Dynamical representations of substituted Sturmian sequences

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1 Introduction

We announce some theorems about Sturmian words in this report. The proofs and details will be published elsewhere. We need some notations. Let \( L \) be an alphabet, i.e., a non-empty finite set of letters. Now, we set \( L = \{0, 1\} \). Let \( W = \bigcup_{n=0}^{\infty}(0,1)^n \), \( W^* = \bigcup_{n=0}^{\infty}(0,1) \cup \{0,1\}^n \). For \( x, y \in [0,1] \) we define \( G(x,y), \tilde{G}(x,y) \in W^* \) by

\[
G(x,y) = G_0(x,y)G_1(x,y)\ldots
\]

\[
\tilde{G}(x,y) = \tilde{G}_0(x,y)\tilde{G}_1(x,y)\ldots
\]

where \( G_j(x,y) = [(j+1)x+y] - [jx+y] \), \( \tilde{G}_j(x,y) = [(j+1)x+y] - [jx+y] \) for each integer \( j \)

and \([u]\) is an integral part of \( u \) and \([u] = -[-u]\) for each \( u \in \mathbb{R} \).

Examples
\( x = \frac{1}{3} \)

\[
G(x,0) = 001001\ldots = (001)_{\infty}
\]

\( x = \sqrt{2} - 1, y = \frac{1}{2} \)

\[
G(x,y) = 0101001010\ldots
\]

For \( w \in L^\mathbb{N} \), we define \( \text{Sub}(w) \) by

\[
\text{Sub}(w) = \{ u \in W \mid u \text{ is a subword of } w \}. \]

A Sturmian word is defined to be a word \( w \in L^\mathbb{N} \) satisfying

\[
|A|_1 - |B|_1 \leq 1
\]

for any \( A, B \in \text{Sub}(w) \) with \(|A| = |B|\), where for \( u \in W \) \(|u|\) is a length of \( u \) and \(|u|_1\) is a number of the occurrences of the letter 1 in \( u \). In this lecture we consider only non periodic Sturmian word.

**Theorem.** (Morse and Hedlund [2]; Coven and Hedlund[3]) \( w \) is Sturmian if and only if \( w \) is equal to \( G(x,y) \) or \( \tilde{G}(x,y) \) for some \( x, y \in [0,1] \).

A transformation \( f \) on \( W^* \) is called substitution on \( W^* \), if \( f \) satisfies following conditions:
1. $f(0), f(1) \in W$,
2. for any $a \in W$ and $b \in W^*$, $f(ab) = f(a)f(b)$.

**Example**

Let $f$ be a substitution on $W^*$ defined by

$$f : \begin{cases} 
0 & \rightarrow 01 \\
1 & \rightarrow 010
\end{cases}$$

Then,

$$f(00101) = f(0)f(0)f(1)f(0)f(1) = 010101001010.$$ 

Let $\alpha$ be a real number. Define right infinite word $\chi(S, \alpha) \in W^*$ for a set $S$ in interval $[0, 1]$ and $\alpha$ by

$$\chi(S, \alpha) = \lambda(S, \alpha; 0)\lambda(S, \alpha; 1) \cdots,$$

where

$$\lambda(S, \alpha; n) = \begin{cases} 
1 & \text{if } \langle n\alpha \rangle \in S, \\
0 & \text{if } \langle n\alpha \rangle \notin S,
\end{cases}$$

where $\langle x \rangle$ is a fractional part of $x$.

**Example of $\chi(S, \alpha)$**

We define a mod 1 semiclosed interval $[x, y]^\sim$ for $0 \leq x, y \leq 1$ by

$$[x, y]^\sim = \begin{cases} 
[x, y) & \text{if } 0 \leq x \leq y, \\
[0, y) \cup [x, 1) & \text{if } 0 \leq y < x.
\end{cases}$$

We can define a mod 1 semiclosed interval $(x, y)^\sim$ in the same manner as above. Our main result is as follows.
Theorem 1. Let $S$ be a Sturmian sequence. Let $F$ be a substitution with $\text{GCD}(|F(0)|, |F(1)|) = 1$. Then, there exist $x, y \in \mathbb{R}$ and integers $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ such that $x$ is irrational and $0 < x < 1$ and

$$\chi(I, x) = F(S),$$

where

$$I = \bigcup_{i=1}^{k} [(m_i x - y), (n_i x - y)]^\circ,$$

or

$$I = \bigcup_{i=1}^{k} [(m_i x - y), (n_i x - y)]^\circ.$$

The converse is also true.

2 An algorithm on inhomogeneous Diophantine approximation

We introduce the following algorithm on inhomogeneous Diophantine approximation to prove main Theorem. Let us define functions $t_0, t_1, t_2$ on $\mathbb{R}^2$ by

$$t_0(x, y) = \left(\frac{x}{1+x}, \frac{y}{1+x}\right),$$

$$t_1(x, y) = \left(\frac{1}{2-x}, \frac{y}{2-x}\right),$$

$$t_2(x, y) = (1-x, 1-y).$$

Let us a domain $X$ by

$$X = \{(x, y)|0 \leq x, y \leq 1 \text{ and } y \neq mx + n \text{ for any integers } m, n\}.$$

We define domains $S_i^0 (i = 0, \ldots, 5)$ by

![Figure of X]

We define transformation $T_0$ on $X$ as follows:

$$T_0(x, y) = \begin{cases} 
    t_0^{-1}(x, y) & \text{if } (x, y) \in S_0^0, \\
    t_1^{-1}(x, y) & \text{if } (x, y) \in S_1^0, \\
    t_2^{-1} \circ t_0^{-1}(x, y) & \text{if } (x, y) \in S_2^0, \\
    t_2^{-1} \circ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_3^0, \\
    t_2^{-1} \circ t_1^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_4^0, \\
    t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_5^0.
\end{cases}$$
We define domains \( S^i_t \) \((i = 0, \ldots, 5)\) as follows:

\[
(0, 1) \quad (1/2, 1) \quad (1, 1)
\]

\[
(0, 0) \quad (1/2, 0) \quad (1, 0)
\]

Figure of \( X \)

We define transformation \( T_1 \) on \( X \) as follows:

\[
T_1(x, y) = \begin{cases} 
\tau_1^{-1}(x, y) & \text{if } (x, y) \in S^1_0, \\
\tau_0^{-1}(x, y) & \text{if } (x, y) \in S^1_1, \\
\tau_2^{-1} \circ \tau_1^{-1}(x, y) & \text{if } (x, y) \in S^1_2, \\
\tau_2^{-1} \circ \tau_1^{-1} \circ \tau_2^{-1}(x, y) & \text{if } (x, y) \in S^1_3, \\
\tau_1^{-1} \circ \tau_2^{-1}(x, y) & \text{if } (x, y) \in S^1_4, \\
\tau_0^{-1} \circ \tau_2^{-1}(x, y) & \text{if } (x, y) \in S^1_5.
\end{cases}
\]

For \((x, y) \in X\) we consider the following binary tree:
We associate $u = \{i_1, i_2, \ldots \} \in \{0, 1\}^\mathbb{N}$ with a path in the tree like the following example:

$$u = i_1, i_2, i_3, \ldots = 0, 1, 0, \ldots$$

(\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (T0) at (0,0) {$T_0(x, y)$};
  \node (T1) at (1,0) {$T_1(x, y)$};
  \node (T0T0) at (0,1) {$T_0T_0(x, y)$};
  \node (T1T0) at (1,1) {$T_1T_0(x, y)$};
  \node (T0T1) at (0,2) {$T_0T_1(x, y)$};
  \node (T1T1) at (1,2) {$T_1T_1(x, y)$};
  \node (x,y) at (0,3) {$(x, y)$};

  \draw[->] (T0) -- (T0T0);
  \draw[->] (T0) -- (T0T1);
  \draw[->] (T1) -- (T1T0);
  \draw[->] (T1) -- (T1T1);
  \draw[->] (T0T0) -- (x,y);
  \draw[->] (T0T1) -- (x,y);
  \draw[->] (T1T0) -- (x,y);
  \draw[->] (T1T1) -- (x,y);
\end{tikzpicture}
\end{figure})

For $u = \{i_1, i_2, \ldots \} \in \{0, 1\}^\mathbb{N}$ and a positive integer $n$, we define $g(u, n, (x, y)) \in X$ by

$$g(u, n, (x, y)) = T_{i_1} \cdots T_{i_n} (x, y).$$

We define a sequence $S(u, (x, y)) = \{j_n\}_{n=1}^\infty \in \{0, 1, 2, 3, 4, 5\}^\mathbb{N}$ which is called the name of $(x, y)$ related to $u \in \{0, 1\}^\mathbb{N}$ as follows: for $n = 1, 2, \ldots$

$$g(u, n - 1, (x, y)) \in S_{j_n}^i.$$
where \( \{j_1, j_2, \ldots \} \) is the name of \((x, y)\) related to \(u\).

We define substitutions \(s_0, s_1, c\) on \(L\) by

\[
\begin{align*}
    s_0 &: \begin{cases} 
    0 &\mapsto 0, \\
    1 &\mapsto 01. 
\end{cases} \\
    s_1 &: \begin{cases} 
    0 &\mapsto 01, \\
    1 &\mapsto 1. 
\end{cases} \\
    c &: \begin{cases} 
    0 &\mapsto 1. 
\end{cases}
\end{align*}
\]

For \(i, k \in \{0, 1\} \times \{0, 1, 2, 3, 4, 5\}\), we define substitutions \(\phi(i, k)\) as follows:

\[
\phi(i, j) = \begin{cases} 
    s_0 & \text{if } (i, j) = (0, 0), \\
    s_1 & \text{if } (i, j) = (0, 1), \\
    s_0c & \text{if } (i, j) = (0, 2), \\
    es_0c & \text{if } (i, j) = (0, 3), \\
    es_1c & \text{if } (i, j) = (0, 4), \\
    es_0 & \text{if } (i, j) = (0, 5), \\
    s_1 & \text{if } (i, j) = (1, 0), \\
    s_0 & \text{if } (i, j) = (1, 1), \\
    s_1c & \text{if } (i, j) = (1, 2), \\
    es_1c & \text{if } (i, j) = (1, 3), \\
    es_0c & \text{if } (i, j) = (1, 4), \\
    es_1 & \text{if } (i, j) = (1, 5). 
\end{cases}
\]

By the theory [1] we have the following important Lemma:

Lemma

\[
\phi(i_1, j_1) \cdots \phi(i_n, j_n)G(g(u, n, (x, y))) = G(x, y).
\]

For substitutions \(f\) and \(g\) on \(W^*\) we say that \(f\) is equivalent to \(g\), if for any \(w \in W\) \(|f(w)| = |g(w)|\).

We have the following Theorem 2.

**Theorem 2** Let \((x, y) \in X\). Let \(u = \{i_1, i_2, \ldots \} \in \{0, 1\}^\mathbb{N}\) be a good path in the previous tree related to \((x, y)\). Let \(I\) be a finite union of intervals \([m_1 \alpha - \beta, m_2 \alpha - \beta]) \sim (m_1, m_2 \in \mathbb{Z})\). Then, there exist a integer \(k \geq 0\) and a substitution \(\psi\) on \(W^*\) which is equivalent to \(\phi(i_1, j_1) \cdots \phi(i_k, j_k)\) such that

\[
\chi(I, x) = \psi(G(y(u, k, (x, y)))),
\]

where \(\{j_1, j_2, \ldots \}\) is the name of \((x, y)\) related to \(u\). The converse also holds.

From Theorem 2 and considering homogeneous cases \((y = mx + n)\), we get Theorem 1.
References


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