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<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1159: 81-86</td>
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<tr>
<td>Issue Date</td>
<td>2000-06</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64210">http://hdl.handle.net/2433/64210</a></td>
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Conjectures about the differential operators in an algorithm for computing the residues.

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Let $X = \mathbb{C}^2$ and fix a coordinate system $z = (x, y)$ of $X$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$. Let $f_1, f_2 \in \mathcal{O}_X$ and $(f_1, f_2)$ be a regular sequence. Denote by $I$ the sheaf of ideal of $\mathcal{O}_X$ generated by $f_1, f_2$. Put $A = \{z \in X | f_1 = f_2 = 0\}$. Assume that at least one zero has multiplicity greater than 1. We denote by $m$ the algebraic local cohomology class associated to the meromorphic function $1/f_1f_2$.

In [4], we gave an algorithm to compute the cohomology class $m$ and the residues. This algorithm has been constructed by the aid of the theory of $\mathcal{D}_X$-module and is based on the properties of the annihilators of $m$.

In this note, we examine the more detailed properties of annihilators which are useful for our algorithm. We use the computer algebra system Kan ([5]) and Risa/Asir ([2]).

1 The operators used in our algorithm

Let $\Omega_X$ be the sheaf of holomorphic differential form on $X$. We assume that the set of common zeros $A$ consists of finitely many points $A_1, \ldots, A_{\nu}$. There is a pairing

$$\text{Res}_{A_\nu} : \Omega_X/I\Omega_X \otimes \mathbb{C} \to \mathbb{C}.$$ 

For $m$, this pairing yields a unique linear mapping $\Omega_X/I\Omega_X \ni \phi(z)dz \mapsto \text{Res}_{A_\nu}(\phi(z)dz, m) \in \mathbb{C}$ defined by the residue of the differential form $\phi(z)dz/f_1f_2$ at $A_\nu$.

Put $V_K = \{\phi(z)dz \in \Omega_X/I\Omega_X | \text{Res}_{A_\nu}(\phi(z)dz, m) = 0, j = 1, \ldots, \nu\}$. Let $\mu_j$ be the multiplicity of $A_j$, $j = 1, \ldots, \nu$ and $\mu = \mu_1 + \cdots + \mu_\nu$. Then, $V_K$ can be regarded as $\mu - \nu$ dimensional vector space. Denote by $\mathcal{A}n$ the ideal generated by differential operators which annihilate $m$. Then we have the following theorem.

**Theorem 1**

$$V_K = \{(R^*\psi(z))dz | R \in \mathcal{A}n, \psi(z)dz \in \Omega_X/I\Omega_X\}.$$ 

Now we give conjectures about the properties of operators $P_1, \ldots, P_k \in \mathcal{A}n$ which we use in our algorithm for computing the residues.

**Conjecture (A)** There exist $P_1, \ldots, P_k \in \mathcal{A}n$ whose adjoint operators act on the vector space $\mathbb{C}[x, y]/I$ and $\text{Im}(P_1^*, \ldots, P_k^*)$ span $V_K$. where $\text{Im}(P_1^*, \ldots, P_k^*)$ stands for the set of images of the adjoint operators $P_j^*$, $j = 1, \ldots, k$ associated to $\mathbb{C}[x, y]/I$.

If there exist operators $P_1, \ldots, P_k \in \mathcal{A}n$ which satisfy the property in the conjecture (A), we have the following conjectures about construction of them.

**Conjecture (C1)** $P_j$'s are first-order differential operators.

Put $P_j = c_{j1}\partial_x + c_{j2}\partial_y + c_{j0}$ where $c_{j0}, c_{j1}, c_{j2} \in \mathbb{C}[x, y]$ and $\partial_x := \partial/\partial x$, $\partial_y := \partial/\partial y$.

**Conjecture (C2)** $(c_{11}, c_{12}, \ldots, c_{k1}, c_{k2}, f_1, f_2) = \sqrt{(f_1, f_2)}$ as the ideal of $\mathbb{C}[x, y]$.

**Conjecture (C3)** $(F_1, F_2, P_1, \ldots, P_k) = \mathcal{A}n$, where $F_j = f_j$, $j = 1, 2$ stands for differential operators of order 0.

**Conjecture (C4)** As for the number of first order differential operators, we have $1 \leq k \leq 2$. 
2 Illustration of conjectures

We use the following procedure to investigate the annihilators $P_j$, $j = 1, \ldots, k$.

(i) Construct annihilators of order zero and of order one.

(ii) Take the gröbner bases $GB$ of operators in (i).

(iii) Find first order operators which generate $GB$ together with 0th order operators. (we shall see the particular case in 2.2.2)

(iv) Verify the condition (1).

These computation can be carried by computer algebra system Kan and Risa/Asir.

2.1 The case $A = \{(0,0)\}$.

2.1.1 Example: $f_1 = x^5$, $f_2 = y^2 + x^4 + x^3$

In this case, $f_1$ and $f_2$ have common zero only at the origin with multiplicity 10.

(i) Computing syzygies on the ring of polynomials, we obtain

$F_1 = x^5$,  
$F_2 = y^2 + x^4 + x^3$, 

as annihilators of $m$ of order zero and

- $2yx \partial_x + (4x^4 + 3x^3) \partial_y - 10y$,  
- $2yx \partial_x + (x^2 + 4y^2) \partial_y + 16y$,  
- $(2x^2 + 2x) \partial_x + (4yx + 3y) \partial_y + 18x + 16$,  
- $2yx \partial_x + (-4x^4 - 3x^3) \partial_y + 10y$,  
- $(-2y^2 x + 2y^2) \partial_x + (4yx^4 - yx^3 - 3y^2) \partial_y - 10x^2 - 10y^2 x$,  
- $2yx \partial_x + (4x^4 + 3x^3) \partial_y - 10y$,  
- $(-2x^2 + 2x) \partial_x + 9y \partial_y - 10x + 48$, 

as annihilators of $m$ of order one (see Section 3).

(ii) The gröbner basis $GB$ of the ideal generated by these operators with respect to the lexicographic order $y > x$ is given by following 8 operators:

$F_1 = x^5$,  
$F_2 = y^2 + x^4 + x^3$,  
$P_1 = (-2x^2 + 2x) \partial_x + 9y \partial_y - 10x + 48$,  
$P_2 = x^5 \partial_x + 5x^2$,  
$P_3 = 2yx \partial_x + (-4x^4 - 3x^3) \partial_y + 10y$,  
$P_4 = 3x^4 \partial_x + (4x^4 + 24x) \partial_y - 20x + 30$,  
$P_5 = 9x \partial_x^2 + (-16x^2 + 12x + 54) \partial_y - 9x^2 \partial_y^2 - 80x + 60$,  
$P_6 = -x \partial_x^2 - 8 \partial_y^2 - 4x \partial_y \partial_x + (4x - 8) \partial_y^2$. 

(iii) We find that the operators $F_1$, $F_2$ and $P_1$ generate $GB$.

(iv) The ideal generated by $f_1$, $f_2$ and the coefficients of $\partial_x$ and $\partial_y$ in $P_1$ is equal to the radical of the ideal $I$, i.e. $(f_1, f_2, -2x^2 + 6x, 9y) = (x, y) = \sqrt{(f_1, f_2)}$.

In fact, we can see that the operator $P_1$ satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism $\Omega_X/I \Omega_X \cong C[x,y]/I$, these operators $P_j$, $j = 1, 2, 3$ act on the 10 dimensional vector space $C[x,y]/I$. Using the gröbner basis with respect to the lexicographic order $y > x$, the monomial basis $MB$ of $C[x,y]/I$ is $MB = \{1, y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4\}$.

Then $Im(P_1)$ is given by
\( P^*_1 = -6x + 33 \mod I, \)
\( P^*_y = -6y + 24y \mod I, \)
\( P^*_x = -4x^2 + 27x \mod I, \)
\( P^*_yx = -4yx^2 + 18yx \mod I, \)
\( P^*_x^2 = -2x^3 + 21x^2 \mod I, \)
\( P^*_yx^2 = -2yx^3 + 12yx^2 \mod I, \)
\( P^*_x^3 = 15x^3 \mod I, \)
\( P^*_yx^3 = 6yx^3 \mod I, \)
\( P^*_x^4 = 9x^4 \mod I, \)
\( P^*_yx^4 = 0 \mod I. \)

From this computation, it follows that \( \dim \text{Im}(P^*_1) = 9. \) The other side, \( \dim \text{Im}(P^*_j) < 9, j = 2, 3. \) Thus, we verify that the operator \( P_1 \) enjoys (A).

The functions \( f_1 \) and \( f_2 \) are semiquasihomogeneous polynomials of degree 10 and 6 with weights \( wt(x) = 2, \) \( wt(y) = 3. \) Put \( wt(\partial_x) = -2 \) and \( wt(\partial_y) = -3. \) Then the operator \( P_1 \) is the semiquasihomogeneous polynomial in \( C[x, y, \partial_x, \partial_y] \) with the quasihomogeneous part \( 3(2x\partial_x + 3y\partial_y + 10 + 6). \) The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of \( f_1 \) and \( f_2 \) as semiquasihomogeneous polynomials.

2.1.2 Example: \( f_1 = x^7, \) \( f_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6) \)

In this case, \( f_1 \) and \( f_2 \) have common zero only at the origin with multiplicity 14.

(i) Computing syzygies on the ring of polynomials, we obtain

\( F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6), \)

as annihilators of \( m \) of order zero and

\( \begin{align*}
-((2x^2 - 2yx^2)\partial_x + ((-5y^2 + 8y^2)x))\partial_y - 24x^2 - 30yx, \\
-((3x^2 + 36yx^2)\partial_x + (94yx^2 + 144yx)\partial_y + 447x^3 + 540yx, \\
-((16y + 37)x^2 - 20yx^2 + 16x)\partial_x + ((-24y^2 - 24yx^2 + (64y^2 + 94yx^2 - 80yx^2 + 40y))\partial_y - 48x^2 + (240y + 447)x^3 - 300yx + 192, \\
-((4y - 2)x^2 - 2yx)\partial_x + ((16y - 5yx - 8y^2))\partial_y + (60y^2 - 24)x - 30y, \\
yx^2\partial_y + 4y^3x\partial_y + 15x^3, \\
-((16y - 37)x^3 + (10y^3 + 55y^2x^2 - 26x))\partial_x + (39x^4 + 39y^3 + (-79y^2 - 94y)x^2 + (40y^2 + 220yx)x - 65y)\partial_y + 78x^4 + (-270y - 447)x^5 + (150y^2 + 825y)x - 312, \\
-((16y^2 + 57y)x^3 - 10x^2 + 6yx))\partial_x + ((-24yx^2 - 24y^2x^3 + (64y^3 + 147yx^2) - 25yx))\partial_y - 48yx^3 + 747x^2 + (480y^2 - 120)x + 42y, \\
-((4y - 2)x^2 - 2yx)\partial_x + ((16y - 5yx - 8y^2))\partial_y + (60y^2 - 24)x - 30y, \\
-((16y^2 - 57y)x^2 + (32y^3 + 114y^2 + 10)x^2 - 26yx + 12y))\partial_x + (24yx^2 - 24y^2x + (-112y - 174yx)x^2 + (128y^3 + 348y^2 + 25yx) - 50y))\partial_y - 84x^2 + 120yx + 747y^2x^2 + (-192yx^2 + 1410y^2 + 120x - 84y^3 - 28y^2, \\
-((16y - 57y)x^3 + (10x^2 - 6yx))\partial_x + (24yx^2 + 24y^2x^3 + (-64y^3 - 174yx^2)x^2 + 25yx))\partial_y + 48yx^3 + 747x^2 + (480y^2 - 120)x + 42y, \\
-((4y - 171y)x^3 + (-16y - 27yx)x^2 + (-18y + 10y)x - 6y^2)\partial_x + (72y^2 + 24y^2 + 24y^2x^3 + (192y^2 - 522yx^4 + 24y^4)x + (64y^4 - 99y^2)x + 25yx^2))\partial_y + 42x^2 + 84yx^3 + (-144y^3 - 345yx^2)x - 84y^3 + 120y
\end{align*} \)

as annihilators of \( m \) of order one.

(ii) The gr"obner basis \( GB \) of the ideal generated by these operators with respect to the lexicographic order \( y > x \) is given by following 10 operators

\( F_1 = x^7, \)
\( F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6), \)
\( P_1 = (21x^3 + 16x)\partial_x + ((-24x^4 + 40y)\partial_y + 147x^2 + 192, \)
\( P_3 = x^4\partial_y + 7x, \)
\( P_5 = y\partial_x + 4x^5\partial_y - 7x^2, \)
\( P_6 = -2yx\partial_x + 5x^5\partial_y + 36x^6 - 16x^4 - 14y, \)
\( P_7 = (-5x^2\partial_y - 24y^4)\partial_x + (96x^2 + 35yx)\partial_y - 168x, \)
\( P_8 = 4x^2\partial_x + (9x^3 + 40y)\partial_x + 162\partial_x + 63x^2 + 56, \)
\( P_9 = 3x^2\partial_x + 24y - 5x\partial_y + (-27x^2 + 36x^2)\partial_y, \)
\( P_{10} = 5x^2\partial_x + 45y^2 - 288x\partial_y + 25x^2\partial_y + (115x^2 + 90x^2 + 240x^2)\partial_y - 2016x. \)

(iii) We find that the operators \( F_1, F_2 \) and \( P_1 \) generate \( GB. \)
\( \text{(iv) Then, the ideal generated by } f_1, f_2 \text{ and the coefficients of } \partial_x \text{ and } \partial_y \text{ in } P_1 \text{ is equal to the radical of the ideal } I, \text{ i.e., } (f_1, f_2, 21x^3 + 16x, -24x^4 + 40y) = (x, y) = \sqrt{(f_1, f_2)}. \)

In fact, we can verify that the operator \( P_1 \) satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism \( \Omega_X/I\Omega_X \cong C[x, y]/I \), these operators \( P_j, j = 1, 2, 3, 4 \) act on the 14 dimensional vector space \( C[x, y]/I \). Using the gröbner basis with respect to the lexicographic order \( y > x \), we have \( MB = \{ y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4, x^5, yx^5, x^6, yx^6 \}. \) Then \( \text{Im}(P_1^*) \) is given by

\[
\begin{align*}
P_1^1 y &= 24x^4 + 84yx^2 + 96y \mod I, \\
P_1^1 x &= 63x^3 + 120x \mod I, \\
P_1^1 yx &= 24x^4 + 63yx^2 + 80yx \mod I, \\
P_1^1 x^2 &= 42x^4 + 104x^2 \mod I, \\
P_1^1 yx^2 &= 24x^6 + 52yx^4 + 64yx^2 \mod I, \\
P_1^1 x^3 &= 21x^5 + 88x^3 \mod I, \\
P_1^1 yx^3 &= 21yx^5 + 48yx^3 \mod I, \\
P_1^1 x^4 &= 72x^4 \mod I, \\
P_1^1 yx^4 &= 32yx^4 \mod I, \\
P_1^1 x^5 &= 56x^5 \mod I, \\
P_1^1 yx^5 &= 16yx^2 \mod I, \\
P_1^1 x^6 &= 40x^6 \mod I, \\
P_1^1 yx^6 &= 0 \mod I.
\end{align*}
\]

From this computation, it follows that \( \dim \text{Im}(P_1^*) = 13 \). The other side, \( \dim \text{Im}(P_1^*) < 13, j = 2, 3, 4 \).

The functions \( f_1 \) and \( f_2 \) are semiquasihomogeneous polynomials of degree 14 and 10 with weights \( wt(x) = 2, wt(y) = 5 \). Put \( wt(\partial_x) = -2 \) and \( wt(\partial_y) = -5 \). Then the operator \( P_1 \) is the semiquasihomogeneous polynomial in \( C[x, y, \partial_x, \partial_y] \) with quasihomogeneous part \( 8(2x\partial_x + 5y\partial_y + 14 + 10) \). The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of \( f_1 \) and \( f_2 \) as semiquasihomogeneous polynomials.

### 2.2 In the case that \( A \) consists of several points

#### 2.2.1 Example:

\( f_1 = (x^2 + y^2)^2 + 3x^2y - y^3, f_2 = x^2 + y^2 - 1 \)

In this case, \( A = \{(0, 1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)\} \) with multiplicities 2 at each points.

(i) Computing syzygies on the ring of polynomials, we obtain

\( F_1 = 16x^2 - 24x^4 + 9x^2, F_2 = 4x^4 - 5x^2 - y + 1 \)

as annihilators of \( m \) of order zero and

- \((x^2 + y^2 - 1)\partial_y + 2y, \)
- \((x^2 + y^2 - 1)\partial_x + 2x, \)
- \((2y^2 - 2y + 2x + y - 1)x\partial_y + 6y^2 + 3y - 3, \)
- \((2y^2 - 2y + 2x + y - 1)x\partial_y + (-2y^2 + y + 1)x\partial_y + (-6y + 3)x, \)
- \((2y^2 - y)^2 - y\partial_y + (-2y^2 + y + 1)x\partial_y + 3y + 3, \)
- \((-2y^2 - y)x\partial_y + (2y + 1)x^2\partial_y - 6y^2 - 2y - 3, \)
- \((2y + 1)x\partial_y + (-2x^2 - 4y^2 + y + 3)\partial_y - 6y + 5, \)

as annihilators of \( m \) of order one.

(ii) The gröbner basis \( GB \) of these operators with respect to the lexicographic order \( y > x \) is given by following 6 operators:

\( F_1 = 16x^2 - 24x^4 + 9x^2, F_2 = 4x^4 - 5x^2 - y + 1 \)

\( P_1 = (4x^2 - 3x)\partial_y + (8x^4 - 6x^2)\partial_y - 16x^4 + 36x^2 - 6, P_2 = (-16x^4 + 24x^3 - 9x)\partial_y - 96x^4 + 96x^2 - 18, \)

\( P_3 = (8x^4 - 6x^2)\partial_y + ((12x^3 - 9x)\partial_y + 64x^3 - 12x)\partial_y + (48x^2 - 18)\partial_y + 96x^2 + 12, P_4 = (4x^2 - 3x)\partial_y + (48x^2 - 12)\partial_y^2 + ((12x^3 + 9x)\partial_y^2 + (24x^2 - 30)\partial_y + 144x)\partial_y + ((-48x^2 - 18)\partial_y^2 + 96x^2 - 60)\partial_2 + 96. \)

(iii) We find that the operators \( F_1, F_2 \) and \( P_1 \) generate \( GB \).
(iv) The ideal generated by $f_1, f_2$ and the coefficients of $\partial_x$ and $\partial_y$ in $P_1$ is equal to the radical of the ideal $I$, i.e., $(f_1, f_2, 4x^3 - 3x, 8x^3 - 6x^2) = (4x^3 - 3x, 2x^2 + y - 1) = \sqrt{(f_1, f_2)}$.

In fact, we can verify that the operator $P_1$ satisfies the property in the conjecture (A) and the other first order operators are not as follows. Under the isomorphism $\Omega X/\mathcal{I} \Omega X \cong \mathbb{C}[x, y]/I$, the operators $P_j$, $j = 1, 2$ act on the 6 dimensional vector space $\mathbb{C}[x, y]/I$. Using the gröbner basis with respect to the lexicographic order $y > x$, we have $MB = \{1, x, x^2, x^3, x^4\}$. Then $\text{Im}(P_1^*)$ is given by

$P_1^*x^3 = -16x^4 + 24x^3 - 3 \mod I$,  
$P_1^*x^2 = -16x^4 + 20x^3 \mod I$,  
$P_1^*x^2 = -8x^4 + 12x^3 \mod I$,  
$P_1^*x^2 = -12x^4 + 15x^3 \mod I$,  
$P_1^*x^2 = -6x^4 + 9x^3 \mod I$,  
$P_1^*x^2 = -9x^4 + 45/4x^3 \mod I$.

From this computation, it follows that $\text{dim Im}(P_1^*) = 3(= 6 - 3)$. The other side, $\text{dim Im}(P_2^*) = 1 < 3$.

Put $I_1 = \{(4x^2 - 3)^2, 4x^3 - 4y - 5\}$ and $I_2 = \{(x^2, y - 1)\}$. Then $I_1 = I_1 \cap I_2$. Let $m_1$ be the cohomology class with support at $V(I_1)$ and $m_2$ the cohomology class with support at $V(I_2)$ which satisfy $m = m_1 + m_2$. From the ideals $(4x^2 - 3)^2, 4x^3 - 4y - 5, P_1)$ and $(x^2, y - 1, P_1)$, we obtain $R_1 = \langle 12x^2 y + 6x \partial_x + (18y + 9) \partial_y + 12y + 42 \rangle$ as an annihilator of first order of $m_1$ and $R_2 = x \partial_x + 2$ as an annihilator of first order of $m_2$. These operators satisfy the localization of the property in the conjecture (A) to $\mathcal{O}_X/I_1, j = 1, 2$.

2.2.2 Example: $f_1 = x^6 + (y^2 - 3)x^4 + (y^2 + y + 3)x^2 + y^6 - y^4 + y^2 - 1$, $f_2 = x^6 + (3y^2 - 3)x^4 + (3y^2 + 3)x^2 + y^6 - y^4 + 3y^2 - 1$

In this case, $A$ consists of $\{(x, y)|x^2 - y^2 + 3x^2 - x^2 - 1 = x^6 + 2x^2 - y^2 = 0\}$ with multiplicity $1$, $(0, 1)$ with multiplicity $2$, $(0, -1)$ with multiplicity $6$, and $(-1, 0)$ with multiplicity $6$.

(i) Computing syzygies on the ring of polarizations, we obtain $F_1 = -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 + 3y^2 + 3$, $F_2 = x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2$ as annihilators of $m$ of order zero and 26 operators of order one.

(ii) The gröbner basis $GB$ of these operators with respect to the lexicographic order $y > x$ is given by following 5 operators:

$F_1 = -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 + 3y^2 + 3$,  
$F_2 = x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2$,  
$F_3 = (x^{16} + 4x^{14} - 9x^{12} + 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2$,  
$P_2 = (yx^{10} - y^2x^9 + 3yx^8 - yx^7 + y^2x^6 - y^3x^5 + y^4x^4 - y^5x^3 + y^6x^2 - y^7x^1 + y^8x^0 + 3yx^9 + y^2x^8 - yx^7 + y^3x^6 - y^4x^5 + y^5x^4 - y^6x^3 + y^7x^2 - y^8x^1 + y^9x^0)$,  
$P_3 = (yx^{10} - y^2x^9 + 3yx^8 - yx^7 + y^3x^6 - y^4x^5 + y^5x^4 - y^6x^3 + y^7x^2 - y^8x^1 + y^9x^0 + 3yx^9 + y^2x^8 - yx^7 + y^3x^6 - y^4x^5 + y^5x^4 - y^6x^3 + y^7x^2 - y^8x^1 + y^9x^0)$

(iii) In this case, we need four operators $F_1$, $F_2$, $P_1$, and $P_2$ to generate $GB$.

(iv) Then the ideal generated by $f_1$, $f_2$, and the coefficients of $\partial_x$ and $\partial_y$ in $P_1$ and $P_2$ is equal to the radical of the ideal $I$, i.e., $(f_1, f_2, 13x^{11} + 26x^9 - 52x^7 + 52x^5 - 26x^3 + 13x, yx^{10} - y^8x^8 + 3yx^6 + y^4x + y^2x^2) = (-x^{11} + 2x^9 - 4x^7 + 4x^5 - 2x^3 + x, -yx^9 + yx^7 - 3yx^5 + yx^3 - yx, -2x^{10} + 3x^8 - 6x^6 + 5x^4 - x^2 - y^2 + 1) = \sqrt{(f_1, f_2)}$.

In fact, we can verify that the operators $P_1$ and $P_2$ satisfy the property in the conjecture (A). Under the isomorphism $\Omega X/\mathcal{I} \Omega X \cong \mathbb{C}[x, y]/I$, the operators $P_1$ and $P_2$ act on the 32 dimensional vector space $\mathbb{C}[x, y]/I$. And it follows that the vector space $\text{Im}(P_1^*, P_2^*)$ is 12 dimension.

Put $I_1 = (x^4 + (y^2 + 1)x^2 - y^2 + 1, 2x^4 - x^2 + y^2 + 2, x^4 + 2x^2 - y^2)$, $I_2 = (x^2, y - 1)$, $I_3 = (x^2, y + 1)$, $I_4 = ((x - 1)^3 y^2), I_5 = ((x + 1)^3 y^2)$. Then $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5$. Let $m_1$ be the cohomology class with support at $V(I_j)$, $j = 1, 2, 3, 4, 5$, which satisfy $m = m_1 + m_2 + m_3 + m_4 + m_5$. From the ideals generated by $P_1$, $P_2$, and $I_j$, we obtain the annihilators of each $m_j$. For $m_3$ and $m_5$, we have $x \partial_x + 2$. Concerning to $m_4$, we have $\langle (x - 1)^3 y^2, (12x - 12) \partial_x - x^2 - 44x - 7, y \partial_y + 2 \rangle$ as annihilators of $m_4$. Note that $I_4$ is generated by $(x + 1)^3 y^2$ and $y^3$, both are univariate polynomials. For such a case, we need two first order differential operators. In the same way, we have $\langle (x + 1)^3 y^2, (12x + 12) \partial_x - x^2 - 44x - 7, y \partial_y + 2 \rangle$ as
annihilators of $m_5$. Note that since the ideal $I_1$ is simple, $m_1$ does not require any first order differential operators.

3 Construction of annihilators of first order

We can find annihilators of first order by the computations of syzygies. Put $P = a \partial_x + b \partial_y + c$ where $a, b, c \in \mathbb{C}[x, y]$. If there exist $u_{11}, u_{12}, u_{21}$ and $u_{22}$ which satisfy $-af_{1x} - bf_{1y} = u_{11}f_1 + u_{12}f_2$ and $-af_{2x} - bf_{2y} = u_{21}f_1 + u_{22}f_2$, $P$ annihilates the cohomology class associated to the meromorphic function $1/f_1f_2$ with $c = -u_{11} - u_{22}$. In other words, $(a, b, u_{11}, u_{12}, u_{21}, u_{22})$ is a syzygy of $\begin{pmatrix} -f_{1x} \\ -f_{2x} \end{pmatrix}$, $\begin{pmatrix} -f_{1y} \\ -f_{2y} \end{pmatrix}$, $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} f_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ f_1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ f_2 \end{pmatrix}$. Thus, we can obtain the first order differential operators annihilating the cohomology class $m$ with respect to the given meromorphic function by using Kan. This observation is due to T. Oaku ([3]) and the algorithm has been implemented by him.

If these conjectures are right, we can compute the algebraic local cohomology group as left $D_X$-module without any information on the $b$-function. Then, we will be able to obtain more efficient algorithm for computing the residues.

References


