<table>
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<th>Discrepancy of Some Special Sequences (Analytic Number Theory and Related Topics)</th>
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<tr>
<td>Author(s)</td>
<td>Goto, Kazuo; Ohkubo, Yukio</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1160: 94-101</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64241">http://hdl.handle.net/2433/64241</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Discrepancy of Some Special Sequences

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Abstract

We obtained some results concerned with the discrepancy of the sequence \((\alpha n + \beta \log n)_{n=1}^{\infty}, \alpha \not\in \mathbb{Q}, \beta \neq 0\).

First we give some definitions.

Definition 1. Let \((x_n), n = 1, 2, \ldots, \) be a sequence of \(\mathbb{R}\). Then the discrepancy of \((x_n)\) is defined by

\[
D_N(x_n) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_{[a, b)}(x_n) - (b - a) \right|
\]

where \(\chi_{[a, b)}(x)\) is the characteristic function mod 1 of \([a, b)\), that is, \(\chi_{[a, b)}(x) = 1\) for \(\{x\} = x - [x] \in [a, b)\) and \(\chi_{[a, b)}(x) = 0\) otherwise.
Definition 2. An irrational number $\alpha$ is said to be of constant type if there exists a constant $c > 0$ such that $\|\alpha h\| \geq c/h$ holds for all integers $h > 0$, where $\|x\| = \min\{\{x\}, 1 - \{x\}\}$ for $x \in \mathbb{R}$.

Definition 3. An irrational number $\alpha$ is said to be of type $\eta$ if $\eta$ is the infimum of all real numbers $\tau$ for which there exists a positive constant $c = c(\tau, \alpha)$ such that $h \tau \|\alpha h\| \geq c$ holds for all positive integers $h$.

For an integer $s \geq 1$, let $U^s = \{(t_1, \ldots, t_s) \in \mathbb{R}^s : 0 \leq t_i \leq 1$ for $1 \leq i \leq s\}$ be the $s$-dimensional unit cube. We set

$$\chi(x, y) = \begin{cases} 1 & \text{if } \{x_i\} < y_i \ (i = 1, \ldots, s), \\ 0 & \text{otherwise} \end{cases}$$

for $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ and $y = (y_1, \ldots, y_s) \in U^s$.

Definition 4. For $N \in \mathbb{N}$, the discrepancy of the sequence $(x_n), n = 1, 2, \ldots$, in $\mathbb{R}^s$ is defined by

$$D_N(x_n) = \sup_{y \in U^s} \left| \frac{1}{N} \sum_{n=1}^{N} \chi(x_n, y) - y_1 \cdots y_s \right|.$$ 

where $y = (y_1, \ldots, y_s) \in U^s$.

Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$. Suppose that $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Z}$.

Definition 5. For a real number $\eta$, the vector $\alpha$ is said to be of type $\eta$ if $\eta$ is the infimum of all real numbers $\tau$ for which there exists a positive constant $c = c(\tau, \alpha)$
such that

\[ r(h)^\tau \| h \cdot \alpha \| \geq c \]

holds for all lattice points \( h \neq 0 \) in \( \mathbb{R}^s \), where \( r(h) = \prod_{i=1}^{s} \max\{1, |h_i|\} \), \( h = (h_1, \ldots, h_s) \).

From Minkowski’s linear form theorem, we have \( \eta \geq 1 \).

**Definition 6.** The vector \( \alpha \) is said to be of constant type if there exists a positive constant \( c \) such that

\[ r(h) \| h \cdot \alpha \| \geq c \]

holds for all lattice points \( h \neq 0 \) in \( \mathbb{R}^s \).

1 Theorems and Examples

Tichy and Turnwald[4] proved the following:

**Theorem 1.** For any \( \epsilon > 0 \)

\[ D_N(\omega) \ll_{\alpha, \beta} N^{-\frac{1}{\eta+1} + \epsilon}, \]

provided \( \alpha \) is an irrational number of finite type \( \eta \geq 1 \).

In 1999, Ohkubo[3] improved Theorem 1 as follows:

In 1999, Ohkubo[3] improved Theorem 1 as follows:
Theorem 2. If $\alpha$ is an irrational number of finite type $\eta \geq 1$, then for any $\epsilon > 0$

$$D_N(\omega) \ll \beta N^{-\frac{3}{2(\eta+1/2)}+\epsilon},$$

and also if $\alpha$ is an irrational number of constant type, then

$$D_N(\omega) \ll \beta N^{\frac{3}{2}} \log N.$$

We found out an another proof of Theorem 2 and an extension to the multidimensional case as follows (see [1]):

Theorem 3. Let $\epsilon$ be an arbitrary positive number, $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ and $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{R}^s$ with $\beta \neq 0$. If $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Z}$ and $\alpha$ is of finite type $\eta$, then we have

$$D_N(n\alpha + (\log n)\beta) \ll \beta N^{-\frac{3}{2(\eta+1/2)}+\epsilon}.$$

Theorem 4. Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ and $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{R}^s$ with $\beta \neq 0$. If $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Z}$ and $\alpha$ is of constant type, then we have

$$D_N(n\alpha + (\log n)\beta) \ll \beta N^{-\frac{3}{2}} (\log N)^s.$$ 

Recently, we also generalized Theorem 2.

Theorem 5. Let $f(x)$ be twice differentiable for $x \geq 1$. Suppose also that there exists an irrational number $\alpha$ of finite type $\eta$ such that either

$$f'(x) > \alpha, f''(x) < 0 \quad \text{or} \quad f'(x) < \alpha, f''(x) > 0 \quad \text{for} \quad x \geq 1$$
and \( f'(x) = \alpha + O(|f''(x)|^{1/2}) \). Then for any \( \epsilon > 0 \)

\[
D_N(f(n)) \ll N^{-\frac{1}{\pi_1/2} + \epsilon}.
\]

**Proof.** Let \( h \) be a positive integer. Applying \([5, \text{p.74, Lemma 4.7}]\), we get

\[
\left| \sum_{n=1}^{N} e^{2\pi i h f(n)} \right| \ll \sum_{A-1/2 < \nu < B+1/2} \left| \int_{1}^{N} e^{2\pi i (h f(x) - \nu x)} dx \right| + \log(B - A + 2),
\]

where \( A = hf'(N) \) and \( B = hf'(1) \). We set \( g(x) = h\{f(x) - \alpha x\} \). Using integration by parts, we have

\[
\int_{1}^{N} e^{2\pi i (h f(x) - \nu x)} dx = \int_{1}^{N} e^{2\pi i (h a - \nu x + g(x))} dx = \int_{1}^{N} e^{2\pi i (h a - \nu x)} e^{2\pi i g(x)} dx
\]

\[
= \left[ \frac{e^{2\pi i (h a - \nu x)}}{2\pi i (h a - \nu)} e^{2\pi i g(x)} \right]_{1}^{N} - \frac{1}{h a - \nu} \int_{1}^{N} g'(x) e^{2\pi i ((h a - \nu) x + g(x))} dx.
\]

Hence,

\[
\int_{1}^{N} e^{2\pi i (h f(x) - \nu x)} dx \ll \frac{1}{|h a - \nu|} + \frac{1}{|h a - \nu|} \left| \int_{1}^{N} g'(x) e^{2\pi i ((h a - \nu) x + g(x))} dx \right|.
\]

We suppose that

\( f'(x) > \alpha \) and \( f''(x) < 0 \) for \( x \geq 1 \).

From \([6, \text{p.226, Lemma 10.5}]\) and the hypothesis, it follows that

\[
\left| \int_{1}^{N} g'(x) e^{2\pi i ((h a - \nu) x + g(x))} dx \right| \ll h^{1/2} \max_{1 \leq x \leq N} \left\{ \frac{f'(x) - \alpha}{|f''(x)|^{1/2}} \right\} + 1 \ll h^{1/2}.
\]
Hence we have

\[
\left| \sum_{n=1}^{N} e^{2\pi i h f(n)} \right| \ll h^{1/2} \sum_{A^{-1/2} < \nu < B^{1/2}} \frac{1}{|h\alpha - \nu|} + \log(B - A + 2)
\]

\[
\ll h^{1/2} \left\{ \frac{1}{||h\alpha||} + \int_{||h\alpha||}^{h} \frac{1}{x} dx \right\}
\]

\[
+ \log[h\{f'(1) - f'(N)\}] + 2]
\]

\[
\ll h^{1/2} \left\{ \frac{1}{||h\alpha||} + \log[h\{f'(1) - \alpha\} + 2] \right\}.
\]

Applying Erdős-Turán inequality and \(\sum_{h=1}^{m} \frac{1}{h^{1/2}||h\alpha||} \ll m^{\eta-1/2+\delta}\) (see [2, p.123, Lemma 3.3]), for any positive integer \(m\), we obtain

\[
D_N(f(n)) \ll \frac{1}{m} + \frac{1}{N} \left\{ \sum_{h=1}^{m} \frac{1}{h^{1/2}||h\alpha||} + \sum_{h=1}^{m} \frac{\log[h\{f'(1) - \alpha\} + 2]}{h^{1/2}} \right\}
\]

\[
\ll \frac{1}{m} + \frac{1}{N} (m^{\eta-1/2+\delta} + m^{1/2} \log m)
\]

\[
\ll \frac{1}{m} + \frac{1}{N} m^{\eta-1/2+\delta},
\]

for any \(\delta > 0\).

Choosing \(m = \left[ N^{1/\eta+1/2} \right] \), we have

\[
D_N(f(n)) \ll N^{-\frac{1}{\eta+1/2}} + N^{-\frac{1}{\eta+1/2} + \frac{\delta}{\eta+1/2}} \ll N^{-\frac{1}{\eta+1/2} + \epsilon}.
\]

In the case \(f'(x) < \alpha, f''(x) > 0\) for \(x \geq 1\), the proof runs along the same lines as above.

\[\square\]

**Remark 1.** The following was shown by van der Corput: If \(f(x), x \geq 1\), is differentiable for sufficiently large \(x\) and \(\lim_{x \to \infty} f'(x) = \alpha(\text{irrational})\), then the sequence \((f(n))\) is uniformly distributed mod 1 (see [2, p.28, Theorem 3.3 and p.31,
Exercises 3.5]). If the function $f(x)$ in Theorem 5 also satisfies the condition
\[ \lim_{x \to \infty} f''(x) = 0, \text{ then } \lim_{x \to \infty} f'(x) = \alpha. \]
Therefore, Theorem 5 gives a quantitative aspect of van der Corput's result.

Examples. $f(x) = \alpha x + \beta \log \log x$, or $f(x) = \alpha x + \beta \log x$.

Acknowledgement

The authors are deeply indebted to Professor S.Akiyama for his many helpful comments.

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