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THE NONRELATIVISTIC LIMIT OF THE NONLINEAR KLEIN–GORDON EQUATION

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ABSTRACT. In this paper we consider the nonrelativistic limit of the nonlinear Klein–Gordon equation. We study how the solutions of the nonlinear Klein–Gordon equation converge toward the corresponding solutions of the nonlinear Schrödinger equation when the speed of light tends to infinity. Especially we consider the rate of convergence. We use Strichartz's estimate for the Klein–Gordon equation.

1. INTRODUCTION

We consider the nonlinear (and linear) Klein–Gordon equation in space-time $\mathbb{R}^{n+1}$

$$\frac{\hbar^2}{2mc^2}u'' - \frac{\hbar^2}{2m}\Delta u + \frac{mc^2}{2}u + \lambda|u|^{\gamma-1}u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1}$$

where $\hbar$ is the Planck constant, m is the mass of particle, c is the speed of light, and u'' is the second time derivative, and $\lambda > 0$. When $n = 3$ and $\gamma = 3$, the equation (1.1) was introduced by Schiff [1] as the equation of classical neutral scalar mesons. If $\lambda = 0$, the equation (1.1) is the linear Klein–Gordon equation.

Substituting

$$u = ve^{-imc^2t/\hbar},$$

we obtain from (1.1) the following nonlinear Klein–Gordon equation for v:

$$\frac{\hbar}{2mc^2}v'' - i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

The aim of this paper is to study this equation, particularly in the limit $c \to \infty$. We regard the procedure of taking limit $c \to \infty$ as "nonrelativistic limit." Formally, the limit equation is

$$-i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

This is the nonlinear Schrödinger equation. So we expect that solutions of the nonlinear Klein–Gordon equation converge as $c \to \infty$ toward the corresponding solutions of the nonlinear Schrödinger equation. We may think of the Klein–Gordon equation as a relativistic generalization for the Schrödinger equation. From this relation, we have a particular interest in the convergence of solutions of two equations. In this paper we study this problem in detail. For simplicity, we set $A = -\Delta$, $\varepsilon = 1/c^2$, $f(v) = \lambda|v|^{\gamma-1}v$, and $\hbar = 2m = 1$. Given initial data, we rewrite the equations in question as

$$\varepsilon v'' - iv' + Av + f(v) = 0,\ v(0) = v_{0\varepsilon},\ v'(0) = v_{1\varepsilon}, \tag{1.2}$$

$$-iv' + Av + f(v) = 0,\ v(0) = v_{00}. \tag{1.3}$$

We denote by v_ε and v_0 the solution of (1.2) and (1.3), respectively.

We investigate how v_ε converges to v_0 as $\varepsilon \to 0$. There are a few results on the problem. The convergence in several modes has been proved, see [2] [3]. In [15], we have proved the convergence in $L^\infty(0,T;L^2)$. In this paper, we consider the rate of this convergence. When ε tends to 0, how rapidly does v_ε converge toward v_0 ? We show in Theorem 1 the upper bound of the order for nonlinear case. For linear case, we give the upper bound as well as the lower bound in Theorem 2.

This paper is constructed as follows. In Section 2, we state the main theorem. In Section 3, we give Strichartz's estimate for the Klein–Gordon equation. Using this estimate, we prove the main theorem in Section 4.

We close this section by giving several notation. We abbreviate $L^q(\mathbb{R}^n)$ to L^q and $L^r(I;L^q(\mathbb{R}^n))$ to L^rL^q , where I is a time interval. We denote by $H^{s,p}$ and $B^s_{p,l}$ the Sobolev space and Besov space of order s, respectively. For any p with $1 < p < \infty$, p' stands for its Hölder conjugate, i.e. $p' = p/(p-1)$.

2. Main Theorem

We state our main theorem.

Theorem 1. (Nonlinear Case)
Let $n = 3$, $\lambda > 0$ and $1 < \gamma < 21/5$. We assume that

$$v_{0\varepsilon} \in H^1,\ v_{1\varepsilon} \in L^2, \tag{2.1}$$

$$v_{00} \in H^1, \tag{2.2}$$

$$\sup_{\varepsilon>0}(\|v_{0\varepsilon}\|_{H^1} + \varepsilon^{1/2}\|v_{1\varepsilon}\|_{L^2}) < \infty, \tag{2.3}$$

$$\|v_{0\varepsilon} - v_{00}\|_{L^2} \leq c\varepsilon^{1/4}. \tag{2.4}$$

Then for every $T > 0$, there exists c such that

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \leq c\varepsilon^{1/4}. \tag{2.5}$$

Remark 1.
In [15], we have shown only convergence of the LHS of (2.5) without specific rate. Theorem 1 gives an upper bound of the rate of this convergence.

Theorem 2. (Linear Case)
Let $\lambda = 0$. We assume (2.1),(2.2),(2.3), and (2.4).
Then for every $T > 0$, there exists c such that

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \leq c\varepsilon^{1/4}. \tag{2.6}$$

Moreover, for any $\alpha \geq 1/4$, $\delta > 0$, there exist $v_{0\varepsilon}$ and v_{00} such that

$$\|v_{0\varepsilon} - v_{00}\|_{L^2} \leq c\varepsilon^\alpha, \tag{2.7}$$

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \geq c\varepsilon^{1/4+\delta}. \tag{2.8}$$

3. Strichartz's type estimate for the Klein–Gordon equation

In this section we study the space–time integrability properties of solutions of the free Klein–Gordon equation for the proof of Theorem 1. To this end we construct Strichartz's estimate involving the parameter ε for equation (1.2). From Duhamel principle, the solution v_ε of (1.2) satisfies the integral equation,

$$v_\varepsilon(t) = I_\varepsilon(t)v_{0\varepsilon} + J_\varepsilon(t)v_{1\varepsilon} - \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)f(v_\varepsilon(s))ds, \tag{3.1}$$

where

$$\begin{aligned} I_\varepsilon(t) &= e^{\frac{it}{2\varepsilon}}(\cos tA_\varepsilon - \frac{i}{2\varepsilon}A_\varepsilon^{-1}\sin tA_\varepsilon), \\ J_\varepsilon(t) &= e^{\frac{it}{2\varepsilon}}A_\varepsilon^{-1}\sin tA_\varepsilon, \\ A_\varepsilon &= \frac{1}{\varepsilon}\Big(\varepsilon A + \frac{1}{4}\Big)^{1/2}. \end{aligned}$$

We investigate the operator $J_\varepsilon(t)$.

Proposition 3. For any interval $I \subset \mathbb{R}$ with $0 \in \bar{I}$, $u \in C_0(I \times \mathbb{R}^n)$ and pair (q', r') such that

$$1 - \frac{1}{r'} = \frac{n}{2}\Big(\frac{1}{q'} - \frac{1}{2}\Big) \quad , \quad \frac{1}{2} \leq \frac{1}{q'} \leq \frac{1}{2} + \frac{2}{n+2}, \tag{3.2}$$

the following estimate holds :

$$\Big\|\int_0^t \frac{1}{\varepsilon}J_\varepsilon(t-s)u(s)ds\Big\|_{L^\infty(I;L^2)} \leq c\|u\|_{L^{r'}(I;L^{q'})}, \tag{3.3}$$

where c is independent of u, I, and ε.

Proof of Proposition 3.
We introduce the results on decay of solution of Klein–Gordon equation, (see [13]). For any $1 < q' \leq 2 \leq q < \infty$, the following inequality holds :

$$\|\sin((I - \Delta)^{1/2}t)u\|_{L^q(\mathbb{R}^n)} \leq ct^{-n(1/2-1/q)}\|u\|_{H^{(n+2)(1/2-1/q),q'}(\mathbb{R}^n)}. \tag{3.4}$$

We investigate the operator $K_\varepsilon(t) = e^{itA_\varepsilon}$ first, and then the operator $J_\varepsilon(t)$. We define

$$\mathfrak{K}_\alpha(t) = e^{it(A+\alpha)^{1/2}}.$$

For $\beta > 0$, we define $(U_\beta f)(x) = f(\beta x)$ and we use the facts that $U_\beta^{-1} = U_{1/\beta}$, that $\beta^{n/p}U_\beta$ is an isometry on L^p and that

$$\mathfrak{K}_\alpha(t) = U_{\alpha^{1/2}}\mathfrak{K}_1(\alpha^{1/2}t)U_{\alpha^{1/2}}^{-1}.$$

Therefore we have

$$\begin{aligned} K_\varepsilon(t) &= \mathfrak{K}_{1/4\varepsilon}(t/\varepsilon^{1/2}) \\ &= U_{(4\varepsilon)^{-1/2}}\mathfrak{K}_1(t/(2\varepsilon))U_{(4\varepsilon)^{-1/2}}^{-1}. \end{aligned}$$

From this identity and (3.4), we obtain,

(3.5)
$$\begin{aligned} \|K_\varepsilon(t)u\|_{L^q} &= \|U_{(4\varepsilon)^{-1/2}}\mathfrak{K}_1(t/(2\varepsilon))U_{(4\varepsilon)^{-1/2}}^{-1}u\|_{L^q} \\ &= c\varepsilon^{n/(2q)}\|\mathfrak{K}_1(t/(2\varepsilon))U_{(4\varepsilon)^{-1/2}}^{-1}u\|_{L^q} \\ &\le c\varepsilon^{n/(2q)}(t/(2\varepsilon))^{-n(1/2-1/q)}\|U_{(4\varepsilon)^{-1/2}}^{-1}u\|_{H^{(n+2)(1/2-1/q),q'}} \\ &= c\varepsilon^{n/(2q)}(t/(2\varepsilon))^{-n(1/2-1/q)}\|(I-\Delta)^{(1/2)(n+2)(1/2-1/q)}U_{(4\varepsilon)^{-1/2}}^{-1}u\|_{L^{q'}} \\ &= c\varepsilon^{n/(2q)}(t/(2\varepsilon))^{-n(1/2-1/q)}\|U_{(4\varepsilon)^{-1/2}}^{-1}(I-4\varepsilon\Delta)^{(1/2)(n+2)(1/2-1/q)}u\|_{L^{q'}} \\ &= c\varepsilon^{n/(2q)}(t/(2\varepsilon))^{-n(1/2-1/q)}\varepsilon^{-n/(2q')}\|(4\varepsilon A+1)^{(1/2)(n+2)(1/2-1/q)}u\|_{L^{q'}} \\ &= ct^{-n(1/2-1/q)}\|(4\varepsilon A+1)^{(1/2)(n+2)(1/2-1/q)}u\|_{L^{q'}}. \end{aligned}$$

Thus

(3.6)
$$\left\|\int_0^t K_\varepsilon(t-s)u(s)ds\right\|_{L^q} \le c\int_0^t |t-s|^{-n(1/2-1/q)}\|(4\varepsilon A+1)^{(1/2)(n+2)(1/2-1/q)}u\|_{L^{q'}}ds.$$

The Hardy-Littlewood-Sobolev inequality in time implies

$$\left\|\int_0^t K_\varepsilon(t-s)u(s)ds\right\|_{L^rL^q} \le c\|(4\varepsilon A+1)^{(1/2)(n+2)(1/2-1/q)}u\|_{L^{r'}L^{q'}}, \tag{3.7}$$

with

$$2/r = n\,(1/2-1/q).$$

We denote by (,) the L^2 scalar product and estimate

(3.8)

$$\begin{aligned}
&\left\|\int_0^t K_\varepsilon(t-s)u(s)ds\right\|_{L^\infty L^2}^2\\
&=\sup_t\left(\int_0^t e^{i(t-s)A_\varepsilon}u(s)ds \quad , \int_0^t e^{i(t-s')A_\varepsilon}u(s')ds'\right)\\
&=\sup_t\int_0^t ds'\left(\int_0^t e^{i(s'-s)A_\varepsilon}(4\varepsilon A+1)^{-(1/4)(n+2)(1/2-1/q)}u(s)ds \quad , (4\varepsilon A+1)^{(1/4)(n+2)(1/2-1/q)}u(s')\right)\\
&\leq c\|\int_0^t K_\varepsilon(s'-s)(4\varepsilon A+1)^{-(1/4)(n+2)(1/2-1/q)}u(s)ds\|_{L^rL^q}\|(4\varepsilon A+1)^{(1/4)(n+2)(1/2-1/q)}u\|_{L^{r'}L^{q'}}\\
&\leq c\|(4\varepsilon A+1)^{(1/4)(n+2)(1/2-1/q)}u\|_{L^{r'}L^{q'}}^2
\end{aligned}$$

We used the Holder inequality in space and time at third inequality. Last inequality is from (3.7). We consider the operator J_ε. We know

$$\left\|\frac{1}{\varepsilon}J_\varepsilon(t)u\right\|_{L^2}=c\|(4\varepsilon A+1)^{-1/2}K_\varepsilon(t)u\|_{L^2}.$$

Therefore we have from (3.8)

$$\left\|\int_0^t\frac{1}{\varepsilon}J_\varepsilon(t-s)u(s)ds\right\|_{L^\infty L^2}\leq c\|(4\varepsilon A+1)^{(1/4)(n+2)(1/2-1/q)-1/2}u\|_{L^{r'}L^{q'}} \tag{3.9}$$

The following inequality holds for any $\alpha>0,\ 1<p<\infty$,

$$\|(4\varepsilon A+1)^{-\alpha}u\|_{L^p}\leq c\|u\|_{L^p}, \tag{3.10}$$

here c is independent of u and ε.

Therefore we have

$$\left\|\int_0^t\frac{1}{\varepsilon}J_\varepsilon(t-s)u(s)ds\right\|_{L^\infty L^2}\leq c\|u\|_{L^{r'}L^{q'}}, \tag{3.11}$$

with

$$(1/4)(n+2)(1/2-1/q)-1/2\leq 0.$$

This is expected estimate.

4. Proof of The Main Theorem

At first, we recall some properties of the solutions of nonlinear Klein–Gordon equation and nonlinear Schrödinger equation. From the assumption (2.1), there exists a unique solution v_ε of (1.2) such that (see [14])

$$v_\varepsilon\in L^\infty(0,T;H^1)\ \cap\ L^{r(q)}(0,T;B_{q,2}^{1-\sigma(q)}),$$

with

$$\frac{2\sigma(q)}{n+1}=\frac{2}{r(q)(n-1)}=\frac{1}{2}-\frac{1}{q}\quad ,\qquad 2\leq q<\infty,\quad n\leq 3.$$

Moreover by the assumption (2.3) and the energy conservation for (1.2), we obtain

$$\sup_{\varepsilon>0}(\|v_\varepsilon\|_{L^\infty(0,T;H^1)}+\|v_\varepsilon\|_{L^{r(q)}(0,T;B_{q,2}^{1-\sigma(q)})})<\infty. \tag{4.1}$$

For the case of equation (1.3) , there exists a unique solution (see [8])

$$v_0 \in L^\infty(0,T;H^1) \cap L^{s(p)}(0,T;W^{1,p}),$$

with

$$\frac{2}{s(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right), \qquad 2 \le p < \frac{2n}{n-2}.$$

From the conservation laws of energy and charge for (1.3) , we obtain

$$\|v_0\|_{L^\infty(0,T;H^1)} + \|v_0\|_{L^{s(p)}(0,T;W^{1,p})} < \infty. \tag{4.2}$$

Proof of Theorem 1.

We consider the case of space dimension 3. The solution v_ε of (1.2) satisfies (3.1) . The solution v_0 of (1.3) satisfies

$$v_0(t) = I_0(t)v_{00} - i\int_0^t I_0(t-s)f(v_0(s))ds, \tag{4.3}$$

with

$$I_0(t) = e^{-iAt}.$$

We study $v_\varepsilon - v_0$. Subtracting (4.3) from (3.1) yields

$$v_\varepsilon(t) - v_0(t) = \sum_{i=1}^{5} P_\varepsilon^{(i)}(t), \tag{4.4}$$

with

$$P_\varepsilon^{(1)}(t) = (I_\varepsilon(t) - I_0(t))v_{00}, \tag{4.5}$$

$$P_\varepsilon^{(2)}(t) = I_\varepsilon(t)(v_{0\varepsilon} - v_{00}), \tag{4.6}$$

$$P_\varepsilon^{(3)}(t) = J_\varepsilon(t)v_{1\varepsilon}, \tag{4.7}$$

$$P_\varepsilon^{(4)}(t) = \int_0^t (iI_0(t-s) - \frac{1}{\varepsilon}J_\varepsilon(t-s))f(v_0(s))ds, \tag{4.8}$$

$$P_\varepsilon^{(5)}(t) = \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)(f(v_0(s)) - f(v_\varepsilon(s)))ds. \tag{4.9}$$

We investigate $\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)}$,

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \le \sum_{i=1}^{5} \|P_\varepsilon^{(i)}\|_{L^\infty(0,T;L^2)}. \tag{4.10}$$

With respect to $P_\varepsilon^{(5)}$, we use Proposition 3 to have

$$\|P_\varepsilon^{(5)}\|_{L^\infty(0,T;L^2)} \le c\|f(v_\varepsilon) - f(v_0)\|_{L^{r'}(0,T;L^{q'})}, \tag{4.11}$$

where

$$1 - \frac{1}{r'} = \frac{3}{2}\left(\frac{1}{q'} - \frac{1}{2}\right), \qquad \frac{1}{2} \le \frac{1}{q'} \le \frac{9}{10}.$$

The Hölder inequality implies

$$\|f(v_\varepsilon) - f(v_0)\|_{L^{r'}L^{q'}} \le c(\|v_\varepsilon\|_{L^aL^b}^{\gamma-1} + \|v_0\|_{L^aL^b}^{\gamma-1})\|v_\varepsilon - v_0\|_{L^\infty L^2}, \tag{4.12}$$

where

$$\frac{1}{q'} = \frac{\gamma - 1}{b} + \frac{1}{2} \quad , \qquad \frac{1}{r'} = \frac{\gamma - 1}{a}. \tag{4.13}$$

We use the following embedding results,

$$B^{1-\sigma}_{q,2} \subset L^b, \qquad \frac{1}{b} = \frac{1}{q} - \frac{1-\sigma}{n},$$

$$W^{1,q} \subset L^b, \qquad \frac{1}{b} = \frac{1}{q} - \frac{1}{n} \quad .$$

From this results and (4.1),(4.2), we estimate

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^8L^8} + \|v_0\|_{L^8L^8} < \infty. \tag{4.14}$$

Considering (4.13), if $\gamma < 21/5$, we can take $a < 8$, and

$$\|v_\varepsilon\|_{L^aL^8} + \|v_0\|_{L^aL^8} \leq T^{1/a-1/8}(\|v_\varepsilon\|_{L^8L^8} + \|v_0\|_{L^8L^8}). \tag{4.15}$$

Thus we obtain

$$\|P^{(5)}_\varepsilon\|_{L^\infty(0,T;L^2)} \leq cT^{(1/a-1/8)(\gamma-1)}\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)}.$$

We have from (4.10),

$$(1 - cT^{(1/a-1/8)(\gamma-1)})\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \leq \sum_{i=1}^{4} \|P^{(i)}_\varepsilon\|_{L^\infty(0,T;L^2)} \tag{4.16}$$

For sufficiently small T, we have

$$\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} \leq c\sum_{i=1}^{4} \|P^{(i)}_\varepsilon\|_{L^\infty(0,T;L^2)}. \tag{4.17}$$

So we have to study the rate of convergence for $P^{(i)}_\varepsilon, i = 1, 2, 3, 4$.

For $P^{(1)}_\varepsilon$, we rewrite $\cos tA_\varepsilon, \sin tA_\varepsilon$ with $e^{itA_\varepsilon}, e^{-itA_\varepsilon}$ and rearrange,

$$\begin{aligned} \|(I_\varepsilon(t) - I_0(t))v_{00}\|_{L^\infty L^2} &\leq \|\{(1/2)(1 + (4\varepsilon A + 1)^{-1/2})e^{\frac{it}{2\varepsilon}-itA_\varepsilon} - e^{-itA}\}v_{00}\|_{L^\infty L^2} \\ &\quad + \|(1/2)(1 - (4\varepsilon A + 1)^{-1/2})e^{\frac{it}{2\varepsilon}+itA_\varepsilon}v_{00}\|_{L^\infty L^2} \\ &\leq \|(e^{\frac{it}{2\varepsilon}-itA_\varepsilon+itA} - 1)v_{00}\|_{L^\infty L^2} \\ &\quad + \|(1 - (4\varepsilon A + 1)^{-1/2})v_{00}\|_{L^2} \\ &= \|(e^{ita_\varepsilon} - 1)v_{00}\|_{L^\infty L^2} + \|b_\varepsilon v_{00}\|_{L^2}, \end{aligned} \tag{4.18}$$

here we have set

$$a_\varepsilon = 1/(2\varepsilon) - A_\varepsilon + A,$$
$$b_\varepsilon = 1 - (4\varepsilon A + 1)^{-1/2}.$$

We study the operator $a_\varepsilon, b_\varepsilon$. From the Parseval relation, we have

$$\|Au\|_{L^2} = \||\xi|^2\tilde{u}\|_{L^2}.$$

We set $\tilde{a}_\varepsilon = 1/(2\varepsilon) - 1/(2\varepsilon)(4\varepsilon|\xi|^2+1)^{1/2} + |\xi|^2$ and estimate

$$
\begin{aligned}
|e^{it\tilde{a}_\varepsilon} - 1| &\le 2, \\
|e^{it\tilde{a}_\varepsilon} - 1| &= |i\tilde{a}_\varepsilon \int_0^t e^{is\tilde{a}_\varepsilon} ds| \\
&\le |\tilde{a}_\varepsilon t| \\
&= t\left|4\varepsilon|\xi|^4/((4\varepsilon|\xi|^2+1)^{1/2}+1)^2\right| \\
&\le 4t\varepsilon|\xi|^4.
\end{aligned}
$$

Thus

$$|e^{it\tilde{a}_\varepsilon} - 1| \le 2^{1-\theta}(4t\varepsilon|\xi|^4)^\theta, \qquad 0 \le \theta \le 1.$$

Considering assumption (2.2) , we set $\theta = 1/4$,

$$
\begin{aligned}
\|(e^{it\tilde{a}_\varepsilon} - 1)v_{00}\|_{L^\infty L^2} &\le c\|t^{1/4}\varepsilon^{1/4}|\xi|v_{00}\|_{L^\infty L^2} \\
&\le cT^{1/4}\varepsilon^{1/4}\||\xi|v_{00}\|_{L^2} \qquad (4.19)\\
&\le c\varepsilon^{1/4}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|1-(4\varepsilon|\xi|^2+)^{-1/2}| &\le 2, \\
|1-(4\varepsilon|\xi|^2+1)^{-1/2}| &= \left|4\varepsilon|\xi|^2/((4\varepsilon|\xi|^2+1)^{-1/2}+1)(4\varepsilon|\xi|^2+1)\right| \\
&\le 4\varepsilon|\xi|^2,
\end{aligned}
$$

then

$$|\tilde{b}_\varepsilon| = |1-(4\varepsilon|\xi|^2+1)^{-1/2}| \le (4\varepsilon|\xi|^2)^{1/2}2^{1/2}.$$

From this, we have

$$\|b_\varepsilon v_{00}\|_{L^2} \le c\varepsilon^{1/2}. \qquad (4.20)$$

Thus we have from (4.18),(4.19) and (4.20),

$$\|P_\varepsilon^{(1)}\|_{L^\infty L^2} \le c\varepsilon^{1/4}. \qquad (4.21)$$

From (2.4) and $\sup_{t\in[0,T],\varepsilon>0} \|I_\varepsilon(t)\|_{\mathfrak{L}(L^2)} < \infty$, we have

$$
\begin{aligned}
\|P_\varepsilon^{(2)}\|_{L^\infty L^2} &\le C\|v_{0\varepsilon} - v_{00}\|_{L^2} \qquad (4.22)\\
&\le c\varepsilon^{1/4}.
\end{aligned}
$$

The assumption (2.3) especially for $v_{1\varepsilon}$ implies

$$
\begin{aligned}
\|P_\varepsilon^{(3)}\|_{L^\infty L^2} &= \|2e^{\frac{it}{2\varepsilon}} \sin tA_\varepsilon\ \varepsilon(4\varepsilon A+1)^{-1/2}v_{1\varepsilon}\|_{L^\infty L^2} \qquad (4.23)\\
&= \|2\varepsilon(4\varepsilon A+1)^{-1/2}v_{1\varepsilon}\|_{L^2} \\
&\le c\varepsilon\|v_{1\varepsilon}\|_{L^2} \\
&\le c\varepsilon^{1/2}.
\end{aligned}
$$

In order to estimate $P_\varepsilon^{(4)}$, we show that $f(v_0) = \lambda|v_0|^{\gamma-1}v_0 \in L^1H^1$.

From $|\nabla f(v_0)| \le c|v_0|^{\gamma-1}|\nabla v_0|$, we have

$$\begin{aligned}\|f(v_0)\|_{H^1} &\le c(\|f(v_0)\|_{L^2} + \|\nabla f(v_0)\|_{L^2}) \\ &\le c(\|v_0\|_{L^2} + \|\nabla v_0\|_{L^2})\ \|v_0\|_{L^\infty}^{\gamma-1}.\end{aligned} \tag{4.24}$$

From (4.2), v_0 satifies

$$v_0 \in L^{r(q)}W^{1,q} \subset L^{r(q)}L^\infty, \qquad q > 3. \tag{4.25}$$

We continue the estimate as

$$\begin{aligned}\|f(v_0)\|_{L^1H^1} &\le c\int_0^t (\|v_0\|_{L^2} + \|\nabla v_0\|_{L^2})\ \|v_0\|_{L^\infty}^{\gamma-1} ds \\ &\le c(\|v_0\|_{L^\infty L^2} + \|\nabla v_0\|_{L^\infty L^2}) \int_0^t \|v_0\|_{L^\infty}^{\gamma-1} ds \\ &\le c(\|v_0\|_{L^\infty L^2} + \|\nabla v_0\|_{L^\infty L^2})\ \|v_0\|_{L^r L^\infty},\end{aligned} \tag{4.26}$$

provided

$$\gamma - 1 \le r = 4q/(3q-6).$$

Considering $q > 3$, we have for $1 < \gamma < 5$,

$$\|f(v_0)\|_{L^1H^1} < \infty. \tag{4.27}$$

We rewrite $P_\varepsilon^{(4)}$ as

$$\begin{aligned}P_\varepsilon^{(4)} =& \int_0^t (ie^{-iA(t-s)} - i(4\varepsilon A + 1)^{-1/2} e^{i(\frac{1}{2\varepsilon} - A_\varepsilon)(t-s)}) f(v_0(s)) ds \\ &+ i\int_0^t (4\varepsilon A + 1)^{-1/2} e^{i(\frac{1}{2\varepsilon} + A_\varepsilon)(t-s)} f(v_0(s)) ds \end{aligned} \tag{4.28}$$

$$= I_1 + iI_2. \tag{4.29}$$

Regarding I_1 , the same arguement with $P_\varepsilon^{(1)}$ and (4.27) proves

$$\|I_1\|_{L^\infty L^2} \le c\varepsilon^{1/4}. \tag{4.30}$$

The convergence of $\|I_1\|_{L^\infty L^2}$ is obtained by a technique from the *Riemann–Lebesgue* Theorem. We define, with the characteristic function $X_{[0,t]}(s)$,

$$g(s) = X_{[0,t]}(s)(4\varepsilon A + 1)^{-1/2} e^{i(\frac{1}{2\varepsilon} + A_\varepsilon)t} f(v_0(s)).$$

We have

$$\begin{aligned}I_2 &= \int_{-\infty}^{\infty} e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)s} g(s) ds \\ &= \int_{-\infty}^{\infty} e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)(s+\pi\varepsilon)} g(s+\pi\varepsilon) ds \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (g(s) + g(s+\pi\varepsilon) e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)\pi\varepsilon}) e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)s} ds \\ &= \frac{1}{2}\Big(\int_{-\infty}^{\infty} (g(s) - g(s+\pi\varepsilon)) e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)s} ds \\ &\quad + \int_{-\infty}^{\infty} g(s+\pi\varepsilon)(1 + e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)\pi\varepsilon}) e^{-i(\frac{1}{2\varepsilon} + A_\varepsilon)s} ds\Big) \\ &= \frac{1}{2}(I_{2,1} + I_{2,2}).\end{aligned} \tag{4.31}$$

For $I_{2,2}$, we have

$$\left|1+e^{-i(\frac{1}{2\varepsilon}+\frac{1}{2\varepsilon}(4\varepsilon|\xi|^2+1)^{1/2})\pi\varepsilon}\right| \le c\varepsilon|\xi|^2,$$

then

$$\|I_{2,2}\|_{L^\infty L^2} \le c\varepsilon^{1/2}. \tag{4.32}$$

We utilize Proposition 3 for $I_{2,1}$,

(4.33)

$$\begin{aligned}
&\|I_{2,1}\|_{L^\infty L^2}\\
&= \Big\| \int_{\infty}^{\infty} (4\varepsilon A+1)^{-1/2} e^{i(\frac{1}{2\varepsilon}+A_\varepsilon)(t-s)} \Big(X_{[0,t]}(s)f(v_0(s)) - X_{[0,t]}(s+\pi\varepsilon)f(v_0(s+\pi\varepsilon))\Big)ds\Big\|_{L^\infty L^2}\\
&\le c\Big\|X_{[0,t]}(s)f(v_0(s)) - X_{[0,t]}(s+\pi\varepsilon)f(v_0(s+\pi\varepsilon))\Big\|_{L^{r'}(0,T;L^{q'})}\\
&= c\|f(v_0(s)) - f(v_0(s+\pi\varepsilon))\|_{L^{r'}(0,t-\pi\varepsilon;L^{q'})} + c\|f(v_0(s))\|_{L^{r'}(t-\pi\varepsilon,t;L^{q'})}\\
&= I_{2,1,1} + I_{2,1,2}.
\end{aligned}$$

Concerning $I_{2,1,2}$, we estimate

$$\begin{aligned}
I_{2,1,2} &= \Big(\int_{t-\pi\varepsilon}^{t} \|v_0(s)\|_{\gamma q'}^{\gamma r'}\Big)^{1/r'}\\
&\le c\|v_0\|^{\gamma}_{L^\infty L^{\gamma q'}}\ \varepsilon^{1/r'}.
\end{aligned}$$

Considering (3.2), we have $1/r' > 1/4$. For arbitrary $1<\gamma<5$, there exists q' such that $2 \le \gamma q' \le 6$. Therefore we obtain

$$I_{2,1,2} \le c\varepsilon^{1/4}. \tag{4.34}$$

By the Hölder inequality, we have

$$I_{2,1,1} \le c(\|v_0(s)\|^{\gamma-1}_{L^{10}L^{10}} + \|v_0(s+\pi\varepsilon)\|^{\gamma-1}_{L^{10}L^{10}})\|v_0(s)-v_0(s+\pi\varepsilon)\|_{L^aL^b}, \tag{4.35}$$

where

$$\frac{1}{q'} = \frac{\gamma-1}{10}+\frac{1}{b}, \quad \frac{1}{r'} = \frac{\gamma-1}{10}+\frac{1}{a}, \tag{4.36}$$

with $a = 4(\gamma+1)/(\gamma-1)$, $b = 3(\gamma+1)/(\gamma+2)$. Investigating under (3.2), there exist (q', r'), for $1<\gamma<5$, satisfying (4.36). From (4.2) and embedding results, we have

$$I_{2,1,1} \le c\|v_0(s)-v_0(s+\pi\varepsilon)\|_{L^aL^b}. \tag{4.37}$$

We now introduce another property of the solution of the nonlinear Schröndinger equation (see [12])

$$v_0 \in B^{1/2,a}_{\infty}(I;L^b). \tag{4.38}$$

An equivalent norm of the space is

$$\|u\|_{B^{1/2,a}_\infty(I;L^b)} = \sup_{0<\tau<\delta} \tau^{-1/2}\|u(s)-u(s+\tau)\|_{L^r(I_{\delta'};L^b)}, \tag{4.39}$$

where δ and δ' are sufficiently small and

$$I_{\delta'} = \{s|\ s, s+\tau \in I\}.$$

Therefore we have obtained

(4.40) $$I_{2,1,1} \le c\varepsilon^{1/2},$$

and therefore

(4.41) $$\|P_\varepsilon^{(4)}\|_{L^\infty L^2} \le c\varepsilon^{1/4}.$$

REFERENCES

[1] SCHIFF L. I., *Nonlinear meson theory of nuclear forces*, Phys. Rev., **84** (1951), 1–9.

[2] TSUTSUMI M., *Nonrelativistic approximation of nonlinear Klein–Gordon equation in two space dimension*, Nonlinear Analysis,Theory, Methods, Application, **8** (1984), 637–643.

[3] NAJMAN B., *The nonrelativistic limit of the nonlinear Klein–Gordon equation*, Nonlinear Analysis, Theory, Methods, Application, Vol. 15, **3** (1990), 217–228.

[4] BREZIS H., GALLOUET T., *Nonlinear Schrödinger evolution equations*, Nonlinear Analysis, Theory, Methods, Application, **4** (1980), 677–681.

[5] FATTORINI H. O., *Second Order Linear Differential Equations in Banach Spaces*, North–Holland, Amsterdam, (1985).

[6] MARSHALL B., STRAUSS W., WAINGER S., *$L^p - L^q$ estimates for the Klein– Gordon equation*, J. Math. pures Appl, **59** (1980), 417–440.

[7] YAJIMA K., *Existence of solutions for Schrödinger evolution equations*, Commun. Math. Phys., **110** (1987), 415–426.

[8] KATO T., *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré, Physique théorique , **46** (1987), 113–129.

[9] LIONS J. L., *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris (1969).

[10] GINIBRE J., VELO G., *Time decay of finite energy solutions of the non linear Klein–Gordon and Schrödinger equations*, Ann. Inst. Henri Poincaré ,Physique théorique , **43**, n4 (1985), 399–442.

[11] GINIBRE J., VELO G., *Smoothing properties and retarded estimates for some dispersive evolution equations*, Commun. Math. Phys., **144** (1992), 163–188.

[12] PECHER H., *Solution of semilinear Schrödinger equations in H^s*, Ann. Inst. Henri Poincaré, Physique théorique , **67** (1997), 259–296.

[13] PECHER H., *Nonlinear small data scattering for the wave and Klein–Gordon equation*, Mathematische Zeitschrift, **185** (1984), 261–270.

[14] WANG B., *On existence and scattering for critical and subcritical nonlinear Klein–Gordon equation in H^s*, Nonlinear Analysis, Theory, Methods, Application, **31** (1998), 573–587.

[15] MACHIHARA S., *Nonrelativistic limit of the nonliner Klein–Gordon equation*, preprint.