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Gain of regularity for semilinear Schrödinger equations

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1 Introduction

This note is based on [3] and presents a few improvements of it. We are concerned with local existence and gain of regularity of solutions to the initial value problem for semilinear Schrödinger equations of the form

$$\partial_t u - i\Delta u = f(u, \partial u) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

where u is a complex-valued and unknown function of $(t,x)\in\mathbb{R}\times\mathbb{R}^n$, $x=(x_1,\ldots,x_n)$, $i=\sqrt{-1}$, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, $\partial=(\partial_1,\ldots,\partial_n)$, $\Delta=\partial_1^2+\cdots+\partial_n^2$, n is the spatial dimension, and the nonlinear term $f(u,v)$ is a smooth function on $\mathbb{R}^2\times\mathbb{R}^{2n}$ satisfying

$$f(u,v) = O(|u|^2+|v|^2) \quad \text{near} \quad (u,v)=0. \tag{1.3}$$

The existence of time local solutions to (1.1)-(1.2) was studied in [1], [2], [17], [24] and [25]. The equation (1.1) cannot be treated by the standard energy method, because the nonlinear term contains ∂u. So-called loss of derivatives occurs. To overcome this difficulty, smoothing effect of dispersive-type equations (see [4], [5], [6], [8], [11], [18], [19], [23], [27], [32] and [35] for instance) effectively applied to (1.1). More precisely, the sharp smoothing estimate of $e^{it\Delta}$ (see [24]) or the theory of Schrödinger-type equations (see [9], [10] and [28, Lecture VII] for instance) makes (1.1)-(1.2) to be solvable locally in time. In fact we proved the local existence of smooth solutions to (1.1)-(1.2) by diagonalizing a 2×2 system for ${}^t[u,\bar{u}]$ modulo bounded operators and applying Doi's pseudodifferential operator discovered in [9] to it. See [1] and [2]. More recently, in [25], Kenig, Ponce and Vega succeeded in removing the ellipticity condition on the principal part required in [1] and [2]. The drawback of [1], [2] and [25] is that the initial data are required to be extremely smooth because the method of proof is based on pseudodifferential calculus of operators with smooth coefficients.

We are interested in the gain of regularity associated with the spatial decay of the initial data as well. Such phenomena are generally observed in solutions to various dispersive-type equations. See [7], [12], [13], [14], [15], [16], [20], [21], [22], [31] and [33]. Hayashi, Nakamitsu and Tsutsumi ([14] and [15]), and Doi ([12]) studied this problem for (1.1) in which $f(u, \partial u)$ was independent of ∂u and $\partial \bar{u}$. In [14] and [15] gauge invariance (see (1.5)) was assumed, and an operator $J = (J_1, \dots, J_n)$ defined by

$$J_k u = x_k u + 2it\partial_k u = e^{i|x|^2/4t} 2it\partial_k (e^{-i|x|^2/4t} u),$$

was effectively used, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. The operator J satisfies good commutation relations $[\partial_t - i\Delta, J] = 0$ and $[\partial_j, J_k] = \delta_{jk}$, and acts on nonlinear terms with gauge invariance as if it were a usual differentiation ∂, where $\delta_{jk} = 1$ if $j = k$, 0 otherwise. In [12] Doi developed their idea and made strong use of microlocal analysis and paradifferential calculus when the nonlinear term was a holomorphic function of u. Recently, Hayashi, Naumkin and Pipolo ([16]), and Pipolo ([30]) studied this problem for (1.1) in *one* space dimension. Their nonlinear term $f(u, \partial_1 u)$ depends not only on $(u, \bar{u})$ but also on $(\partial_1 u, \partial_1 \bar{u})$, and is gauge invariant. It is very interesting to mention that they constructed the modified operator only from a multiplier and the Hilbert transformation, and that to eliminate the loss of derivatives, they obtained one kind of the ordinary Gårding inequalities for singular integral operators of order one.

There are two purposes in this paper. One is to improve the local existence theorems in [1] and [2] from the viewpoint of the smoothness of the initial data. Another is to observe the gain of regularity without restrictions on the spatial dimension. To state our results, we recall several function spaces and notation. Let m and l be real numbers. We set $\langle x \rangle = \sqrt{1 + |x|^2}$ and $\langle D \rangle = (1 - \Delta)^{1/2}$. $H^{m,l}$ is the set of all tempered distributions on $\mathbb{R}^n$ satisfying

$$\|u\|_{m,l} = \left(\int_{\mathbb{R}^n} |\langle x \rangle^l \langle D \rangle^m u(x)|^2 dx \right)^{1/2} < +\infty.$$

In particular, we put $H^m = H^{m,0}$, $\|\cdot\|_m = \|\cdot\|_{m,0}$, $L^2 = H^0$ and $\|\cdot\| = \|\cdot\|_0$. We often deal not only with scalar-valued functions but also with vector-valued ones, and we use the same notation of norms for them. In a similar way, $(\cdot, \cdot)$ denotes the inner product of scalar-valued or vector-valued L^2-functions. Any confusion will not occur. Let X be a Fréchet space, and let I be an interval in $\mathbb{R}$. $C^k(I; X)$ denotes the set of all X-valued C^k-functions on I for $k = 0, 1, 2, \dots$. For any real number s, $[s]$ denotes the largest integer less than or equal to s. We now present our main results.

Theorem 1.1 (Local existence for quadratic equations). *Assume* (1.3). *Let θ be a real number greater than $n/2 + 3$, and let δ be also a real number greater than one. Then for any $u_0 \in H^\theta \cap H^{\theta-\delta,\delta}$ there exist a positive time T depending on $\|u_0\|_\theta + \|u_0\|_{\theta-\delta,\delta}$ and a unique solution u to* (1.1)-(1.2) *belonging to* $C([-T, T]; H^\theta \cap H^{\theta-\delta,\delta})$.

Theorem 1.2 (Local existence for cubic equations). *Assume that $f(u, v)$ is cubic, that is,*

$$f(u, v) = O(|u|^3 + |v|^3) \quad \textit{near} \quad (u, v) = 0. \tag{1.4}$$

Let θ be a real number greater than $n/2+3$. Then for any $u_0 \in H^\theta$ there exist a positive time T depending on $\|u_0\|_\theta$ and a unique solution u to (1.1)-(1.2) *belonging to* $C([-T,T];H^\theta)$.

Theorem 1.3 (Gain of regularity). *Assume that $f(u,v)$ is cubic and gauge invariant, that is, for any $(u,v) \in \mathbb{C} \times \mathbb{C}^n$ and for any $\sigma \in \mathbb{R}$*

$$f(e^{i\sigma}u, e^{i\sigma}v) = e^{i\sigma} f(u,v). \tag{1.5}$$

Let θ be a real number greater than $n/2+3$, and let m be a positive integer. Then for any $u_0 \in H^{\theta,m}$ there exist a positive time T depending on $\|u_0\|_\theta$ and a unique solution u to (1.1)-(1.2) *belonging to $C([-T,T];H^\theta)$. Moreover u satisfies*

$$\langle x \rangle^{-|\alpha|} \partial^\alpha u \in C([-T,T] \setminus \{0\}; H^\theta) \tag{1.6}$$

for $|\alpha| \leqslant m$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1,2,\dots\}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Note that if $f(u,v)$ is smooth, quadratic and gauge invariant, then $f(u,v)$ is cubic. We would like to emphasize that the existence time T in Theorem 1.3 is independent of m. Therefore we can say that the solution to (1.1)-(1.2) gains regularity according to the decay of the initial data.

Remark 1.1. Suppose that $f(u,v)$ can be split into $f(u,v) = f_0(u) + f_1(u,v)$, where $f_1(u,v)$ satisfys the gauge condition (1.5) and $f_0(u)$ does not. Then Theorem 1.3 holds provided $m=1$.

Our idea of proof is basically the developed version of that of [1] and [2]. We see (1.1) as a system for ${}^t[J^\alpha u, \overline{J^\alpha u}]_{|\alpha| \leqslant m}$. For this reason, we study the L^2-well-posedness for linear systems corresponding to nonlinear ones. To eliminate the loss of derivatives, we make use of block diagonalization and Doi's operator. Our basic tools are pseudodifferential operators with nonsmooth coefficients.

This paper is organized as follows. In Section 2 we introduce pseudodifferential operators with nonsmooth coefficients and prepare lemmas needed later. In Section 3 we study well-posedness of linear systems. Finally, in Sections 4, we remark how to apply the linear theory developed in Section 3 to proving Theorems 1.1, 1.2 and 1.3.

2 ΦDOs with nonsmooth coefficients

We here introduce classes of pseudodifferential operators whose coefficients have limited smoothness. Such an operator was originated by Nagase in [29]. Since then, the theory about it has advanced and has applied to studying nonlinear partial differential equations. See [34] and references therein. Let $S^m_{\rho,\delta}$ be the set of all symbols of m-th order classical pseudodifferential operators of the type ρ, δ. We set $S^m = S^m_{1,0}$ for short. See [26].

Definition 2.1 (Nonsmooth symbols). Let m be a real number, and let s be a nonnegative number. A function $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a symbol belonging to a class $\mathscr{B}^s S^m$ if

$$\|p\|_{\mathscr{B}^s S^m, l} = \sum_{|\alpha| \leqslant l} \sup_{\xi \in \mathbb{R}^n} \left(\langle \xi \rangle^{-m+|\alpha|} \|p^{(\alpha)}(\cdot,\xi)\|_{\mathscr{B}^s}\right) < +\infty$$

for all nonnegative integer l, where $\mathscr{B}^s$ denotes the Banach space of all $C^{[s]}$-functions $\phi(x)$ on $\mathbb{R}^n$ satisfying

$$\|\phi\|_{\mathscr{B}^s} = \sum_{|\alpha| \leqslant [s]} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \phi(x)| + \sum_{|\alpha| = [s]} \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|\partial^\alpha \phi(x) - \partial^\alpha \phi(y)|}{|x-y|^{s-[s]}} < +\infty.$$

For the sake of convenience, we often use $D = -i\partial$ below. If a symbol $p(x,\xi)$ is given, then a pseudodifferential operator $P = p(x,D)$ is defined by

$$\begin{aligned} Pu(x) &= \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x,\xi) u(y)\, dy d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \hat{u}(\xi)\, d\xi \end{aligned}$$

for $u \in \mathscr{S}$, where $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$, $\hat{u}$ is the Fourier transform of u, and $\mathscr{S}$ denotes the Schwartz class on $\mathbb{R}^n$. Conversely, if an operator P is given, then its symbol $\sigma(P)(x,\xi)$ is determined by $\sigma(P)(x,\xi) = e^{-ix\cdot\xi} P e^{ix\cdot\xi}$. In addition, we will often need the L^2-boundedness theorem for pseudodifferential operators with nonsmooth coefficients.

Theorem 2.1 (Nagase [29, Theorem A]). *Assume that $p(x,\xi)$ satisfies*

$$\begin{gathered} |p^{(\alpha)}(x,\xi)| \leqslant C_\alpha \langle \xi \rangle^{-|\alpha|}, \\ |p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)| \leqslant C_\alpha \langle \xi \rangle^{-|\alpha|+\tau} |x-y|^\sigma \end{gathered}$$

for $|\alpha| \leqslant n+1$ with $0 \leqslant \tau < \sigma \leqslant 1$. Then $p(x,D)$ is L^2-bounded, that is, there exists a constant C_1 depending only on n, σ and τ such that

$$\|p(x,D)u\| \leqslant C_1 C(p) \|u\|$$

for any $u \in L^2$, where

$$\begin{aligned} C(p) = &\sum_{|\alpha| \leqslant n+1} \sup_{x,\xi \in \mathbb{R}^n} \left(\langle \xi \rangle^{|\alpha|} |p^{(\alpha)}(x,\xi)|\right) \\ &+ \sum_{|\alpha| \leqslant n+1} \sup_{\substack{x,y,\xi \in \mathbb{R}^n \\ x \neq y}} \left(\langle \xi \rangle^{|\alpha|-\tau} \frac{|p^{(\alpha)}(x,\xi) - p^{(\alpha)}(y,\xi)|}{|x-y|^\sigma}\right). \end{aligned}$$

Nagase proved Theorem 2.1 by the approximation of nonsmooth symbols by smooth ones. This is said to be symbol smoothing. We will observe that symbol smoothing is a strong method to deal with nonsmooth symbols below.

We now introduce symbol smoothing. Let $p(x,\xi)$ be a symbol belonging to $\mathscr{B}^s S^m$, and let $\rho(x) \in \mathscr{S}$ be a Friedrichs' mollifier satisfying

$$\operatorname{supp}\rho \subset \{|x| \leqslant 1\}, \quad \rho(x) = \rho(-x) \geqslant 0, \quad \int_{\mathbb{R}^n} \rho(x)dx = 1.$$

We put $\rho_{\alpha,\beta}(x) = x^\beta \partial^\alpha \rho(x)$ for short. Since $\rho(x)$ is an even function, it follows that $\rho_{\alpha,\beta}(x) = (-1)^{|\alpha+\beta|}\rho_{\alpha,\beta}(-x)$. We set

$$\begin{aligned} p^\sharp(x,\xi) &= \int_{\mathbb{R}^n} \rho(y)p(x - \langle\xi\rangle^{-1/s}y,\xi)dy \\ &= \langle\xi\rangle^{n/s}\int_{\mathbb{R}^n} \rho(\langle\xi\rangle^{1/s}y)p(x-y,\xi)dy \\ &= \langle\xi\rangle^{n/s}\int_{\mathbb{R}^n} \rho(\langle\xi\rangle^{1/s}(x-y))p(y,\xi)dy, \\ p^\flat(x,\xi) &= p(x,\xi) - p^\sharp(x,\xi). \end{aligned}$$

Then $p(x,\xi)$ is decomposed as $p(x,\xi) = p^\sharp(x,\xi) + p^\flat(x,\xi)$, and $p^\sharp(x,\xi)$ and $p^\flat(x,\xi)$ are the smooth principal part and the lower order term of $p(x,\xi)$ respectively. More precisely, the properties of symbol smoothing are the following.

Lemma 2.2. *Let m and s be real numbers satisfying $1 < s \leqslant 2$. Assume that $p(x,\xi)$ belongs to $\mathscr{B}^s S^m$. Then for any multi-indices α and β*

$$\begin{aligned} |p^{\sharp(\alpha)}_{(\beta)}(x,\xi)| &\leqslant C_{\alpha\beta}\|p\|_{\mathscr{B}^{[s]}S^m,|\alpha|}\langle\xi\rangle^{m-|\alpha|+(|\beta|-[s])_+/s}, \\ |p^{\flat(\alpha)}(x,\xi)| &\leqslant C_\alpha\|p\|_{\mathscr{B}^s S^m,|\alpha|}\langle\xi\rangle^{m-1-|\alpha|}, \\ |p^{\flat(\alpha)}(x,\xi) - p^{\flat(\alpha)}(y,\xi)| &\leqslant C_\alpha\|p\|_{\mathscr{B}^s S^m,|\alpha|}\langle\xi\rangle^{m-1+(s-1)/s-|\alpha|}|x-y|^{s-1}, \end{aligned}$$

where $\tau_+ = \tau$ if $\tau > 0$, 0 otherwise.

Using the symbol smoothing, we obtain the fundamental theorem for algebra and the sharp Gårding inequality.

Lemma 2.3. *Let s be a real number greater than one. Assume that $p_j(x,\xi)$ belongs to $\mathscr{B}^s S^j$ for $j = 0,1$. Set*

$$q(x,\xi) = p_0(x,\xi)p_1(x,\xi), \quad r(x,\xi) = \overline{p_1(x,\xi)}.$$

Then

$$p_0(x,D)p_1(x,D) \equiv p_1(x,D)p_0(x,D) \equiv q(x,D), \tag{2.1}$$

$$p_1(x,D)^* \equiv r(x,D) \tag{2.2}$$

modulo L^2-bounded operators, where $p_1(x,D)^*$ is the formal adjoint of $p_1(x,D)$.

Lemma 2.4 (The sharp Gårding inequality). *Assume that* $p(x,\xi) = [p_{ij}(x,\xi)]_{i,j=1,\dots,m}$ *is an* $m \times m$ *matrix of symbols belonging to the class* $\mathscr{B}^2 S^1$*, and assume that there exists a nonnegative constant* R *such that*

$$p(x,\xi) + {}^t\overline{p(x,\xi)} \geqslant 0$$

for $|\xi| \geqslant R$*. Then there exist a positive constant* C_1 *and a positive integer* l *such that for any* $u \in (\mathscr{S})^m$

$$\operatorname{Re}(p(x,D)u,u) \geqslant -C_1 \sum_{i,j=1}^{m} \|p_{ij}\|_{\mathscr{B}^2 S^1, l} \|u\|^2. \tag{2.3}$$

Roughly speaking, Lemmas 2.3 and 2.4 show that pseudodifferential operators of at most order one with C^2-coefficients can be seen as classical ones of the type $1,0$. Since $p_j^\sharp(x,\xi)$ belongs to $S^j_{1,1/s}$, an asymptotic formula for $p_j^\sharp(x,\xi)$ implies (2.1) and (2.2) provided $s > 1$. To prove (2.3), we regard that $p^\sharp(x,\xi)$ is in $(S^1)^{m^2}$ and apply the Friedrichs symmetrization to it in the sprit of [26, Chapter 3, §4].

3 Linear systems with nonsmooth coefficients

Roughly speaking, Theorems 1.1 and 1.2 are the local existence theorems of the system for ${}^t[u,\bar{u}]$, and Theorem 1.3 is that for ${}^t[[J^\alpha u]_{|\alpha|\leqslant m}, [\overline{J^\alpha u}]_{|\alpha|\leqslant m}]$. So, this section is devoted to studying the well-posedness of the initial value problem for $2l \times 2l$ systems of Schrödinger-type equations of the form

$$\mathscr{L}w = g(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{3.1}$$

$$w(0,x) = w_0(x) \quad \text{in} \quad \mathbb{R}^n, \tag{3.2}$$

where w is a $\mathbb{C}^{2l}$-valued and unknown function, $g(t,x)$ and $w_0(x)$ are given functions, T is a positive time, l is a positive integer, and the operator L is defined as follows:

$$L = I_{2l}\partial_t - iE_{2l}\Delta + \sum_{k=1}^{n} B^k(t,x)\partial_k + C(t,x),$$

I_p is the $p \times p$ identity matrix ($p = 1,2,3\dots$),

$$E_{2l} = I_l \oplus [-I_l] = \begin{bmatrix} I_l & 0 \\ 0 & -I_l \end{bmatrix},$$

$B^k(t,x) = [b^k_{ij}(t,x)]_{i,j=1,\dots,2l}$, and $C(t,x) = [c_{ij}(t,x)]_{i,j=1,\dots,2l}$. In [1] and [2] we studied the case of $l = 1$ by diagonalization and Doi's method. We here develop the idea of [1] and [2] and solve (3.1)-(3.2).

Lemma 3.1. *Assume that for $i, j = 1, \dots, 2l$ and for $k = 1, \dots, n$*

$$b_{ij}^k(t,x) \in C([0,T]; \mathscr{B}^2) \cap C^1([0,T]; \mathscr{B}^0),$$

$$c_{ij}(t,x) \in C([0,T]; \mathscr{B}^0),$$

and assume that there exists a nonnegative function $\phi(t,s)$ on $[0,T] \times \mathbb{R}$ such that

$$\phi(t,s) \in C([0,T]; \mathscr{B}^2(\mathbb{R})),$$

$$\sup_{t\in[0,T]} \int_{-\infty}^{+\infty} \phi(t,s)ds + \sup_{\substack{t\in[0,T] \\ \tau\in\mathbb{R}}} \left| \int_0^\tau \partial_t \phi(t,s)ds \right| < +\infty,$$

$$\sum_{k=1}^n \sum_{i,j=1}^l (|b_{ij}^k(t,x)| + |b_{(l+i)(l+j)}^k(t,x)|) \leqslant \phi(t, x_p) \tag{3.3}$$

for $(t,x) \in [0,T] \times \mathbb{R}^n$ and for $p = 1, \dots, n$. Then (3.1)-(3.2) *is L^2-well-posed, that is, for any $w_0 \in (L^2)^{2l}$ and for any $g \in L^1(0,T; (L^2)^{2l})$ there exists a unique solution w to* (3.1)-(3.2) *belonging to $C([0,T]; (L^2)^{2l})$.*

Lemma 3.1 is basically proved by a energy inequality and duality argument. For the sake of convenience, we denote the $l \times l$ block diagonal part of $B^k(t,x)$ by $B^{k,\text{diag}}(t,x)$, that is,

$$B^{k,\text{diag}}(t,x) = [b_{ij}^k(t,x)]_{i,j=1,\dots,l} \oplus [b_{ij}^k(t,x)]_{i,j=l+1,\dots,2l}.$$

We here introduce pseudodifferential operators as follows:

$$\Lambda(t) = I_{2l} - \frac{i}{2} \sum_{k=1}^n E_{2l}(B^k(t,x) - B^{k,\text{diag}}(t,x))\partial_k(1-\Delta)^{-1},$$

$$K(t) = [I_l k_1(t,x,D)] \oplus [I_l k_1'(t,x,D)],$$

$$k_1(t,x,\xi) = e^{-p(t,x,\xi)}, \quad k_1'(t,x,\xi) = e^{p(t,x,\xi)},$$

$$p(t,x,\xi) = \sum_{j=1}^n \int_0^{x_j} \phi(t,s)ds\xi_j \langle\xi\rangle^{-1}.$$

The block diagonalization is accomplished by $\Lambda(t)$, and Doi-type operator $K(t)$ eliminates the loss of derivatives. We make use of them in a transformation $w \mapsto K(t)\Lambda(t)w$. This is automorphic on $(L^2)^{2l}$. Applying $K(t)\Lambda(t)$ to $\mathscr{L}$, we have

$$K(t)\Lambda(t)\mathscr{L} \equiv (I_{2l}\partial_t - iE_{2l}\Delta + Q(t))K(t)\Lambda(t)$$

modulo L^2-bounded operators, where

$$\sigma(Q(t)) = \sum_{j=1}^n (2I_{2l}\phi(t,x_j)\xi_j^2 + iB^{j,\text{diag}}(t,x)\xi_j).$$

It follows from (3.3) that $\sigma(Q(t)) + {}^t\overline{\sigma(Q(t))} \geqslant 0$ for $|\xi| \geqslant 1$. Then, using the sharp Gårding inequlity (2.3), we can obtain the energy inequality for $K(t)\Lambda(t)w$.

4 Proof of Theorems 1.1, 1.2 and 1.3

Finally, we remark how to apply Lemma 3.1 to the proof of Theorems 1.1, 1.2 and 1.3. When we make use of Lemmas 2.3 and 2.4, we require

$$u \in C([-T,T];\mathscr{B}^3),$$

so that

$$f(u,\partial u) \in C([-T,T];\mathscr{B}^2).$$

Then, in view of the Sobolev embedding, we require

$$u \in C([-T,T];H^\theta), \quad \theta > n/2+3.$$

In order to make use of Lemma 3.1, we set

$$\phi(t,s) = \begin{cases} M\langle x\rangle^{-\delta}, \\ M\displaystyle\sum_{j=1}^{n}\int_{\mathbb{R}^{n-1}} |\langle D\rangle^{(n+1)/2+\varepsilon}u(t,s,\hat{x}_j)|^2 d\hat{x}_j, \\ M\displaystyle\sum_{j=1}^{n}\sum_{|\beta|\leqslant m}\int_{\mathbb{R}^{n-1}} |\langle D\rangle^{(n+1)/2+\varepsilon}J^\beta u(t,s,\hat{x}_j)|^2 d\hat{x}_j, \end{cases}$$

for Theorems 1.1, 1.2 and 1.3 respectively, where $M \gg 1$, $0 < \varepsilon \ll 1$ and $\hat{x}_j = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$.

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