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Asymptotic expansion of singular solutions and characteristic polygon of linear partial differential equations with holomorphic coefficients

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Abstract

Consider the equation $P(z,\partial)u(z)=f(z)$ in a neighbourhood of $z=0$, where $u(z)$ admits singularities on the surface $K=\{z_0=0\}$ and $f(z)$ has the asymptotic expansion of Gevrey type with respect to z_0 as $z_0\to 0$. We study the possibility of asymptotic expansion of $u(z)$. We define the characteristic polygon of $P(z,\partial)$ with respect to K and characteristic indices γ_i $(0\leq i\leq p)$. We discuss the behaviour of $u(z)$ in a neighbourhood of K, by using these notions. The main result is a generalization of that in [2]. The details of this paper is in [4] and will be appeared elsewhere.

KEY WORDS: complex partial differential equations, solutions with asymptotic expansion

§1 Notations and Characteristic Polygon.

The coordinate of $\mathbb{C}^{d+1}$ is denoted by $z=(z_0,z')=(z_0,z_1,\cdots,z_d)\in\mathbb{C}\times\mathbb{C}^d$. $|z|=\max\{|z_i|;\ 0\leq i\leq d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\cdots,\xi_d)$. The differentiation is denoted by $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial')=(\partial_0,\partial_1,\cdots,\partial_d)$. $\alpha=(\alpha_0,\alpha')=(\alpha_0,\alpha_1,\cdots,\alpha_d)\in\mathbb{N}\times\mathbb{N}^d$ is a multi-index and $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^{n}\alpha_i$.

Let $P(z,\partial)=\sum_{|\alpha|\leq m}a_\alpha(z)\partial^\alpha$ be a linear partial differental operator with holomophic coefficients in a neighbourhood Ω of $z=0$ in $\mathbb{C}^{d+1}$ and $K=\{z_0=0\}$. Let us define the characteristic polygon Σ of $P(z,\partial)$ with respect to the surface K. Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0. Hence if $a_\alpha(z)\not\equiv 0$, $a_\alpha(z)=z_0^{j_\alpha}b_\alpha(z)$ with $b_\alpha(0,z')\not\equiv 0$. Put $e_\alpha=j_\alpha-\alpha_0$. We denote by $\Pi(a,b)$ the set $\{(x,y)\in\mathbb{R}^2;x\leq a,y\geq b\}$. The characteristic polygon of Σ is defined by $\Sigma=$ *the convex hull of* $\cup_\alpha\Pi(|\alpha|,e_\alpha)$. The boundary of Σ consists of a vertical half line $\Sigma(0)$, a horizontal half line $\Sigma(p)$ and $p-1$ segments $\Sigma(i)$ $(1\leq i\leq p-1)$ with slope γ_i, $0=\gamma_p<\gamma_{p-1}<\cdots<\gamma_1<\gamma_0=+\infty$.

Let $\{(k_i,e(i))\in\mathbb{R}^2;0\leq i\leq p-1\}$ be vertices of Σ, where $0\leq k_{p-1}<\cdots<k_i<k_{i-1}<\cdots<k_0=m$. So the endpoints of $\Sigma(i)$ $(1\leq i\leq p-1)$ are

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$(k_{i-1}, e(i-1))$ and $(k_i, e(i))$.

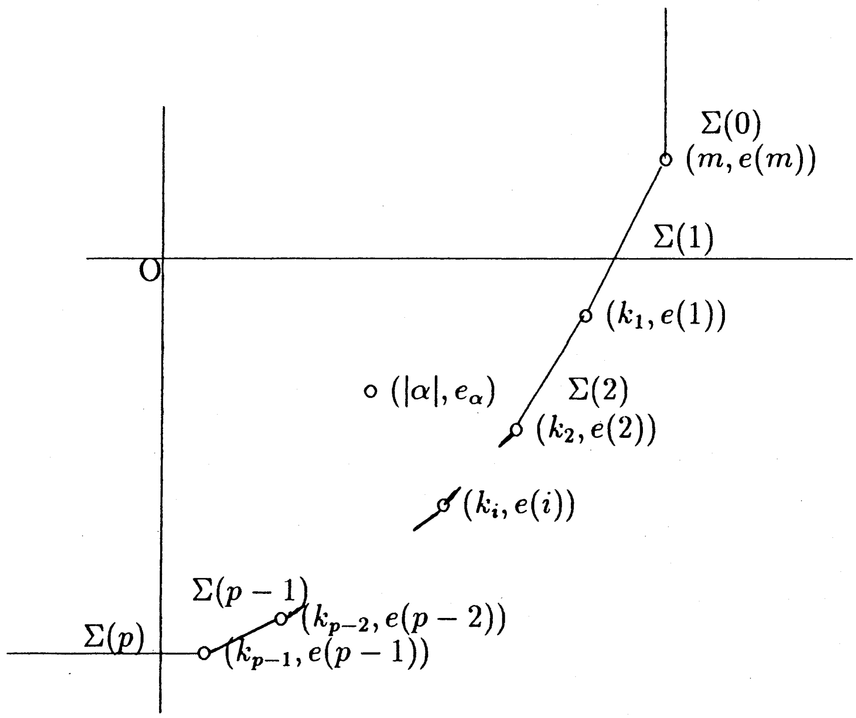

Figure 1: Characteristic polygon

Definition 1 *The slope* γ_i *of* $\Sigma(i)$ *is called the i-th characterisitic index of* $P(z, \partial)$ *with respect to* $K = \{z_0 = 0\}$.

Let us notice the vertices of the polygon Σ and define subsets $\Delta(i)$ and $\Delta_0(i)$ of multi-indices and operators $\mathfrak{P}_i(z, \partial)$ $(0 \leq i \leq p-1)$. Put

$$\begin{cases} \Delta(i) = \{\alpha \in \mathbb{N}^{d+1}; |\alpha| = k_i,\ j_\alpha - \alpha_0 = e(i)\}, \\ l_{k_i} = \max\{|\alpha'| : \alpha \in \Delta(i)\} \end{cases}$$

and

$$(1.1) \qquad \begin{cases} \Delta_0(i) = \{\alpha \in \Delta(i); |\alpha'| = l_{k_i}\}, \\ \mathfrak{P}_i(z, \partial) = \sum_{\alpha \in \Delta_0(i)} z_0^{j_\alpha} b_\alpha(0, z') \partial_0^{\alpha_0} \partial^{\alpha'}. \end{cases}$$

$\mathfrak{P}_i(z,\partial)$ is a partial differential operator with total order k_i and order l_{k_i} with respect to ∂'. We have $e(i) = j_\alpha - \alpha_0 = j_\alpha - k_i + l_{k_i}$ for $\alpha \in \Delta_0(i)$. Hence we can wirte

$$(1.2) \qquad \mathfrak{P}_i(z,\partial) = z_0^{e(i)+k_i-l_{k_i}}\Big(\sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\partial^{\alpha'}\Big)\partial_0^{k_i-l_{k_i}})$$

and define polynomial $\chi_{P,i}(z',\xi')$ in ξ' by

$$(1.3) \qquad \chi_{P,i}(z',\xi') = \sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\xi^{\alpha'}.$$

§2 Function spaces.

Let $\Omega = \Omega_0 \times \Omega'$ be a polydisk with $\Omega_0 = \{z_0 \in C^1;\ |z_0| < R\}$ and $\Omega' = \{z' \in C^d; |z'| < R\}$. Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\};\ |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega'),\ \mathcal{O}(\Omega(\theta)))$ is the set of all holomorphic functions on Ω (*resp.* Ω', $\Omega(\theta)$). We introduce subspaces of $\mathcal{O}(\Omega(\theta))$.

Definition 2 $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ $(0 < \kappa < +\infty)$ *is the set of all* $u(z) \in \mathcal{O}(\Omega(\theta))$ *such that for any* $\varepsilon > 0$ *and any* θ' *with* $0 < \theta' < \theta$

$$(2.1) \qquad |u(z)| \le C\exp(\varepsilon|z_0|^{-\kappa}) \quad \textit{for } z \in \Omega(\theta')$$

holds for a constant $C = C(\varepsilon,\theta')$. *We put* $\mathcal{O}_{(+\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$ *for* $\kappa = +\infty$.

Definition 3 $Asy_{\{\kappa\}}(\Omega(\theta))$ $(0 < \kappa \le +\infty)$ *is the set of all* $u(z) \in \mathcal{O}(\Omega(\theta))$ *such that for any* θ' *with* $0 < \theta' < \theta$ *and any* $N \in \mathbb{N}$

$$(2.2) \qquad |u(z) - \sum_{n=0}^{N} u_n(z')z_0^n| \le AB^N|z_0|^N\Gamma(\frac{N}{\kappa}+1) \quad z \in \Omega(\theta')$$

holds, where $u_n(z') \in \mathcal{O}(\Omega')$, $A = A(\theta')$ *and* $B = B(\theta')$.

We say that $u(z) \in Asy_{\{\kappa\}}(\Omega(\theta))$ has asymptotic expansion with Gevrey exponent κ in $\Omega(\theta)$. $u(z) \in Asy_{\{+\infty\}}(\Omega(\theta))$ means that $u(z)$ is holomorphic at $z = 0$.

§3 Theorem

First we give a condition on $P(z,\partial)$ treated in this paper.

Condition-i $j_\alpha = 0$ *for all* $\alpha \in \Delta_0(i)$.

If $P(z,\partial)$ satisfies Condition-i, then $\mathfrak{P}_i(z,\partial) = (\sum_{\alpha\in\Delta_0(i)} b_\alpha(0,z')\partial^{\alpha'})\partial_0^{k_i-l_{k_i}}$. We have

Theorem 4 *Suppose that* $P(z,\partial)$ *satisfies Condition-i and* $\chi_{P,i}(0,\hat{\xi}') \neq 0$, $\hat{\xi}' = (1,0,\cdots,0)$. *Let* $u(z) \in \mathcal{O}_{(\gamma_i)}(\Omega(\theta))$ *be a solution of*

$$P(z,\partial)u(z) = f(z) \in Asy_{\{\gamma_i\}}(\Omega(\theta)) \tag{3.1}$$

satisfying

$$\partial_1^h u(z_0,0,z'') \in Asy_{\{\gamma_i\}}(\Omega(\theta)\cap\{z_1=0\}) \quad for \quad 0 \le h \le l_{k_i}-1. \tag{3.2}$$

Then there is a polydisk W *centered at* $z=0$ *such that* $u(z) \in Asy_{\{\gamma_i\}}(W(\theta))$.

We studied in [1] and [2] similar problems for the case $i = p-1$ and $l_{k_{p-1}} = 0$. We gave in [2] a simple proof of the same result as Theorem 4 for this case. We show Theorem 4 by modifying the discussion in [2]. When Condition-i does not hold, solutions become less regular and we studied in [3] the behaviours of solutions under the condition that $i = p-1$ and $l_{k_{p-1}} = 0$ but Condition-(p-1) does not necessarily hold.

§4 Example

We give an example. Let us consider

$$P(z,\partial) = \partial_1^5 + \partial_1^3\partial_0 + \partial_0^2, \quad z = (z_0,z_1) \in \mathbb{C}^2. \tag{4.1}$$

We have

$$\begin{cases} \gamma_0 = +\infty, \quad \gamma_1 = 1, \quad \gamma_2 = 1/2, \quad \gamma_3 = 0, \\ \chi_{P,0}(z',\xi_1) = \xi_1^5, \quad \chi_{P,1}(z',\xi_1) = \xi_1^3, \quad \chi_{P,2}(z',\xi_1) = I. \end{cases}$$

Obviously $P(z,\partial)$ satisfies *Condition-i* and $\chi_{P,i}(z',1) \neq 0$ for $i = 0,1,2$. So it follows from Theorem 4 that there is a polydisk W centered at $z=0$ such that

$$i = 0 : u(z) \in \mathcal{O}_{(+\infty)}(\Omega(\theta)),\ \partial_1^h u(z_0, 0) \in Asy_{\{+\infty\}}(\Omega_0(\theta))\ (0 \le h \le 4),$$
$$f(z) \in Asy_{\{+\infty\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{+\infty\}}(W(\theta)),$$
$$i = 1 : u(z) \in \mathcal{O}_{(1)}(\Omega(\theta)),\ \partial_1^h u(z_0, 0) \in Asy_{\{1\}}(\Omega_0(\theta))\ (0 \le h \le 2),$$
$$f(z) \in Asy_{\{1\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1\}}(W(\theta)),$$
$$i = 2 : u(z) \in \mathcal{O}_{(1/2)}(\Omega(\theta)),\ f(z) \in Asy_{\{1/2\}}(\Omega(\theta)) \Rightarrow u(z) \in Asy_{\{1/2\}}(W(\theta)).$$

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