partial_t^m u = f(t, x; (\partial_t^j \partial_x^\alpha u)_{j+|\alpha| \leq m-1})$$

(1)

where $t \in \mathbb{C}$, $x \in \mathbb{C}^d$ and the coefficients are holomorphic in a neighborhood $\Omega$ of the origin in $\mathbb{C}^{d+1}$.

We give a simple example to explain the motivation, before introducing complicated notations.

Example 1 (Burgers equation).

$$u_{tt} + 2uu_t - u_x = 0$$

(2)

has a formal Laurent series solution

$$u = t^{-1} + gt + \left(\frac{1}{10}g_x - \frac{1}{5}g^2\right)t^3 + \cdots,$$

(3)

where $g = g(x)$ is an arbitrary holomorphic function.

It is easy to obtain such a formal solution (3): First, assume $u$ is of the form

$$u = t^\sigma \sum_{n=0}^{\infty} u_n(x) t^n \quad (u_0 \neq 0),$$

(4)

substitute (4) into the equations and then equate the coefficients of the power of $t$ to 0. We have $\sigma = -1$, $2u_0(1 - u_0) = 0$ and

$$(n + 1)(n - 2)u_n = -2 \sum_{1 \leq i, j \leq n - 1}^n (j - 1)u_i u_j + u_{n-2,x} \quad (n \geq 1).$$

(5)

It is natural to ask whether the formal series (3) converges or not. Of course, it converges to define exact solutions, which are singular on $t = 0$. We have four proofs of its convergence.

(I) Linearization:

By setting

$$u = (\ln w)_t = \frac{u_t}{w},$$
Burgers equation is equivalent to the linear equation
\[ w_{tt} - w_x = 0. \tag{6} \]

Singular solution (3) is given by the initial condition
\[ w|_{t=0} = 0, \quad w_t|_{t=0} = h(x) \]
with a suitably chosen \( h(x) \).

\textbf{(II) Direct estimates:}

\textbf{(III) Leray-Volevich system:}
Let \( u = \frac{1}{\lambda} \), then \( \lambda \) satisfies the equation
\[ \lambda \lambda_{tt} + 2 \lambda_t - 2 \lambda^2 - \lambda \lambda_x = 0. \tag{7} \]
After some calculation, Equation (7) reduces to the following Leray-Volevich system
\[ \begin{cases} 
\lambda_t = 1 - \lambda \lambda_x + \lambda^2 \mu, \\
\mu_t = \lambda \mu_x + \lambda_x \mu - \lambda_{xx}.
\end{cases} \tag{8} \]
The solution (3) is given by the initial condition
\[ \begin{cases} 
\lambda|_{t=0} = 0, \\
\mu|_{t=0} = h(x)
\end{cases} \]
with suitably chosen \( h(x) \).

\textbf{(IV) Fuchsian differential equations:}
Let \( \sigma = -1 \) and put
\[ a_N = \sum_{n=0}^{N} u_n t^{n+\sigma}, \quad w_N = \sum_{n=0}^{\infty} u_{N+1+n} t^{n+1}. \tag{9} \]
Then \( u = a_N + t^{N+\sigma} w_N \) and
\[ a_{N,t} + 2a_N a_{N,t} - a_x = t^{\sigma - 2 + N + 1} \times H_N \tag{10} \]
where \( H_N \) is a holomorphic function. Substituting (9) into (2), we obtain from (10)
\[ (t\partial_t + N + 1)(t\partial_t + N - 2)w_N = tA_N + tB_N w_N + tC_N (t\partial_t + N + \sigma)w_N - t^2 v_{N,x} + 2t^N w_N (t\partial_t + N + \sigma)w_N, \tag{11} \]
where \( A_N, B_N \) and \( C_N \) are some holomorphic functions. Now we can apply a theorem by Gérard-Tahara [1].

Consider the following nonlinear differential equation:
\[ (t\partial_t)^m w = F(t, x; (t\partial_t)^j \partial_x^\alpha w)_{(j, \alpha) \in \Lambda}, \tag{12} \]
where $F(t, x; Z)$ is holomorphic in a neighborhood of $(t, x; Z) = (0, 0; 0)$ and satisfies
\begin{align}
    F(0, x; 0) &\equiv 0, \quad \text{(13)} \\
    \frac{\partial F}{\partial Z_{j,\alpha}}(0, x; 0) &\equiv 0 \quad \text{if } |\alpha| > 0. \quad \text{(14)}
\end{align}

The characteristic polynomial of (12) is
\[
    C(\rho, x) := \rho^m - \sum_{j=0}^{m-1} \frac{\partial F}{\partial Z_{j,0}}(0, x; 0) \rho^j.
\]

**Theorem 1 (Gérard-Tahara).** If $C(n, 0) \neq 0$ for all positive integers $n$, then (12) has a unique formal solution $w = \sum_{n=1}^{\infty} w_n(x) t^n$ with $w(0, x) \equiv 0$, where $w_n(x)$ are holomorphic on a common neighborhood of the origin in $\mathbb{C}^d$. Moreover this power series is convergent and holomorphic near the origin in $\mathbb{C}^d$.

## 2 Characteristic Exponent

We put
\[
    \Lambda := \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^d : j < m, j + |\alpha| \leq m\}
\]
and write (1) as
\[
    \partial_t^m u = f(t, x; \partial^\Lambda u), \quad \text{(16)}
\]
where $f(t, x; Z)$ is holomorphic in $\Omega \times \mathbb{C}\#\Lambda$.

We expand $f$ in $Z$
\[
    f(t, x; Z) = \sum_{\mu \in \mathcal{M}} f_{\mu}(t, x) Z^\mu, \quad \text{(17)}
\]
where $\mathcal{M}$ is a subset of $\mathbb{N}\#\Lambda$.

Let $k_{\mu} \in \mathbb{N}$ be the valuation of $f_{\mu}(t, x)$ in $t$,
\[
    f_{\mu}(t, x) = t^{k_{\mu}} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k. \quad \text{(18)}
\]

**Definition 1.** The characteristic exponent $\sigma_c$ of (16) with respect to the surface $t = 0$ is
\[
    \sigma_c := \sup_{\mu \in \mathcal{M}, |\mu| \geq 2} \frac{\gamma_\mu(\mu) - m - k_{\mu}}{|\mu| - 1}, \quad \text{(19)}
\]
where
\[
    |\mu| := \sum_{(j, \alpha) \in \Lambda} \mu_{j,\alpha}, \quad \gamma_\mu(\mu) := \sum_{(j, \alpha) \in \Lambda} j \cdot \mu_{j,\alpha}.
\]

We assign weights as follows:
\[
    u \rightarrow \sigma \quad \partial_t \rightarrow -1 \quad t \rightarrow 1.
\]

Then the total weight of the right hand side of (16) is $m - \sigma$ and that of the term $f_{\mu}(\partial^\Lambda u)^\mu$ is $|\mu|\sigma - \gamma_\mu(\mu) + k_{\mu}$.
Burgers equation (2):

\[ \sigma - 2 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -1. \]

Example 2 (KdV equation).

\[ u_{ttt} - 6uu_t + u_x = 0 \]  \hspace{1cm} (20)

has

\[ \sigma - 3 = 2\sigma - 1 + 0 \Rightarrow \sigma_c = -2, \]

and Laurent series solutions

\[ u = 2t^{-2} + gt^2 + ht^4 - \frac{1}{24}g_x t^5 + \cdots, \]

where \( g = g(x) \) and \( h = h(x) \) are arbitrary holomorphic functions.

\[ |\mu| |\sigma - \gamma_t(\mu) + k_{\mu}| \]

\[ \sigma_c = -2 \]

\[ m = 3 \]

\[ \sigma - m \]

\[ m \]

Characteristic exponent of KdV equation (20)

Lemma 1.

(i) \( \sigma_c \) is invariant with respect to coordinate change which keeps the variable \( t \).

(ii) \( \sigma_c \leq m_0 \leq m - 1 \), where \( m_0 \) is the order of differentiation with respect to \( t \) in \( f(t,u;\partial^\Lambda u) \).

3 Singular Solutions

We assume

(A-1) \( f(t,x;Z) \) is a polynomial in \( Z \) of degree greater than or equal to 2.

Under (A-1), the characteristic exponent

\[ \sigma_c = \max_{\substack{\mu \in \mathcal{M} \\ |\mu| \geq 2}} \frac{\gamma_t(\mu) - m - k_{\mu}}{|\mu| - 1}, \]  \hspace{1cm} (21)
is a rational number strictly less than $m_0$, and the subset
\[ \mathcal{M}^* := \{ \mu \in \mathcal{M} : |\mu| \sigma - \gamma_{\ell}(\mu) + k_\mu = \sigma - m \}. \]  
(22)
is not empty. We call the nonlinear term corresponding to $\mu$ in $\mathcal{M}^*$ principal nonlinear term.

(A-2) If $\mu \in \mathcal{M}^*$ then $\mu_{j,\alpha} = 0$ for $|\alpha| \geq 1$

We construct a solution to (16) in the form:
\[ u(t, x) := t^{x} \sum_{n=0}^{\infty} u_n(x) t^{n/p}, \]  
(23)
where $p$ is the denominator of the reduced fraction $\sigma_c$.

Substitute (23) into (16), we obtain recursion equations:
\[ \begin{cases} P_c(x; u_0) \cdot u_0 = 0, \\ Q_c(x; u_0; \frac{n}{p}) \cdot u_n = R_n(x; \partial_x u_0, \ldots, \partial_x u_{n-1})_{|\alpha| \leq m}, \end{cases} \]  
(24)
where
\[ P_c(x; \eta) := [\sigma_c; m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \left( \prod_{(j,\alpha) \in \Lambda} \left[ \sigma_j; j \right]^{\mu_{j,\alpha}} \right) \eta^{|\mu|-1}, \]  
(25)
and
\[ Q_c(x; \eta; \rho) := [\rho + \sigma_c; m] - \sum_{\mu \in \mathcal{M}^*} f_{\mu,0}(x) \times \left( \prod_{(j,\alpha) \in \Lambda} \left[ \sigma_j; j \right]^{\mu_{j,\alpha}} \right) \left( \sum_{(j,\alpha) \in \Lambda} \frac{\rho + \sigma_c; j}{[\sigma_c; j]} \right) \eta^{|\mu|-1}. \]  
(26)

Here we have set for $\rho \in \mathbb{R}$ and $j \in \mathbb{N}$,
\[ [\rho; j] := \rho(\rho-1) \cdots (\rho-j+1). \]  
(27)

$P_c(x; \eta)$ and $Q_c(x; \eta; \rho)$ are polynomials in $\eta$ and $\rho$ and depend only on principal nonlinear terms. The order in $\eta$ is $\max_{\mu \in \mathcal{M}^*} |\mu| - 1$ and $m$ in $\rho$.

**Burgers equation (2):**
\[ \sigma_c = -1, \quad P_c(x; \eta) = 2 - 2\eta, \quad Q_c(x; \eta = 1; \rho) = (\rho + 1)(\rho - 2). \]

**KdV equation (20):**
\[ \sigma_c = -2, \quad P_c(x; \eta) = -24 + 12\eta, \quad Q_c(x; \eta = 2; \rho) = (\rho + 1)(\rho - 4)(\rho - 6). \]

**Remark.** If $k_\mu = 0$ for all $\mu \in \mathcal{M}^*$, then we have
\[ Q_c(x; \eta; \rho = -1) = \frac{\sigma_c - m}{\sigma_c} P_c(x, \eta). \]  
(28)
The equation $P_c(X; \eta) = 0$ in $\eta$ has at least one solution $\eta = u_0(x)$ which is holomorphic in a neighborhood of $x = 0$ and $u_0(x) \neq 0$.

One of the following holds for each $n \geq 1$

\[ Q(x; u_0(x); \frac{n}{p}) \neq 0 \]

(a)

\[ Q(x; u_0(x); \frac{n}{p}) \equiv 0, \quad R_n(x, \ldots \partial_x^a u_0, \ldots, \partial_x^n u_{n-1}) \equiv 0 \]

(b)

\[ \left\{ \begin{array}{l} Q(x; u_0(x); \frac{n}{p}) = 0, \quad Q(x; u_0(x); \frac{n}{p}) \neq 0 \\ Q(x; u_0(x); \frac{n}{p}) \end{array} \right. \]

(c)

\[ R_n(x, \ldots \partial_x^a u_0, \ldots, \partial_x^n u_{n-1}) \]

Remark. In case of (a) or (c), $u_n$ is determined uniquely, and in case of (b), $u_n(x)$ may be any holomorphic function.

Remark.

$Q(0; u_0(0); \rho) = 0$

has at most $m$ distinct roots.

**Theorem 2.** Suppose (A-1), (A-2), (A-3), (A-4) are satisfied. Then we can construct a solution to (16) in the form (23). Moreover all formal solutions (23) converge near the origin in $\mathbb{C}_t \times \mathbb{C}_x^{d}$.

We apply a theorem by Gérard-Tahara [1] to prove the convergence of formal solutions. For a positive integer $N$, we put

\[ w_N(t, x) := \sum_{n=0}^{\infty} u_{N+n+1}(x) t^{\frac{n+1}{p}}. \]

**Proposition 1.** If the formal series (23) satisfies the equation (16), then $w_N$ satisfies the following differential equation:

\[ Q_e(x; u_0(x); t\partial_t + \frac{N}{p}) w_N = t^{1/p} \cdot G \left( t^{1/p}, x; \left( (t\partial_t)^j \partial_x^a \bar{w}_N \right)_{(j, a) \in \Lambda} \right), \]

where $G(\tau, x; Z)$ is a polynomial in $Z$ with coefficients holomorphic near the origin in $\mathbb{C}_{\tau, x}^{d+1}$.

Next put $\tau = t^{1/p}$ and

\[ \bar{w}_N(\tau, x) = \sum_{n=0}^{\infty} u_{N+n+1}(x) \tau^{n+1}. \]

Then $\bar{w}_N(0, x) \equiv 0$, and by using the relation $t\partial_t = \frac{1}{p} \tau \partial_{\tau}$, we obtain $\bar{w}_N$ satisfies

\[ Q_e(x; u_0(x); \frac{1}{p} \tau \partial_{\tau} + \frac{N}{p}) \bar{w}_N = \tau \cdot G \left( \tau, x; \left( \left( \frac{1}{p} \tau \partial_{\tau} \right)^j \partial_x^a \bar{w}_N \right)_{(j, a) \in \Lambda} \right). \]

Equation (31) satisfies the conditions (13) and (14), and its characteristic polynomial is

\[ C(\rho, x) = Q_e(x; u_0(x); \frac{1}{p}(\rho + N)). \]

If we take $N$ sufficiently large, then $C(n, 0) \neq 0$ for all positive integers.
4 Prolongation of Solutions

We need to define a modified version of characteristic exponent.

**Definition 2.** For (16), we define $\sigma_c^*$ by

$$\sigma_c^* = \sup_{\mu \in \mathcal{M}, \nu \leq \mu, |\nu| \geq 2} \frac{\gamma_t(\nu) - m - k\mu}{|\nu| - 1}.\quad (33)$$

**Example 3.**

$$u_{tt} + 6uu_t^3 + xu_x^2 + uu_x = 0,$$

has a singular solution with exponent $\sigma_c = \frac{1}{3}$:

$$u = t^{1/3} - \frac{x}{12}t^{2/3} + \frac{x^2}{240}t^{3/3} + \frac{x^3}{5184}t^{4/3} - \cdots,$$

and ones with exponent $\sigma_c^* = \frac{1}{2}$:

$$u = \frac{1}{3}g^2 + \frac{1}{g}t^{1/2} + (-\frac{1}{2g^4} - \frac{x}{6g^2})t + \cdots,$$

where $g = g(x)$ is an arbitrary holomorphic function with $g(0) \neq 0$.

**Lemma 2.**

(i) $\sigma_c \leq \sigma_c^* \leq m_0 (\leq m - 1)$.

(ii) If $\sigma_c \leq 0$, then $\sigma_c = \sigma_c^*$.

**Definition 3.** For $\sigma \in \mathbb{R}$, we define $\delta_c(\sigma)$ by

$$\delta_c(\sigma) := \inf_{\mu \in \mathcal{M}, \nu \leq \mu, |\nu| \geq 2} (|\nu| - 1)|\nu| - \gamma_t(\nu) + m + k\mu$$

$$= \inf_{\mu \in \mathcal{M}, \nu \leq \mu, |\nu| \geq 2} (|\nu|\sigma - \gamma_t(\nu) + k\mu) - (\sigma - m).$$

**Lemma 3.**

(i) $\delta_c(\sigma) \geq 0$ if and only if $\sigma \geq \sigma_c^*$.

(ii) If $\sigma > \sigma_c^*$, then $\delta_c(\sigma) > 0$.

(iii) $\delta_c(m_0) > 0$.

(iv) If $\delta_c(\sigma) > 0$ and $\sigma \leq m_0$, then there is a constant $\delta > 0$ such that

$$|\nu|\sigma - \gamma_t(\nu) + k\mu \geq \sigma - m + \delta$$

for all $\mu \in \mathcal{M}, \nu \leq \mu$. 
**Definition 4.** $u \in \mathcal{O}(\Omega_{-})$ is bounded of order $\sigma$ in $\Omega_{-}$ means that $\exists M > 0$ such that for all $(t, x) \in \Omega_{-}$, if $\sigma \leq 0$

$$|u(t, x)| \leq M|\Re t|^\sigma$$

or if $\sigma > 0$,

$$|\partial_t^j u(t, x)| \leq \begin{cases} M & \text{for } j = 0, 1, \ldots, [\sigma], \\ M|\Re t|^{\sigma - j} & \text{for } j = [\sigma] + 1, \end{cases}$$

**Example 4.** $t^\sigma \cdot h(t, x)$ with a holomorphic function $h(t, x)$ is bounded of order $\sigma$, and $\log t \cdot h(t, x)$ is bounded of order $-\epsilon$ for any $\epsilon > 0$.

**Theorem 3.** If $u \in \mathcal{O}(\Omega_{-})$ satisfies Equation (16) and is bounded of order $\sigma$ in $\Omega_{-}$ with $\delta_\epsilon(\sigma) > 0$, then $u$ is holomorphic in a neighborhood of the origin. Especially if $\sigma > \sigma^*_c$ or $\sigma = m_0$, then $u$ is holomorphic near the origin.

**Corollary 1.** If $u \in \mathcal{O}(\Omega_{-})$ satisfies Equation (16) and the derivatives of $u$ up to order $m_0$ are bounded in $\Omega_{-}$, then $u$ is holomorphic near the origin.

**Remark 1.** Examples 1 and 2 give singular solutions which are bounded of order $\sigma_0 = \sigma^*_c$, and Example 3 gives ones of order $\sigma^*_c$ with $\sigma^*_c > \sigma_0$.

**References**


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