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A localization algorithm for $D$-modules and its application

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First, we review the localization algorithm given in a joint paper with N. Takayama (Kobe) and U. Walther (Minnesota/MSRI) \cite{6} with slightly different reasoning of the correctness. The latter part applies this algorithm to the problem of finding the annihilator ideal of some elementary functions.

\section{A localization algorithm}

We work entirely in the algebraic category. Put $X = \mathbb{C}^n$ and $Y = \{x = (x_1, \ldots, x_n) \in X \mid f(x) = 0\}$ with a nonzero polynomial $f \in \mathbb{C}[x]$. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators on $X$. Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module on $X$ such that $\mathcal{M}$ is holonomic on $X \setminus Y$. Then Kashiwara (\cite{2}) proved that the localization $\mathcal{M}[1/f] := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[1/f]$ of $\mathcal{M}$ by $f$ is a holonomic $\mathcal{D}_X$-module on $X$, where $\mathcal{O}_X$ is the sheaf of regular functions on $X$. (In fact he proved this fact in the analytic category, which is a stronger statement.) Here we remark that starting from an algebraic (i.e., a $\mathcal{D}_X$-) module, the localization is the same both in the algebraic and in the analytic category. More precisely, if we denote by $\mathcal{O}_X^{an}$ and $\mathcal{D}_X^{an}$ the sheaves of holomorphic functions and of holomorphic differential operators on $X$ respectively, then we have an isomorphism

$$\mathcal{O}_X^{an}[1/f] \otimes_{\mathcal{O}_X^{an}} (\mathcal{D}_X^{an} \otimes_\mathcal{D}_X \mathcal{M}) \simeq \mathcal{D}_X^{an} \otimes_\mathcal{D}_X \mathcal{M}[1/f].$$

Our claim is that $\mathcal{M}[1/f]$ is computable if the input, i.e. both $\mathcal{M}$ and $f$ are defined over a computable subfield of $\mathbb{C}$ (e.g. over $\mathbb{Q}$).

Now let us explain our algorithm. Introducing an auxiliary variable $t$, put

$$W := \{(t, x) \in \mathbb{C} \times X \mid tf(x) = 1\}$$

and let $\iota : W \rightarrow \mathbb{C}^{n+1}$ be the natural embedding. Let $p : W \rightarrow X$ be the projection $p(t, x) = x$ and $\varphi : X \setminus Y \rightarrow W$ be the isomorphism defined by $\varphi(x) = (1/f(x), x)$. Let
$j : X \setminus Y \longrightarrow X$ be the natural embedding. Thus we have a commutative diagram

\[
\begin{array}{ccc}
W & \overset{\iota}{\longrightarrow} & \mathbb{C} \times X \\
\varphi \downarrow & & \downarrow p \\
X \setminus Y & \overset{j}{\longrightarrow} & X
\end{array}
\]

Then by using the integration functor (in the algebraic category) we get

\[
\mathcal{M}[1/f] = \int_{j} j^{-1} \mathcal{M} = \int_{p} \int_{\iota} \int_{\varphi} j^{-1} \mathcal{M}.
\]

(See e.g. [7].) Our algorithm simply performs the rightmost successive integration step by step. (For the sake of simplicity we describe the algorithm in the case where $\mathcal{M}$ is generated by one element.) We denote by $D_n = D_n(\mathbb{C})$ the Weyl algebra on $n$ variables $x$ with coefficients in $\mathbb{C}$. Then we can regard $D_n$ as the set of global sections $\Gamma(X, D_X)$ of $D_X$. In general, for a left coherent $D_X$-module $\mathcal{M}$, its global sections $M := \Gamma(X, \mathcal{M})$ is a finitely generated left $D_n$-module and its correspondence yields an equivalence of the category of left coherent $D_X$-modules on $X$ and that of finitely generated left $D_n$-modules since $X$ is affine. The converse correspondence is given by the sheafification (or the ‘localization’) $\mathcal{M} = D_X \otimes_{D_n} M$.

**Algorithm-Theorem 1**

**Input:** A polynomial $f \in \mathbb{Q}[x]$ and a finite subset $\{P_1, \ldots, P_r\}$ of $D_n(\mathbb{Q})$ which generates a left ideal $I$ of $D_n$ such that the sheafification $\mathcal{M}$ of $M := D_n/I$ is holonomic on $X \setminus Y$.

1. (a) Put $\vartheta_i := \partial_i - t^2 f_i \partial_t$ with $f_i := \partial f/\partial x_i$, $\partial_i := \partial/\partial x_i$, and $\partial_t := \partial/\partial t$.

   (b) Compute $\tilde{P}_i := P_i(x, \vartheta_1, \ldots, \vartheta_n)$. More precisely, Writing $P_i$ in the normal form (i.e. make the derivations first and then multiply by polynomials) and substitute $\vartheta_i$ for $\partial_i$ in $P_i$. (Note that $\vartheta_1, \ldots, \vartheta_n$ commute with one another.)

   (c) Let $J$ be the left ideal of $D_{n+1}$ generated by $\tilde{P}_1, \ldots, \tilde{P}_r$ and $1 - tf(x)$ and put $N := D_{n+1}/J$.

2. Compute $N/\partial_t N$ as left $D_n$-module as follows:

   (a) Let $G$ be an involutive basis of $J$ with respect to the weight vector

   \[ w = (1, 0, \ldots, 0; -1, 0, \ldots, 0) \]

   for $(t, x_1, \ldots, x_n; \partial_t, \partial_1, \ldots, \partial_n)$.

   (b) Compute a generator $b(s)$ of the ideal

   \[ \{b(s) \in \mathbb{C}[s] \mid b(t \partial_t) + Q \in J \text{ with some } Q \in D_{n+1} \text{ such that } \operatorname{ord}_w(Q) \leq -1} \]
of $\mathbb{C}[s]$, where $\text{ord}_w(Q)$ denotes the maximum weight of the terms of $Q$ with respect to the weight $w$. $b(s)$ can be computed by eliminating $x$ and $\partial_1, \ldots, \partial_n$ from the highest weight parts (w.r.t. $w$) of elements of $G$. Find the largest nonnegative integer root $k_1$ of $b(s) = 0$. If there is no nonnegative integer root, then we have $M[1/f] = 0$.

(c) In general, for $P \in D_{n+1}$, there exist unique $Q \in D_{n+1}$ and $R \in D_n[t]$ such that

$$P = \partial_t Q + R.$$ 

Let us denote this $R$ by $R = \rho(P)$. Then $R$ can be regarded as a relation among the residue classes $\overline{1}, \overline{t}, \overline{t^2}, \ldots$ in $N/\partial_t N$. Let $L$ be the left $D_n$-submodule of $D_n + tD_n + \cdots + t^{k_1}D_n \cong D_n^{k_1+1}$ generated by

$$\{\rho(t^j P) \mid P \in G, \text{ord}_w(P) + j \leq k_1\}.$$ 

Then $L$ defines a system of linear differential equations for $\overline{1}, \overline{t}, \cdots \overline{t^{k_1-1}}$ in $N/\partial_t N$.

(d) Eliminate $\overline{1}, \overline{t}, \cdots \overline{t^{k_1-1}}$ from $L$ and obtain an ideal $L_0$ of $D_n$ which annihilates $\overline{t^{k_1}}$.

Output: $\mathcal{M}[1/f]$ is isomorphic to the sheafification of $D_n/L_0$. More precisely the ideal $L_0$ is the annihilator ideal of $f^{-k_1-2}u$, which generates $M[1/f]$ (here $u$ is the residue class of 1 in $M$).

Proof: First we have

$$j^{-1} \mathcal{M} = D_X[1/f]/(D_X[1/f]P_1 + \cdots + D_X[1/f]P_r),$$

which is a holonomic $D_X[1/f]$-module. Let $A_W$ be the subring of $D_{n+1}$ generated by $\mathbb{C}[t, x]$ and $\partial_1, \ldots, \partial_n$. Then $A_W(tf(x) - 1)$ is a two-sided ideal of $A_W$ and $D_W := A_W/A_W(1 - tf(x))$ is the set of global sections of the sheaf $D_X$ of algebraic differential operators on $W$ (note that $W$ is affine). Then we have an isomorphism (see [4])

$$\int_{\varphi} j^{-1} \mathcal{M} \cong D_W / (D_W \tilde{P}_1 + \cdots + D_W \tilde{P}_r).$$

Next, the integration along $\iota$ is nothing but the so-called Kashiwara equivalence and in view of Proposition A.1 of [4] we have

$$\int_{\iota} \int_{\varphi} j^{-1} \mathcal{M} = D_{\mathbb{C} \times X} \otimes_{D_{n+1}} N,$$

which is a holonomic $D_{\mathbb{C} \times X}$-module. Next by the definition of the integration we have

$$\int_p N = N/\partial_t N.$$
Let $\mathcal{F}(J)$ be the partial Fourier transform of $J$ with respect to $t$, which is the ring isomorphism of $D_{n+1}$ that sends $t$ to $-\partial_t$, $\partial_t$ to $t$, and leaves $x_i$, $\partial_i$ unchanged. Put $\mathcal{F}(N) := D_{n+1}/\mathcal{F}(J)$. Then we have

$$N/\partial_t N \simeq \mathcal{F}(N)/t\mathcal{F}(N)$$

and the step (2) is nothing but (the Fourier transform of) the restriction algorithm (Theorem 5.7) of [5]. Note that $N/\partial_t N$ is a holonomic $D_n$-module since $N$ is holonomic on $\mathbb{C} \times X$. Thus we have proved that

$$M[1/f] \simeq N/\partial_t N \simeq (D_n)^{k_1+1}/L.$$  \hspace{1cm} (1)

Let us describe the first isomorphism of (1) more explicitly. First note the isomorphism

$$N/\partial_t N \simeq D_{n+1}/(J + \partial_t D_{n+1}).$$

For an arbitrary $P \in D_{n+1}$, there exist unique $R_0, R_1, \ldots \in D_n$ and $S \in D_{n+1}$ such that

$$P = \sum_{j \geq 0} t^j R_j(x, \theta_1, \ldots, \theta_n) + \partial_t S. \hspace{1cm} (2)$$

Then we define

$$\psi(P) := \sum_{j \geq 0} f^{-j-2} R_j(x, \partial_1, \ldots, \partial_n) u$$

$$= \sum_{j \geq 0} R_j \left( x, \partial_1 + (j + 2) \frac{f_1}{f}, \ldots, \partial_n + (j + 2) \frac{f_n}{f} \right) f^{-j-2} u \in M[1/f].$$

Note that since the commutation relation of $x_i$ and $\theta_i$ is the same as that of $x_i$ and $\partial_i$, the above (non-commutative) substitution makes sense irrespective of the actual expression of $R_j$. This defines a left $D_n$-homomorphism $\psi : D_{n+1} \rightarrow M[1/f]$. In fact, we have $\psi(\partial_i P) = \partial_i \psi(P)$ since

$$\psi(\partial t^j R_j(x, \theta_1, \ldots, \theta_n)) = \psi(t^j(\partial_1 + t^2 f_1, \partial_1) R_j(x, \partial_1, \ldots, \partial_n))$$

$$= \psi(t^j(\partial_1 + (j + 2) t f_1) + \partial_t t^{j+2} f_1) R_j(x, \partial_1, \ldots, \partial_n))$$

$$= \psi(t^j(\partial_1 - (j + 2) t f_1) R_j(x, \partial_1, \ldots, \partial_n))$$

$$= f^{-j-2} \partial_t R_j(x, \partial_1, \ldots, \partial_n) u - (j + 2) f^{-j-3} f_i R_j(x, \partial_1, \ldots, \partial_n)$$

$$= \partial_t f^{-j-2} R_j(x, \partial_1, \ldots, \partial_n) u.$$

Since $P_1, \ldots, P_r$ annihilate $u$, we get

$$\psi(t^j \tilde{P}_i) = f^{-j-2} P_i(x, \partial_1, \ldots, \partial_n) u = 0.$$
It is easy to see that $\psi(t^j(1-tf(x)) = 0$. Hence $J + \partial_tD_{n+1}$ is contained in the kernel of $\psi$.

Conversely, suppose that $P$ of the form (2) is contained in the kernel of $\psi$. Then there exist $Q_1(t, x, \partial), \ldots, Q_r(t, x, \partial) \in D_n[t]$ such that

$$\sum_{j\geq 0} t^j R_j(x, \partial_1, \ldots, \partial_n) = \sum_{i=1}^r f^2 Q_i(1/f, x, \partial_1, \ldots, \partial_n) P_i(x, \partial_1, \ldots, \partial_n)$$

holds in $D_n[1/f]$. Then the Hilbert Nullstellensatz assures that

$$\sum_{j\geq 0} t^j R_j(x, \partial_1, \ldots, \partial_n) - \sum_{i=1}^r f^2 Q_i(1/f, x, \partial_1, \ldots, \partial_n) P_i(x, \partial_1, \ldots, \partial_n) \in (1-tf(x))D_{n+1}$$

since $1-tf(x)$ is irreducible. Noting $(1-tf(x))\partial_i \in D_{n+1}(1-tf(x))$, we conclude that $P \in J + \partial_tD_{n+1}$. Thus $\psi$ gives the first isomorphism of (1). This implies that $M[1/f]$ is generated by $f^{-2}u, \ldots, f^{-k_1-2}u$, and hence only by $f^{-k_1-2}u$. This completes the proof.

2 An application to holonomic functions

Let $u$ be a (possibly multivalued) analytic function defined on $\mathbb{C}^n$ minus an algebraic set. Suppose that $u$ is hyperexponential ([1]); i.e., $g_i := \partial_i u/u$ is a rational function for any $i = 1, \ldots, n$. For example, if $f_1, \ldots, f_m, g$ are rational functions and $\alpha_1, \ldots, \alpha_m$ are complex numbers, then

$$u = f_1^{\alpha_1} \cdots f_m^{\alpha_m} \exp(g(x))$$

is a hyperexponential function. Then we can find the annihilator ideal

$$\text{Ann}(u) := \{P \in D_n \mid Pu = 0\}$$

of $u$ exactly by applying the localization algorithm.

**Algorithm-Theorem 2**

Input: Let $u$ be a (possibly) multi-valued analytic function such that $g_i := \partial_i u/u \in \mathbb{Q}(x)$ for any $i = 1, \ldots, n$.

1. Let $g \in \mathbb{Q}[x]$ be the least common multiple of the denominators of $g_1, \ldots, g_n$. Let $f(x)$ be the square-free part of $g$.
2. Put $I := D_n(g\partial_1 - g_1) + \cdots + D_n(g\partial_n - g_n)$.
3. Apply Algorithm-Theorem 1 with input $D_n/I$ and $f$, and let $L_0$ be the output ideal with the integer $k_1$. 


(4) Compute the ideal quotient

\[ L_1 := L_0 : (f^{k_1+2}) = \{ P \in D_n \mid Pf^{k_1+2} \in L_0 \} \]

by syzygy computation through Gröbner basis.

Output: \( L_1 = \text{Ann}(u) \). In particular, \( u \) is a holonomic function, i.e., \( D_n/\text{Ann}(u) \) is a holonomic system.

**Proof:** Put \( \mathcal{L} := D_X u \), which is a sheaf of multivalued analytic functions, and define the sheaf

\[ \text{Ann}(u) := \{ P \in D_X \mid Pu = 0 \} \]

which is the sheafification of \( \text{Ann}(u) \). Then we have \( \mathcal{L} \cong D_X / \text{Ann}(u) \). Let \( \mathcal{M} \) be the sheafification of \( M = D_n/I \). It is easy to see that \( \mathcal{M} \) is a holonomic system of rank one outside of \( Y := \{ x \in X = \mathbb{C}^n \mid f(x) = 0 \} \). This implies that the two sheaves \( \mathcal{M} \) and \( \mathcal{L} \) coincide on \( X \setminus Y \). Hence in view of the Hilbert Nullstellensatz, we have

\[ \mathcal{M}[1/f] = \mathcal{L}[1/f] \]  

(3)

By Algorithm-Theorem 1, \( M[1/f] \) is generated by \( f^{-k_1-2} \overline{1} \) whose annihilator ideal is \( L_0 \). Hence \( L_1 \) is the annihilator ideal of \( \overline{1} \) in \( M[1/f] \).

On the other hand, since \( \mathcal{L} \) is a set of analytic functions, the natural homomorphism

\[ \mathcal{L} \longrightarrow \mathcal{L}[1/f] = \mathcal{O}_X[1/f] \otimes_{\mathcal{O}_X} \mathcal{L} \]

induced by the embedding of \( \mathcal{O}_X \) to \( \mathcal{O}_X[1/f] \) is injective. In fact, this follows from the fact that \( f \cdot : \mathcal{L} \longrightarrow \mathcal{L} \) is injective. By the isomorphism (3), \( \overline{1} \in M[1/f] \) corresponds to \( u \in \mathcal{L} \), and its annihilator ideal in \( \mathcal{L}[1/f] \) is given by \( L_1 \). Since \( \mathcal{L} \) is a submodule of \( \mathcal{L}[1/f] \), the annihilator ideal of \( u \) in \( \mathcal{L}[1/f] \) coincides with that in \( \mathcal{L} \). This implies that \( L_1 = \text{Ann}(u) \). This completes the proof.

**Example 1** Put \( X := \{(x, y, z) \in \mathbb{C}^3 \} \) and \( f(x, y, z) := x^3 - y^2 z^2 \). Let us find the annihilator ideal of the function \( u := \exp(1/f(x)) \). The following computations were performed by computer algebra systems kan/sm1 [9] and Risa/Asir [8] which are connected via open xxx protocol [10]. First let \( I \) be the left ideal of \( D_3 \) generated by

\[ f^2 \partial_z - f_z, \quad f^2 \partial_y - f_y, \quad f^2 \partial_x - f_x \]

with \( \partial_x = \partial/\partial x, \ f_x = \partial f/\partial x \), and so on. By computing the characteristic variety \( \text{Char}(M) \) of \( M := D_3/I \) (see [3] for an algorithm) and by decomposing it to prime (or primary) factors, we know that

\[ \text{Char}(M) \supset \{(x, y, z; \xi, \eta, \zeta) \in T^*\mathbb{C}^3 \mid x = y = 0 \} \cup \{x = z = 0 \}. \]
In particular, $M$ is not holonomic on $\mathbb{C}^3$. Next by using Algorithm-Theorem 1 with $I$ and $f$ as input, we know that $\text{Ann}(u)$ is generated by the following eight operators:

\begin{align*}
36y\partial_y - 36z\partial_z, \\
-24yz^2\partial_x - 36x^2\partial_y, \\
-24y^2z\partial_x - 36x^2\partial_z, \\
-24z^3\partial_x\partial_z - 36x^2\partial_y^2 - 24z^2\partial_x, \\
24x^4\partial_x^2 + 72x^3z\partial_x\partial_z + 54x^2z^2\partial_y^2 + 96x^3\partial_x + 162x^2z\partial_z + 72\partial_z, \\
36y^2z^3\partial_y - 24x^4\partial_x - 72x^3z\partial_x\partial_y - 72, \\
-36y^4\partial_x^2 + 24x^4\partial_x\partial_y + 72x^3z\partial_y\partial_z - 108y^3z\partial_z + 72\partial_y, \\
36z^5\partial_z^2 - 24x^4\partial_x\partial_y^2 - 72x^3z\partial_y\partial_z + 216z^4\partial_x^2 + 216z^3\partial_y - 72\partial_y^2. \\
\end{align*}

We can verify that $D_3/\text{Ann}(u)$ is in fact holonomic and that $I$ is contained in $\text{Ann}(u)$.

### 参考文献


