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ON THE ANALYTICITY AND GEVREY REGULARITY OF SOLUTIONS OF SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH MULTIPLE CHARACTERISTICS

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The aim of this paper is to present new results on the analyticity and Gevrey regularity of solutions of semilinear partial differential equations with multiple characteristics. First let us recall some historical fact in question. The study of the analyticity and Gevrey regularity of solutions of non-linear elliptic equations and systems was initiated by a conjecture of Hilbert. The conjecture states that every solution of an elliptic equation (non-linear) is analytic provided the data is analytic. This conjecture was solved by Bernstein for second order equations in two variables [1], and then generally by several other authors, see for example [2]. Let us mention that a function u is called s-Gevrey ($s\geq 1$), denoted by $u\in G^{s}(\Omega)$, if $u\in C^{\infty}(\Omega)$ and for every compact subset K of Ω there exists a constant $C_{1}(K)$ such that for all multi-indices α we have $\sup_{K}|D^{\alpha}u|\leq C_{1}^{|\alpha|}(K)(\alpha!)^{s}$. Note that when $s=1$ then $G^{1}(\Omega)$ is the space of real analytic functions in Ω and $G^{s}(\Omega)\subset G^{s'}(\Omega)$ if $s\leq s'$. We will consider the following equations: semilinear perturbation of power of the Mizohata operator and semilinear perturbation of the Kohn-Laplacian on the Heisenberg group.

I. Semilinear perturbation of power of the Mizohata operator [3], [4]. For $m\in \mathbb{N}^{+}$ we define $\Xi_{m}=\{(\alpha,\beta,\gamma):\alpha+\beta\leq m,2km\geq\gamma\geq\alpha+(2k+1)\beta-m\}$. For $(x_{1},x_{2})\in \mathbb{R}^{2}$ we will write $\partial_{1}^{\alpha},\partial_{2}^{\beta},\ {}_{\gamma}\partial_{\alpha,\beta}$, instead of $\frac{\partial^{\alpha}}{\partial x_{1}^{\alpha}},\frac{\partial^{\beta}}{\partial x_{2}^{\beta}},x_{1}^{\gamma}\frac{\partial^{\alpha+\beta}}{\partial x_{1}^{\alpha}\partial x_{2}^{\beta}}$. We consider the following equation

$$M_{2k}^{h}u+\varphi(x_{1},x_{2},u,\ldots,{}_{\gamma}\partial_{\alpha,\beta}u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}}=0\ \text{ in }\ \Omega, \tag{1}$$

where k is a positive integer, $M_{2k}=\frac{\partial}{\partial x_{1}}+ix_{1}^{2k}\frac{\partial}{\partial x_{2}}$, the Mizohata operator in $\mathbb{R}^{2}$, see [5], and Ω is a bounded domain with piece-wise smooth boundary in $\mathbb{R}^{2}$. Put $h(2k+1)=r_{0}$. For any integer $r\geq 0$ let $\Gamma_{r}=\Gamma_{r}^{1}\cup\Gamma_{r}^{2}$ where

$$\Gamma_{r}^{1}=\{(\alpha,\beta):\alpha\leq r_{0},2\alpha+\beta\leq r\},\Gamma_{r}^{2}=\{(\alpha,\beta):\alpha\geq r_{0},\alpha+\beta\leq r-r_{0}\}.$$

For any non-negative integer r let us define the norm

$$|u,\Omega|_{r}=\max_{(\alpha_{1},\beta_{1})\in\Gamma_{r}}|\partial_{1}^{\alpha_{1}}\partial_{2}^{\beta_{1}}u,\Omega|+\max_{\substack{(\alpha_{1},\beta_{1})\in\Gamma_{r}\\ \alpha_{1}\geq 1,\beta_{1}\geq 1}}\max_{x\in\bar{\Omega}}\left|\partial_{1}^{h}\left(\partial_{1}^{\alpha_{1}}\partial_{2}^{\beta_{1}}u(x)\right)\right|,$$

[1] S. Bernstein Math. Annal., **59**, p.20-76, 1904.
[2] A. Friedman J. Math. Mech., **7**, p. 43-59, 1958.
[3] N. M. Tri Comm. Partial Differential Equations, 24, p. 325-354, 1999.
[4] N. M. Tri To appear in Rend. Sem. Mat. Universita Politecnico Torino.
[5] S. Mizohata J. Math. Kyoto Univ., **1**, p. 271-302, 1962.

where $|w,\Omega| = \sum_{(\alpha,\beta,\gamma)\in\Xi_{h-1}} \max_{x\in\bar{\Omega}} |{}_\gamma\partial_{\alpha,\beta}w(x)|$.
For $l \in \mathbb{N}^+$ let $\mathbb{H}^l_{loc}(\Omega)$ denote the space of all u such that for any compact K of Ω we have $\sum_{(\alpha,\beta,\gamma)\in\Xi_l} \|{}_\gamma\partial_{\alpha,\beta}u\|_{L^2(K)} < \infty$. We note the following properties of $\mathbb{H}^l_{loc}(\Omega)$

$\mathrm{H}^l_{loc}(\Omega) \subset \mathbb{H}^l_{loc}(\Omega)$ where $\mathrm{H}^l_{loc}(\Omega)$ stands for the standard Sobolev spaces,
$\mathbb{H}^{4k+2}_{loc}(\Omega) \subset \mathrm{H}^2_{loc}(\Omega) \subset C(\Omega)$.

Theorem 1. *Let $l \geq 4k^2+6k+h+1$. Assume that u is a $\mathrm{H}^l_{loc}(\Omega)$ solution of the equation (1) and $\varphi \in G^s$. Then $u \in G^s(\Omega)$.*

The proof of this theorem consists of Theorem 1.1 and Theorem 1.2.

Theorem 1.1. *Let $l \geq 4k^2+6k+h+1$. Assume that u is a $\mathrm{H}^l_{loc}(\Omega)$ solution of the equation (1) and $\varphi \in C^\infty$. Then u is a $C^\infty(\Omega)$ function.*

Proof of Theorem 1.1.

Lemma 1.1 (Grushin). *Assume that $u \in \mathcal{D}'(\Omega)$ and $M^h_{2k}u \in \mathbb{H}^l_{loc}(\Omega)$ then $u \in \mathbb{H}^{l+h}_{loc}(\Omega)$.*

Lemma 1.2. *Let $l \geq 4k^2+6k+h+1$. Assume that $u \in H^l_{loc}(\Omega)$ and $\varphi \in C^\infty$ then $\varphi(x_1, x_2, u, \ldots, {}_\gamma\partial_{\alpha,\beta}u) \in \mathbb{H}^{l-h+1}_{loc}(\Omega)$.*

Proof of Lemma 1.2. It is sufficient to prove that

$$ {}_{\gamma_1}\partial_{\alpha_1,\beta_1}\varphi(x_1, x_2, u, \ldots, {}_\gamma\partial_{\alpha,\beta}u) \in L^2_{loc}(\Omega) \text{ for every } (\alpha_1,\beta_1,\gamma_1) \in \Xi_{l-h+1}. $$

Let us denote $(u, \ldots, {}_\gamma\partial_{\alpha,\beta}u)_{(\alpha,\beta,\gamma)\in\Xi_{h-1}}$ by $(w_1, w_2, \ldots, w_\mu)$ with $\mu \leq 2kh^3$. Since $l \geq 4k^2+6k+h+1$ it follows that $w_1, \ldots, w_\mu \in C(\Omega)$. It is easy to verify that $\partial_1^{\alpha_1}\partial_2^{\beta_1}\varphi(x_1, x_2, u, \ldots, {}_\gamma\partial_{\alpha,\beta}u)$ is a linear combination with positive coefficients of terms of the form

$$ \frac{\partial^k\varphi}{\partial x_1^{k_1}\partial x_2^{k_2}\partial w_1^{k_3}\ldots\partial w_\mu^{k_{\mu+2}}} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}, $$

where $k = k_1+k_2+\ldots+k_{\mu+2} \leq \alpha_1+\beta_1$; $\zeta(\alpha_{1,j},\beta_{1,j})$ may be multivalued functions of $\alpha_{1,j},\beta_{1,j}$; $\alpha_{1,j},\beta_{1,j}$ may be multivalued functions of j, and

$$ \sum_j \alpha_{1,j}\cdot\zeta(\alpha_{1,j},\beta_{1,j}) \leq \alpha_1, \sum_j \beta_{1,j}\cdot\zeta(\alpha_{1,j},\beta_{1,j}) \leq \beta_1. $$

Hence $x_1^{\gamma_1}\partial_1^{\alpha_1}\partial_2^{\beta_1}\varphi(x_1, x_2, u, \ldots, {}_\gamma\partial_{\alpha,\beta}u)$ is a linear combination with positive coefficients of terms of the form

$$ \frac{\partial^k\varphi}{\partial x_1^{k_1}\partial x_2^{k_2}\partial w_1^{k_3}\ldots\partial w_\mu^{k_{\mu+2}}} x_1^{\gamma_1} \prod_{j=1}^{\mu} \prod_{(\alpha_{1,j},\beta_{1,j})} \left(\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}. $$

Therefore Lemma 1.2 is proved if we can show this general terms are in $L^2_{loc}(\Omega)$. If all $\zeta(\alpha_{1,j},\beta_{1,j})$ vanish then it is immediate that $\partial^k\varphi/\partial x_1^{k_1}\partial x_2^{k_2}\partial w_1^{k_3}...\partial w_\mu^{k_{\mu+2}} \in C$, since $\varphi \in C^\infty, w_1,\dots,w_\mu \in C(\Omega)$. Therefore we can assume that there exists at least one of $\zeta(\alpha_{1,j},\beta_{1,j})$ that differs from 0. Choose j_0 such that there exists $\alpha_{1,j_0},\beta_{1,j_0}$ with $\zeta(\alpha_{1,j_0},\beta_{1,j_0}) \geq 1$ and

$$\alpha_{1,j_0} + (2k+1)\beta_{1,j_0} = \max_{\substack{j=1,\dots,\mu \\ \zeta(\alpha_{1,j},\beta_{1,j})\geq 1}} \alpha_{1,j} + (2k+1)\beta_{1,j}.$$

Consider the following possibilities
I) $\zeta(\alpha_{1,j_0},\beta_{1,j_0}) \geq 2$. We then have $\alpha_{1,j}+\beta_{1,j} \leq l-(h-1)-(4k+2)$. Indeed, if $j \neq j_0$ and $\alpha_{1,j}+\beta_{1,j} > l-(h-1)-(4k+2)$ then $\alpha_{1,j_0}+\beta_{1,j_0} \geq 2k$. Therefore

$$l-(h-1)-(4k+2) < \alpha_{1,j}+\beta_{1,j} \leq \alpha_{1,j}+(2k+1)\beta_{1,j} \leq \alpha_{1,j_0}+(2k+1)\beta_{1,j_0} \leq (2k+1)(\alpha_{1,j_0}+\beta_{1,j_0}) \leq 2k(2k+1).$$

Thus $l < (2k+2)(2k+1)+(h-1)$, a contradiction.
If $j = j_0$ and $\alpha_{1,j_0}+\beta_{1,j_0} > l-(h-1)-(4k+2)$ then we have

$$l-(h-1) \geq \alpha_1+\beta_1 \geq 2(\alpha_{1,j_0}+\beta_{1,j_0}) > 2(l-(h-1)-(4k+2)).$$

Therefore $l < (h+1)+4(2k+1)$, a contradiction.
Next define

$$\gamma(\alpha_{1,j},\beta_{1,j}) = \max\{0, \alpha_{1,j}+(2k+1)\beta_{1,j}+(h-1)+(4k+2)-l\}.$$

We claim that $\gamma(\alpha_{1,j},\beta_{1,j}) \leq 2k(l-(h-1)-(4k+2))$. Indeed, if $j \neq j_0$ and $\gamma(\alpha_{1,j},\beta_{1,j}) > 2k(l-(h-1)-(4k+2))$ then

$$(2k+1)(l-(h-1)) \geq \alpha_1+(2k+1)\beta_1 \geq \geq (\alpha_{1,j}+2\alpha_{1,j_0})+(2k+1)(\beta_{1,j_0}+2\beta_{1,j_0}) > 3(2k+1)(l-(h-1)-(4k+2)).$$

Thus $l < (h-1)+3(2k+1)$, a contradiction.
If $j = j_0$ and $\gamma(\alpha_{1,j_0},\beta_{1,j_0}) > 2k(l-(h-1)-(4k+2))$ then it follows that

$$(2k+1)(l-(h-1)) \geq \alpha_1+(2k+1)\beta_1 \geq 2(\alpha_{1,j_0}+(2k+1)\beta_{1,j_0}) > 2(2k+1)(l-(h-1)-(4k+2)).$$

Thus $l < (h-1)+4(2k+1)$, a contradiction.
From all above arguments we deduce that $(\alpha_{1,j},\beta_{1,j},\gamma(\alpha_{1,j},\beta_{1,j})) \in \Xi_{l-(h-1)-(4k+2)}$. Next we claim that $\sum\gamma(\alpha_{1,j},\beta_{1,j})\zeta(\alpha_{1,j},\beta_{1,j}) \leq \gamma_1$. Indeed, if $\sum\gamma(\alpha_{1,j},\beta_{1,j})\zeta(\alpha_{1,j},\beta_{1,j}) > \gamma_1$ then we deduce that

$$\alpha_1+(2k+1)\beta_1-2(l-(h-1)-(4k+2)) \geq \sum\gamma(\alpha_{1,j},\beta_{1,j})\zeta(\alpha_{1,j},\beta_{1,j}) > \gamma_1 \geq \alpha_1+(2k+1)\beta_1-(l-(h-1)).$$

Therefore $l < (h-1)+4(2k+1)$, a contradiction.
Now we have

$$x_1^{\gamma_1}\prod_{j=1}^{\mu}\prod_{(\alpha_{1,j},\beta_{1,j})}\left(\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}=$$

$$x_1^{\tilde{\gamma}_1}\prod_{j=1}^{\mu}\prod_{(\alpha_{1,j},\beta_{1,j})}\left(x_1^{\gamma(\alpha_{1,j},\beta_{1,j})}\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}\in C(\Omega)$$

since $x_1^{\gamma(\alpha_{1,j},\beta_{1,j})}\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\in\mathbb{H}_{loc}^{4k+2}(\Omega)\subset C(\Omega)$.
II) $\zeta(\alpha_{1,j_0},\beta_{1,j_0})=1$ and $\zeta(\alpha_{1,j},\beta_{1,j})=0$ for $j\neq j_0$. We have

$$x_1^{\gamma_1}\prod_{j=1}^{\mu}\prod_{(\alpha_{1,j},\beta_{1,j})}\left(\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\right)^{\zeta(\alpha_{1,j},\beta_{1,j})}=x_1^{\gamma_1}\partial_1^{\alpha_{1,j_0}}\partial_2^{\beta_{1,j_0}}w_{j_0}\in L_{loc}^2(\Omega).$$

III) $\zeta(\alpha_{1,j_0},\beta_{1,j_0})=1$ and there exists $j_1\neq j_0$ such that $\zeta(\alpha_{1,j_1},\beta_{1,j_1})\neq 0$. Define

$$\bar{\gamma}(\alpha_{1,j_0},\beta_{1,j_0})=\max\{0,\alpha_{1,j_0}+(2k+1)\beta_{1,j_0}+(h-1)-l\}.$$

As in part I) we can prove $(\alpha_{1,j},\beta_{1,j},\gamma(\alpha_{1,j},\beta_{1,j}))\in\Xi_{l-(h-1)-(4k+2)}$ for $j\neq j_0$ and $(\alpha_{1,j_0},\beta_{1,j_0},\bar{\gamma}(\alpha_{1,j_0},\beta_{1,j_0}))\in\Xi_{l-(h-1)}$. Therefore $x_1^{\gamma(\alpha_{1,j},\beta_{1,j})}\partial_1^{\alpha_{1,j}}\partial_2^{\beta_{1,j}}w_j\in\mathbb{H}_{loc}^{4k+2}(\Omega)\subset C(\Omega)$ for $j\neq j_0$ and $x_1^{\bar{\gamma}(\alpha_{1,j_0},\beta_{1,j_0})}\partial_1^{\alpha_{1,j_0}}\partial_2^{\beta_{1,j_0}}w_{j_0}\in L_{loc}^2(\Omega)$. We also have $\sum_{j\neq j_0}\gamma(\alpha_{1,j},\beta_{1,j})\zeta(\alpha_{1,j},\beta_{1,j})+\bar{\gamma}(\alpha_{1,j_0},\beta_{1,j_0})\leq\gamma_1$ as in part I). Now the desired result follows from the decomposition of the general terms. □

(End of the Proof of Theorem 1.1) $u\in\mathbb{H}_{loc}^l(\Omega), l\geq 4k^2+6k+h+1\Longrightarrow\varphi(x_1,x_2,u,...,_\gamma\partial_{\alpha,\beta}u)\in\mathbb{H}_{loc}^{l-h+1}(\Omega)$ (by Lemma 1.2). Therefore by Lemma 1.1 we have $u\in\mathbb{H}_{loc}^{l+1}(\Omega)$. Repeat the argument again and again we finally arrive at $u\in\mathbb{H}_{loc}^{l+m}(\Omega)$ for any $m\in\mathbb{N}^+$, i. e. $u\in\cap_l\mathbb{H}_{loc}^l(\Omega)=C^\infty(\Omega)$. Finally note that $u\in \mathrm{H}_{loc}^l(\Omega)\Longrightarrow u\in\mathbb{H}_{loc}^l(\Omega)$.□

Theorem 1.2. *Let u be a C^∞ solution of the equation (1) and $\varphi\in G^s$. Then $u\in G^s(\Omega)$.*

Proof of Theorem 1.2. The proof of Theorem 1.2 will follow the line of [3]. Let us define

$$F_{2k}^h(x_1,x_2,y_1,y_2)=\frac{1}{2\pi(h-1)!}\frac{(x_1-y_1)^{h-1}}{\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1}+i(x_2-y_2)}.$$

For $j=1,...,h-1$ we have

$$M_{2k}^jF_{2k}^h=\frac{1}{2\pi(h-j-1)!}\frac{(x_1-y_1)^{h-j-1}}{\frac{x_1^{2k+1}-y_1^{2k+1}}{2k+1}+i(x_2-y_2)}\quad\text{and}\quad M_{2k}^hF_{2k}^h=\delta(x-y).$$

Lemma 1.3 (Green' formula). *If $u, v \in C^l(\bar{\Omega})$ where l is any positive integer, then*

$$\int_\Omega u M_{2k}^l v dx_1 dx_2 = \int_\Omega (-1)^l v M_{2k}^l u dx_1 dx_2 + \\ + \int_{\partial\Omega} \left(\sum_{j=0}^{l-1} (-1)^j M_{2k}^j u M_{2k}^{l-j-1} v \right) (n_1 + i x_1^{2k} n_2) ds.$$

where $n = (n_1, n_2)$ is the outward unit normal vector to Ω.

Lemma 1.4 (Representation formula). *Assume that $u \in C^h(\bar{\Omega})$ then we have*

$$u(x) = \int_\Omega (-1)^h F_{2k}^h(x,y) M_{2k}^h u(y) dy_1 dy_2 + \\ + \int_{\partial\Omega} \left(\sum_{j=0}^{h-1} (-1)^j M_{2k}^j u M_{2k}^{h-j-1} F_{2k}^h(x,y) \right) (n_1 + i y_1^{2k} n_2) ds.$$

Lemma 1.5 (Friedman). *There exists a constant C_1 such that if $g(\xi)$ is a positive monotone decreasing function, defined in the interval $0 \le \xi \le 1$ and satisfying*

$$g(\xi) \le \frac{1}{8^{12^k}} g\Big(\xi\big(1 - \frac{6^k}{N}\big)\Big) + \frac{C}{\xi^{N-r_0-1}} \quad (N \ge r_0 + 2, C > 0),$$

then $g(\xi) < CC_1/\xi^{N-r_0-1}$.

Proposition 1.1. *Assume that $\varphi \in G^s$. Then there exist constants $\tilde{H}_0, \tilde{H}_1, C_2, C_3$ such that for every $H_0 \ge \tilde{H}_0, H_1 \ge \tilde{H}_1, H_1 \ge C_2 H_0^{2r_0+3}$ if*

$$|u, \Omega|_q \le H_0 H_1^{(q-r_0-2)}((q - r_0 - 2)!)^s, \quad 0 \le q \le N+1, r_0 + 2 \le N$$

then

$$\max_{x\in\bar{\Omega}} \left| \partial_1^{\alpha_1} \partial_2^{\beta_1} \varphi(x_1, x_2, u, \ldots,_\gamma \partial_{\alpha,\beta} u) \right| \le C_3 H_0 H_1^{N-r_0-1}((N-r_0-1)!)^s; (\alpha_1, \beta_1) \in \Gamma_{N+1}$$

(Continuing the Proof of Theorem 1.2) It suffices to consider the case $(0,0) \in \Omega$. Let us define a distance $\rho\big((y_1, y_2), (x_1, x_2)\big) = \max\Big(\frac{|x_1^{2k+1} - y_1^{2k+1}|}{2k+1}, |x_2 - y_2|\Big)$. For two sets S_1, S_2 the distance between them is defined as $\rho(S_1, S_2) = \inf_{x\in S_1, y\in S_2} \rho(x,y)$. Let V^T be the *closed* cube with edges of size (in the ρ metric) $2T$, which are parallel to the coordinate axes and centered at $(0,0)$. Denote by V_δ^T the *closed* subcube which is homothetic with V^T and such that the distance between its boundary and the boundary of V^T is δ. We shall prove by induction that if T is small enough then there exist constants H_0, H_1 with $H_1 \geq C_2 H_0^{2r_0+3}$ such that

$$(2) \qquad |u, V_\delta^T|_m \leq H_0 \ \text{ for } \ 0 \leq m \leq \max\{r_0+2, 6^k+1\}$$

and

$$(3) \quad |u, V_\delta^T|_m \leq H_0 \left(\frac{H_1}{\delta}\right)^{m-r_0-2} ((m-r_0-2)!)^s \ \text{ for } \ m \geq \max\{r_0+2, 6^k+1\}$$

and δ sufficiently small. Hence the Gevrey regularity of u follows. (2) follows easily from the C^∞ smoothness assumption on u. Assume that (3) holds for $m = N$. We shall prove it for $m = N+1$. Fix $(x_1, x_2) \in V_\delta^T$ and then define $\sigma = \rho((x_1, x_2), \partial V^T)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}$ denote the cube with center at (x_1, x_2) and edges of length $2\tilde{\sigma}$ which are parallel to the coordinate axes. Differentiating ${}_\gamma\partial_{\alpha,\beta}$ the equation (1) and then using Lemma 1.4 with $\Omega = V_{\tilde{\sigma}}$, Proposition 1.1 and the inductive assumptions we can prove

Lemma 1.6. *Assume that $(\alpha, \beta, \gamma) \in \Xi_{h-1}$ and $(\alpha_1, \beta_1) \in \Gamma_{N+1}$. Then if $\alpha_1 \geq 1, \beta_1 \geq 1$ there exists a constant C_4 such that*

$$\max_{x \in V_\delta^T} \left| {}_\gamma\partial_{\alpha,\beta}(\partial_1^{\alpha_1}\partial_2^{\beta_1} u(x))\right| \leq C_4 \Bigg(T^{\frac{1}{2k+1}} \left\|u, V_{\delta(1-1/N)}\right\|_{N+1} + \\ + H_0 \left(\frac{H_1}{\delta}\right)^{N-r_0-1} (N-r_0-1)! \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\Big)\Bigg).$$

Lemma 1.7. *Assume that $(\alpha, \beta, \gamma) \in \Xi_{h-1}$. Then there exists a constant C_5 such that*

$$\max_{x \in V_\delta^T} \left| {}_\gamma\partial_{\alpha,\beta}(\partial_2^{N+1} u(x))\right| \leq C_5 \Bigg(T^{\frac{1}{2k+1}} \left\|u, V^T_{\delta(1-6^k/N)}\right\|_{N+1} + \\ + H_0 \left(\frac{H_1}{\delta}\right)^{N-r_0-1} (N-r_0-1)! \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\Big)\Bigg).$$

Lemma 1.8. *Assume that $(\alpha, \beta, \gamma) \in \Xi_{h-1}$. Then there exists a constant C_6 such that*

$$\max_{x \in V_\delta^T} \left| {}_\gamma\partial_{\alpha,\beta}(\partial_1^{N-r_0+1} u(x))\right| \leq C_6 \Bigg(T^{\frac{1}{2k+1}} \left\|u, V^T_{\delta(1-1/N)}\right\|_{N+1} + \\ + H_0 \left(\frac{H_1}{\delta}\right)^{N-r_0-1} (N-r_0-1)! \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1}\Big)\Bigg).$$

Lemma 1.9. *Assume that* $(\alpha_1, \beta_1) \in \Gamma_{N+1} \backslash \Gamma_N, \alpha_1 \geq 1, \beta_1 \geq 1$. *Then there exists a constant* C_7 *such that*

$$\max_{x \in V_\delta^T} | \partial_1^h (\partial_1^{\alpha_1} \partial_2^{\beta_1} u(x)) | \leq C_7 \Bigg(T^{\frac{1}{2k+1}} \| u, V_{\delta(1-6^k/N)}^T \|_{N+1} + + H_0 \left(\frac{H_1}{\delta} \right)^{N-r_0-1} (N - r_0 - 1)! \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1} \Big) \Bigg).$$

(End of the Proof of Theorem 1.2) Put $|u, V_\delta^T|_{N+1} = g(\delta)$. Using Lemmas 1.6-1.9 we can show that there exists a constant C_8 such that

$$g(\delta) \leq C_8 \Bigg(T^{\frac{1}{2k+1}} g\Big(\delta(1-6^k/N)\Big) + H_0 \left(\frac{H_1}{\delta} \right)^{N-r_0-1} ((N-r_0-1)!)^s \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1} \Big) \Bigg).$$

Choosing $T \leq \left(1/8^{12^k} C_8\right)^{2k+1}$ then by Lemma 1.5 we deduce that

$$g(\delta) \leq C_9 H_0 \left(\frac{H_1}{\delta} \right)^{N-r_0-1} ((N - r_0 - 1)!)^s \Big(T^{\frac{1}{2k+1}} + \frac{1}{H_1} \Big).$$

Choosing $T \leq (1/2C_9)^{2k+1}$ and $H_1 \geq 2C_9$ (in addition to $H_1 \geq C_2 H_0^{2r_0+3}$) we have

$$g(\delta) = |u, V_\delta^T|_{N+1} \leq H_0 \left(\frac{H_1}{\delta} \right)^{N-r_0-1} ((N - r_0 - 1)!)^s . \Box$$

Example. If $h = 3$ we have the following statement : if u is a $\mathrm{H}_{loc}^{4k^2+6k+4}(\Omega)$ solution of the equation $M_{2k}^3 u + \left(x_1^{4k} \frac{\partial^2 u}{\partial x_2^2}\right)^5 e^{x_1^{2k-1} \frac{\partial u}{\partial x_2}} \cos\left(\frac{\partial^2 u}{\partial x_1^2}\right) = 0$, then u is analytic in Ω.

II. Semilinear perturbation of Kohn-Laplacian on the Heisenberg Group [6]. First let us recall some basic facts about the Kohn-Laplacian $\Box_b$ on the Heisenberg group. Let $(x, y, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \in R^{2n+1}$. The Heisenberg group (of degree n) $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ endowed with the following group action

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy')).$$

Let us define the following vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, T = \frac{\partial}{\partial t}; j = 1, \ldots, n,$$

$$Z_j = \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \bar{Z}_j = \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}.$$

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[6] N. M. Tri Math. Note Inst. Math., Univ. Tsukuba.

Then the subbundle $T_{1,0}$ of $\mathbb{C}T\mathbb{H}^n$ spanned by $Z_1, \ldots, Z_n$ define a CR structure on $\mathbb{H}^n$. We will use the volume element on $\mathbb{H}^n$ as $dxdydt$, which *differs* from that of [7] by a factor 2^{-n}. Now on $\mathbb{H}^n$ with the above CR structure and metric we can define the $\bar{\partial}_b$-complex: $\bar{\partial}_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1})$ and its formal adjoint $\vartheta_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q-1})$, where $\Lambda^{p,q} = (\Lambda^p T^*_{1,0}) \otimes (\Lambda^p \bar{T}^*_{1,0})$. Finally the Kohn-Laplacian can be defined as $\Box_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q})$. In specific basis $\Box_b$ can be diagonalized with elements $\mathcal{L}_{n,\lambda}$ on the diagonal. Here $\mathcal{L}_{n,\lambda}$ is a second order differential operator of the form

$$\mathcal{L}_{n,\lambda} = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j) + i\lambda T = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2 + Y_j^2) + i\lambda T; \lambda \in \mathbb{C}.$$

When $\pm\lambda \neq n, n+2, n+4, \ldots$ we say that λ is *admissible*. Now we would like to investigate the Gevrey regularity of solutions of the following equation

$$\mathcal{L}_{n,\lambda} u + \psi(x, y, t, u, Z_1 u, ..., Z_n u, \bar{Z}_1 u, ..., \bar{Z}_n u) = 0 \ \text{ in } \ \Omega, \tag{4}$$

where, in this part, Ω denotes a bounded domain in $\mathbb{H}^n$ with piece-wise smooth boundary. For $l \in \mathbb{N}^+$ let $S^l_{loc}(\Omega)$ denote the space of all u such that for any compact K of Ω we have $\sum_{I \leq l} \|L_{i_1} \ldots L_{i_I} u\|_{L^2(K)} < \infty$, where each of $L_{i_1}, \ldots, L_{i_I}$ is one of $Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n$. We will use the following property $S^l_{loc}(\Omega) \subset C(\Omega)$ provided $l > n+1$. In the future we will need to work on the double $\mathbb{H}^n \times \mathbb{H}^n$. Assume that we have a differential operator $\mathcal{P}(x, y, t, D_x, D_y, D_t) = \sum_{|\alpha|+|\beta|+\gamma \leq m} a_{\alpha,\beta,\gamma}(x, y, t) D^{\alpha,\beta,\gamma}_{x,y,t}$, then we write $\mathcal{P}'$ for the operator $\sum_{|\alpha|+|\beta|+\gamma \leq m} a_{\alpha,\beta,\gamma}(x', y', t') D^{\alpha,\beta,\gamma}_{x',y',t'}$. If $u(x, y, t)$ is a function on $\mathbb{H}^n$ then $\mathcal{P}'$ acts on u as $\mathcal{P}' u(x', y', t')$. If $F(x, y, t, x', y', t')$ is a function on the double $\mathbb{H}^n \times \mathbb{H}^n$ then $\mathcal{P}'$ acts on F as $\mathcal{P}' F(x, y, t, x', y', t')$.

Theorem 2. *Let $l \geq 2n+4$ and λ be admissible. Assume that u is a $S^l_{loc}(\Omega)$ solution of the equation (4) and $\psi \in G^s, s \geq 2$ then $u \in G^s(\Omega)$.*

The proof of this theorem follows the line of the proof of Theorem 1.

Theorem 2.1. *Let $l \geq 2n + 4$ and λ be admissible. Assume that u is a $S^l_{loc}(\Omega)$ solution of the equation (4) and $\psi \in C^\infty$. Then u is a $C^\infty(\Omega)$ function.*

Proof of Theorem 2.1.

Lemma 2.1 (Folland-Stein). *Assume that $u \in \mathcal{D}'(\Omega)$, λ is admissible and $\mathcal{L}_{n,\lambda} u \in S^l_{loc}(\Omega)$ then $u \in S^{l+2}_{loc}(\Omega)$.*

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[7] G. B. Folland, E. M. Stein Comm. Pure Appl. Math., **27**, p. 429-522, 1974.

Lemma 2.2. *Let* $l \geq 2n+4$. *Assume that* $u \in S^l_{loc}(\Omega)$ *and* $\psi \in C^\infty$. *Then* $\psi(x,y,t,u,Z_1u,...,Z_nu,\bar{Z}_1u,...,\bar{Z}_nu) \in S^{l-1}_{loc}(\Omega)$.

Proof of Lemma 2.2. It suffices to prove that

$$Z_{i_1}Z_{i_2}\dots Z_{i_I}\psi(x,y,t,u,Z_1u,...,Z_nu,\bar{Z}_1u,...,\bar{Z}_nu) \in L^2_{loc}(\Omega) \text{ for every } I \leq l-1.$$

Using the fact that $l \geq 2n+4$ we deduce that $u, Z_1u,...,Z_nu,\bar{Z}_1u,...,\bar{Z}_nu \in C(\Omega)$. We have $Z_{i_1}Z_{i_2}\dots Z_{i_I}\psi(x,y,t,u,Z_1u,...,Z_nu,\bar{Z}_1u,...,\bar{Z}_nu)$ is a linear combination with positive coefficients of terms of the form

$$\frac{\partial^k\psi}{\partial x^{k_1}\partial y^{k_2}\partial t^{k_3}\partial w_1^{k_4}...\partial w_{2n+1}^{k_{2n+4}}}\prod_{j=1}^{2n+1}\prod_{J_j}\left(Z_{i_1}Z_{i_2}\dots Z_{i_{J_j}}w_j\right)^{\zeta(J_j)},$$

where $(w_1,w_2,...,w_{2n+1})$ denotes $(u,Z_1u,\dots,Z_nu,\bar{Z}_1u,\dots,\bar{Z}_nu)$, $k = |k_1|+|k_2|+\dots+k_{2n+4} \leq I; J_j$ may be multivalued functions of $j; \zeta(J_j)$ may be multivalued functions of J_j , and $\sum_j J_j\zeta(J_j) \leq I \leq l-1$. Therefore Lemma 2.2 is proved if we can show this general terms are in $L^2_{loc}(\Omega)$. If all $\zeta(J_j)$ vanish then it is immediate that $\partial^k\psi/\partial x^{k_1}\partial y^{k_2}\partial t^{k_3}\partial w_1^{k_4}...\partial w_{2n+1}^{k_{2n+4}} \in C(\Omega)$, since $\psi \in C^\infty, w_1,\dots,w_{2n+1} \in C(\Omega)$. Therefore we can assume that there exists at least one of $\zeta(J_j)$ that differs from 0. Choose j_0 such that there exists J_{j_0} with $\zeta(J_{j_0}) \geq 1$ and $J_{j_0} = \max_{\substack{j=1,\dots,2n+1\\ \zeta(J_j)\geq 1}} J_j$.

Consider the following possibilities

I) $\zeta(J_{j_0}) \geq 2$. We then have $J_j \leq [(l-1)/2]$ for every j, here [.] denotes the integer part of the argument. Indeed, if $j \neq j_0$ and $J_j > [(l-1)/2]$ then $J_{j_0} \geq [(l-1)/2]$. Therefore $J_j + J_{j_0} > l-1$, a contradiction. If $j = j_0$ and $J_{j_0} > [(l-1)/2]$ then we have $\zeta(J_{j_0})J_{j_0} > l-1$, a contradiction. Hence we have $Z_{i_1}Z_{i_2}\dots Z_{i_{J_j}}w_j \in S^{n+2}_{loc}(\Omega) \subset C(\Omega)$ for every j. It follows that $\prod_{j=1}^{2n+1}\prod_{J_j}\left(Z_{i_1}Z_{i_2}\dots Z_{i_{J_j}}w_j\right)^{\zeta(J_j)} \in C(\Omega) \subset L^2_{loc}(\Omega)$.

II) $\zeta(J_{j_0}) = 1$ and $\zeta(J_j) = 0$ for $j \neq j_0$. We have

$$\prod_{j=1}^{2n+1}\prod_{(J_j)}\left(Z_{i_1}Z_{i_2}\dots Z_{i_{J_j}}w_j\right)^{\zeta(J_j)} = Z_{i_1}Z_{i_2}\dots Z_{i_{J_{j_0}}}w_{j_0} \in L^2_{loc}(\Omega).$$

III) $\zeta(J_{j_0}) = 1$ and there exists $j_1 \neq j_0$ such that $\zeta(J_j) \neq 0$. As in part I) we can prove $J_j \leq [(l-1)/2]$ and therefore $Z_{i_1}Z_{i_2}\dots Z_{i_{J_j}}w_j \in S^{n+2}_{loc}(\Omega) \subset C(\Omega)$ for $j \neq j_0, \zeta(J_j) \leq 1$ and $Z_{i_1}Z_{i_2}\dots Z_{i_{J_{j_0}}}w_{j_0} \in L^2_{loc}(\Omega)$. Now the desired result follows. □

(End of the Proof of Theorem 2.1) By Lemma 2.2 from $u \in S^l_{loc}(\Omega), l \geq 2n+4$ we deduce that $\psi(x,y,t,u,Z_1u,...,Z_nu,\bar{Z}_1u,...,\bar{Z}_nu) \in S^{l-1}_{loc}(\Omega)$. Therefore by Lemma 2.1 we deduce that $u \in S^{l+1}_{loc}(\Omega)$. Repeat the argument again and again we finally arrive at $u \in S^{l+m}_{loc}(\Omega)$ for every positive m, i. e. $u \in \cap_l S^l_{loc}(\Omega) = C^\infty(\Omega)$.□

Theorem 2.2. *Let λ be admissible and u be a $C^\infty(\Omega)$ solution of the equation (4), $\psi \in G^s, s \geq 2$. Then $u \in G^s(\Omega)$.*

Proof of Theorem 2.2. Denote $\Gamma(\frac{n+\lambda}{2})\Gamma(\frac{n-\lambda}{2})A_-^{-\frac{n+\lambda}{2}}A_+^{-\frac{n-\lambda}{2}} / \left(2^{2-n}\pi^{n+1}\right)$ by $F_{n,\lambda}$, where

$$A_- :=|x-x'|^2+|y-y'|^2-i(t-t'+2yx'-2y'x),$$
$$A_+ :=|x-x'|^2+|y-y'|^2+i(t-t'+2yx'-2y'x).$$

If λ is admissible then we have $\mathcal{L}_{n,\lambda}F_{n,\lambda}(x,y,t,x',y',t') = \delta(x-x', y-y', t-t')$. Let $(\nu_1^1,\ldots,\nu_n^1,\nu_1^2,\ldots,\nu_n^2,\tau)$ be the unit outward normal to Ω. Define the complex outward normal vector $(\nu,\bar{\nu},\tau)$ to Ω with components $\nu_j = (\nu_j^1 - i\nu_j^2)/2, \bar{\nu}_j = (\nu_j^1 + i\nu_j^2)/2$.

Lemma 2.3 (Green's formula). *If $u, v \in C^2(\Omega)\cap C^1(\bar{\Omega})$, then*

$$\int_\Omega v\mathcal{L}_{n,\lambda}u\,dxdydt = \int_\Omega u\mathcal{L}_{n,-\lambda}v\,dxdydt + \frac{1}{2}\int_{\partial\Omega}(uB_0v - vB_\lambda u)\,dS,$$

where $B_\lambda = \sum_{j=1}^n \left((\nu_j + i\bar{z}_j\tau)\bar{Z}_j + (\bar{\nu}_j - iz_j\tau)Z_j\right) - 2i\lambda\tau$ is an operator defined on $\partial\Omega$.

Lemma 2.4 (Representation Formula). *If $u \in C^2(\Omega)\cap C^1(\bar{\Omega})$ and λ is admissible then we have*

$$u(x,y,t) = \int_\Omega F_{n,\lambda}\mathcal{L}'_{n,\lambda}u(x',y',t')\,dx'dy'dt' + \frac{1}{2}\int_{\partial\Omega}(F_{n,\lambda}B'_\lambda u - uB'_0F_{n,\lambda})\,dS',$$

where $B'_\lambda = \sum_{j=1}^n \left((\nu_j + i\bar{z}'_j\tau)\bar{Z}'_j + (\bar{\nu}_j - iz'_j\tau)Z'_j\right) - 2i\lambda\tau$.

For any non-negative integer r and a function $u \in C^\infty(\bar{\Omega})$ let us define the norm

$$\|u,\Omega\|_r = \sum_{\substack{|\alpha|+|\beta|+\gamma\leq r+1\\ \gamma\leq r}} \max_{(x,y,t)\in\bar{\Omega}} |Z^\alpha\bar{Z}^\beta T^\gamma u(x,y,t)|,$$

where $Z^\alpha\bar{Z}^\beta T^\gamma u(x,y,t)$ stands for $Z_1^{\alpha_1}\bar{Z}_1^{\beta_1}\ldots Z_n^{\alpha_n}\bar{Z}_n^{\beta_n}T^\gamma u(x,y,t)$.

Lemma 2.5 (Tartakoff [8]). *A function $u \in C^\infty(\Omega)$ will belong to $G^s(\Omega)$ if for every compact subset K of Ω there exist constants $C_2(K), C_3(K)$ such that, for all positive integer r we have*

$$\|u,K\|_r \leq C_2(K)C_3^r(K)(r!)^s.$$

Now we would like to recall the following version of lemma of Friedman.

[8] D. Tartakoff Acta. Math., **145**, p. 177-204, 1980.

Lemma 2.6. *There exists a constant C_{10} such that if $g(\xi)$ is a positive monotone decreasing function, defined in the interval $0 \leq \xi \leq 1$ and satisfying*

$$g(\xi) \leq \frac{1}{100} g\Big(\xi\big(1-\frac{1}{N}\big)\Big) + \frac{C}{\xi^{2N-2}} \quad (N \geq 4, C > 0),$$

then $g(\xi) < CC_{10}/\xi^{2N-2}$.

Proposition 2.1. *Assume that $\psi(x, y, t, u, Z_1 u, ..., Z_n u, \bar{Z}_1 u, ..., \bar{Z}_n u) \in G^s, s \geq 1$. Then there exist constants $\tilde{H}_{0*}, \tilde{H}_{1*}, C_{11}, C_{12}$ such that for every $H_0 \geq \tilde{H}_{0*}, H_1 \geq \tilde{H}_{1*}, H_1 \geq C_{11} H_0$ if*

$$\|u, \Omega\|_q \leq H_0 H_1^{2q-4}((q-2)!)^s, \quad 2 \leq q \leq N+1$$

then

$$\max_{(x,y,t)\in\bar{\Omega}} \left|Z^\alpha \bar{Z}^\beta T^\gamma \psi\right| \leq C_{12} H_0 H_1^{2N-2}((N-1)!)^s$$

for every (α, β, γ) such that $|\alpha| + |\beta| + \gamma = N + 1$.

(Continuing the Proof of Theorem 2.2) Let us define a distance $\mathbf{d}\big((x, y, t), (x', y', t')\big) = \max_{j=1,\ldots,n}\Big(|x_j - x'_j|, |y_j - y'_j|, |t - t'|/4\sqrt{n}\Big)$. $\mathbf{d}(S_1, S_2) = \inf_{(x,y,t)\in S_1,(x',y',t')\in S_2} \mathbf{d}((x, y, t), (x', y', t'))$ is the distance between two sets S_1, S_2. Let $\tilde{V}^R(R \leq 1/\sqrt{2n}), \tilde{V}^R_\delta$ be the *closed* cube and subcube defined in the same maner (in the metric $\mathbf{d}$) as in part I. We shall prove by induction that if R is small enough then there exist constants H_0, H_1 with $H_1 \geq C_{11} H_0$ such that

$$\|u, \tilde{V}^R_\delta\|_m \leq H_0 \quad \text{for} \quad 0 \leq m \leq 4 \tag{5}$$

and

$$\|u, \tilde{V}^R_\delta\|_m \leq H_0 \left(\frac{H_1}{\delta}\right)^{2m-4} ((m-2)!)^s \quad \text{for} \quad m \geq 5, \tag{6}$$

and δ small. For a technical reason, together with (5), (6) we will also need to prove a little better estimate than (6) for $T^m u$, namely

$$\max_{(x,y,t)\in\tilde{V}^R_\delta} |T^m u| \leq \frac{H_0\delta}{m-1} \left(\frac{H_1}{\delta}\right)^{2m-4} ((m-2)!)^s, m \geq 5. \tag{7}$$

Again, (5) follows easily from the C^∞ smoothness assumption on u. Assume (6), (7) hold for $m = N$. We shall prove them for $m = N + 1$. Let us fix $(x, y, t) \in \tilde{V}^R_\delta$ and then define $\sigma = \mathbf{d}((x, y, t), \partial\tilde{V}^R)$ and $\tilde{\sigma} = \sigma/N$. Let $V_{\tilde{\sigma}}$ denote the *closed* cube with center at (x, y, t) and edges of length $2\tilde{\sigma}$ which are perpendicular to the coordinate axes. Differentiating $Z^\alpha Z^\beta$ the equation (4) and then using Lemma 2.4 with $\Omega = \tilde{V}_{\tilde{\sigma}}$, Proposition 2.1 and the inductive assumptions we can prove

Lemma 2.7. *Assume that $|\alpha|+|\beta|+\gamma = N+2$ and $|\alpha|+|\beta| \geq 2$. Then there exists a constant C_{13} such that*

$$\max_{(x,y,t)\in V_\delta^R} |Z^\alpha \bar{Z}^\beta T^\gamma u| \leq C_{13}\left(R\|u, V_{\delta'}^R\|_{N+1} + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s \right).$$

Lemma 2.8. *There exist constants C_{14}, C_{15} such that*

$$\max_{(x,y,t)\in V_\delta^R} \{|Z_1 T^{N+1}u|, \ldots, |Z_n T^{N+1}u|, |\bar{Z}_1 T^{N+1}u|, \ldots, |\bar{Z}_n T^{N+1}u|\} \leq$$

$$\leq C_{14}\left(R\|u, V_{\delta'}^R\|_{N+1} + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s \right),$$

$$\max_{(x,y,t)\in V_\delta^R} |T^{N+1}u| \leq \frac{C_{15}\delta}{N}\left(\|u, V_{\delta'}^R\|_{N+1} + H_0\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s \right).$$

(End of the proof of Theorem 2.2) Put $\|u, \tilde{V}_\delta^R\|_{N+1} = g^*(\delta)$. Using Lemmas 2.7 and 2.8 we can show that there exists a constant C_{16} such that

$$g^*(\delta) \leq C_{16}\left(Rg^*\left(\delta(1-1/N)\right) + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s \right).$$

Choosing $R \leq 1/100C_{16}$ then by Lemma 2.6 we deduce that

$$g^*(\delta) \leq \frac{C_{17}H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s.$$

If H_1 is chosen to be big enough such that $H_1 \geq C_{17}$ (in addition to $H_1 \geq C_{11}H_0$) we arrive at

$$g^*(\delta) = \|u, V_\delta^R\|_{N+1} \leq H_0\left(\frac{H_1}{\delta}\right)^{2N-2} ((N-1)!)^s.$$

Finally we complete the proof of Theorem 2.2 by using Lemma 2.5.□