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On the Small Amplitude Solutions to the Derivative Nonlinear Schrödinger Equations in Multi-Space Dimensions

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1 Introduction

In this proceeding, we consider the initial value problem for the nonlinear Schrödinger equation with nonlinear term of derivative type, i.e.,

$$\text{(NLS)}\begin{cases} i\partial_t u = -\Delta u + F(u, \bar{u}, \nabla u, \nabla \bar{u}) \\ u|_{t=0} = u_0, \end{cases}$$

where u is a complex valued unknown function of $(t, x) \in \mathbf{R}^1 \times \mathbf{R}^n$ $(n \geq 2)$ and $\bar{u}$ is the complex conjugate of u. Δ is the Laplacian in the n dimensional space and $\nabla u = (\partial_1 u, \cdots, \partial_n u)$. The nonlinear term F is the polynomial on $\mathbf{C}^{2+2n}$ of degree $\rho = 2$ or 3, i.e.,

$$F(z_1, z_2, z_3, \cdots, z_{2+2n}) = \sum_{|\alpha|=\rho} C_\alpha z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \cdots z_{2+2n}^{\alpha_{2+2n}},$$

where $C_\alpha \in \mathbf{C}$ and $\alpha = (\alpha_1, \cdots, \alpha_{2+2n})$.

There are many results known about (NLS). Kenig-Ponce-Vega [11] showed the well-posedness for small initial data by applying the smoothing effects of linear solutions. Chihara [4] proved the problem for large initial data. His idea is based on the energy method through the pseudo differential operator technique. These results are shown by imposing large regularity on the initial data. (Recently, Chihara [3] has solved (NLS) under the condition $u_0 \in H^{s,0}(\mathbf{R}^n)$ with $s > n/2 + 3$.)

Our concern in this work is to solve (NLS) for the initial data with small regularity. We obtain the following results. (One can see the notations in theorems just after the statements.)

Theorem 1.1 (the case $\rho = 3$) *let $s > (n+3)/2$. then, for $\phi \in H^{s,0}(\mathbf{R}^n)$ with $||\phi||_{s,0}$ sufficiently small, there exists a unique solution u to (NLS) on $[0,T]$ (T depends on $||\phi||_{s,0}$) such that*

$$u \in C([0,T]; H^{s,0}(\mathbf{R}^n)). \tag{1}$$

Theorem 1.2 (the case $\rho = 2$) *Let $s > (n+3)/2, s' > (n+2)/2$ and $t' > 1/2$ satisfy $s > s' + t'$. Then, for $\phi \in H^{s,0}(\mathbf{R}^n) \cap H^{s',t'}(\mathbf{R}^n)$ with $||\phi||_{s',t'}$ sufficiently small, there exists a unique solution u to (NLS) on $[0,T]$ (T depends on $||\phi||_{s',t'}$) such that*

$$u \in C([0,T]; H^{s,0}(\mathbf{R}^n) \cap H^{s',t'}(\mathbf{R}^n)). \tag{2}$$

The solutions in Theorem 1.1 and 1.2 gain the regularity in the following sense.

Theorem 1.3 *The solutions in Theorem 1.1 and 1.2 satisfy*

$$||\partial_j^{s+1/2} u||_{L^\infty_{x_j}(L^2_{T,\widehat{x}_j})} < \infty \quad \textit{for } 1 \le j \le n. \tag{3}$$

Notations.

In the above theorems, the function spaces $L^p_{x_j}(L^r_{T,\widehat{x}_j})$ and $H^{\sigma,\tau}(\mathbf{R}^n)$ are defined by

$$L^p_{x_j}(L^r_{T,\widehat{x}_j}) = \{u; ||u||^p_{L^p_{x_j}(L^r_{T,\widehat{x}_j})} = \int_{\mathbf{R}} \left(\int_0^T \int_{\mathbf{R}^{n-1}} |u(t,x)|^r dt d\widehat{x}_j \right)^{p/r} dx_j < \infty\},$$

where $\widehat{x}_j = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$.

$$H^{\sigma,\tau}(\mathbf{R}^n) = \{f \in \mathcal{S}'; ||f||_{\sigma,\tau} = ||\langle x \rangle^\tau \langle D \rangle^\sigma f||_{L^2_x} < \infty\},$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$ and $\langle D \rangle^\sigma f = \mathcal{F}^{-1} \langle \xi \rangle^\sigma \mathcal{F} f$ ($\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier and inverse Fourier transform, respectively). $\partial_j^\sigma f = \mathcal{F}^{-1} |\xi_j|^{\sigma - [\sigma]} \xi_j^{[\sigma]} \mathcal{F} f$. $[\sigma]$ is the largest integer which does not exceed σ. $U(t)\phi = \exp(it\Delta)\phi$ and $G(t)F = \int_0^t U(t-s)F(s)ds$.

We consider the initial value problem (NLS) by solving the integral equation :

$$\begin{aligned} u(t) &= \Phi(u)(t) \\ &\equiv U(t)u_0 - iG(t)F(u, \bar{u}, \nabla u, \nabla \bar{u}). \end{aligned} \tag{4}$$

Since the nonlinear term contains some derivatives, it causes, so called, the loss of derivative. Because of this difficulty, it is impossible to estimate the second term in (4) by the unitarity and Strichartz' type estimates of $U(t)$ ([2], [17] and [19]). To overcome the loss of derivative, we make use of the smoothing properties of $U(t)$ and $G(t)$, i.e.,

Lemma 1.4 (Hayashi-Hirata [8]) *Let* $\phi \in L^2(\mathbf{R}^n)$ *and* $F \in L^1_{x_j}(L^2_{T,\widehat{x}_j})$. *Then, we have*

$$\begin{aligned} \|\partial_j^{1/2} U(\cdot)\phi\|_{L^\infty_{x_j}(L^2_{T,\widehat{x}_j})} &\le C\|\phi\|_{L^2}, & (5)\\ \|\partial_j^{1/2} G(\cdot)F\|_{L^\infty_T(L^2_x)} &\le C\|F\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}, & (6)\\ \|\partial_j G(\cdot)F\|_{L^\infty_{x_j}(L^2_{T,\widehat{x}_j})} &\le C\|F\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}. & (7) \end{aligned}$$

To introduce the maximal functions, we demonstrate the estimates of $\Phi(u)$ in (4). For simplicity, we consider the case $F(u,\bar{u},\nabla u,\nabla\bar{u}) = \partial u\partial u\partial u$. Applying (6) in Lemma 1.4 to $\Phi(u)$, we can show that

$$\begin{aligned} \|\partial_j^s \Phi(u)\|_{L^\infty_T(L^2_x)} &\le \|u_0\|_{s,0} + C\|\partial_j^{s-1/2}(\partial u\partial u\partial_k u)\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}\\ &\le \|u_0\|_{s,0} + C\|\partial u\partial u\partial_j^{s-1/2}\partial_k u\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}\\ &\qquad +\|(\text{some lower order derivatives})\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}. \qquad (8) \end{aligned}$$

Note that, to obtain the last inequality, we use Leibniz' rule for fractional derivatives ([5] and [12]). Since, by Hölder's inequality, we can show that

$$\begin{aligned} &\|\partial u\partial u\partial_j^{s-1/2}\partial_k u\|_{L^1_{x_j}(L^2_{T,\widehat{x}_j})}\\ &\le \|\partial u\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}\|\partial u\|_{L^2_{x_k}(L^\infty_{T,\widehat{x}_k})}\|\partial_k\partial_j^{s-1/2}u\|_{L^\infty_{x_k}(L^2_{T,\widehat{x}_k})}, \qquad (9) \end{aligned}$$

we need to estimate $\|\partial\Phi(u)\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}$ in order to complete the contraction mapping principle.

$$\begin{aligned} &\|\partial\Phi(u)\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}\\ &\le \|U(\cdot)\partial u_0\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})} + \|G(\cdot)\partial F\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}. \qquad (10) \end{aligned}$$

Hence, it is important to control the $L^2_{x_j}(L^\infty_{\widehat{x}_j})$-norm of maximal functions for $U(t)\partial u_0$ and $G(t)\partial F$, where we call $\|U(\cdot)\phi(x)\|_{L^\infty_T}$ and $\|G(\cdot)F(x)\|_{L^\infty}$ the maximal functions for $U(t)\phi$ and $G(t)F$, respectively. In the proof of Theorem 1.1 and 1.2, the estimates of maximal functions play an important role to determine the regularity of u_0.

Remark 1.1. When the nonlinear term is quadratic, we need to estimate the weighted $L^2_{x_j}(L^\infty_{\widehat{x}_j})$-norm of maximal functions, i.e., $\|\langle x\rangle^\tau U(\cdot)\phi\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}$ and $\|\langle x\rangle^\tau G(\cdot)\phi\|_{L^2_{x_j}(L^\infty_{T,\widehat{x}_j})}$ for $\tau > 1/2$.

Remark 1.2. We are not allowed to estimate $\|\partial_k\partial_j^{s-1/2}u\|_{L^\infty_{x_k}(L^2_{T,\widehat{x}_k})}$ in (9) so that

$$\|\partial_k\partial_j^{s-1/2}u\|_{L^\infty_{x_k}(L^2_{T,\widehat{x}_k})} \le CT^\delta\|\partial_k\partial_j^{s-1/2}u\|_{L^\infty_{x_k}L^r_T L^2_{\widehat{x}_k}}$$

for some $r > 2$, since we want to use (7) in Lemma 1.4. This is the reason we need to impose the smallness on u_0.

We shall introduce the statements about the estimates of maximal functions in the forthcoming section.

2 Estimates of Maximal Functions

In this section, we introduce some inequalities concerned with the maximal functions and the outline of the proofs. There has been several kinds of estimates for maximal functions (see [14], [15] and [18]). Our main result is

Theorem 2.1 *Let $n/2 < \sigma$ and $0 < T < 1$. Then, for $\phi \in H^{\sigma,0}(\mathbf{R}^n)$, we have*

$$\|U(\cdot)\phi\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \leq C\|\phi\|_{\sigma,0}. \tag{11}$$

In addition, let $n/2 < \sigma' < \sigma$ and $1/2 < \tau < 1$. Then, we have

$$\|\langle x\rangle^\tau U(\cdot)\phi\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \leq C\|\phi\|_{\sigma',\tau} + CT^{1/2}\|\phi\|_{\sigma+\tau,0}. \tag{12}$$

As a corollary of Theorem 2.1, we obtain

Corollary 2.2 *Under the same conditions as in Theorem 2.1, we have*

$$\|G(\cdot)F\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \leq C\|F\|_{L^1_T(H^{\sigma,0})}, \tag{13}$$

$$\|\langle x\rangle^\tau G(\cdot)F\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \leq C\|F\|_{L^1_T(H^{\sigma',0})} + CT^{1/2}\sup_{1\leq k\leq n}\|\partial_k^{\sigma+\tau-1/2}F\|_{L^1_{x_k}(L^2_{T,\widehat{x_k}})}. \tag{14}$$

To prove Theorem 2.1, we need several lemmas.

Lemma 2.3 *Let $\sigma > n/2$. Then, we have*

$$\|\langle D\rangle^{-2\sigma}\int_0^T U(\cdot - s)F(s)ds\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \leq C\|F\|_{L^2_{x_j}(L^1_{T,\widehat{x_j}})}. \tag{15}$$

Therefore, it follows that

$$\|\langle D\rangle^{-\sigma}\int_0^T U(-s)F(s)ds\|_{L^2_x} \leq C\|F\|_{L^2_{x_j}(L^1_{T,\widehat{x_j}})}. \tag{16}$$

Proof of Lemma 2.3. Note that the integral kernel of $\langle D\rangle^{-2\sigma}U(t-s)$ is

$$K(t-s,x-y)=(2\pi)^{-n}\int\langle\xi\rangle^{-2\sigma}\exp\left(-i(t-s)\xi^2+i(x-y)\cdot\xi\right)d\xi.$$

Since $2\sigma>n$, there exists no singularity at $t=s$ and we have

$$|K(t-s,x-y)|\le C\langle x-y\rangle^{-2\sigma}.$$

Hence, by Young's inequality, we obtain (15).

We next prove (16). By (15), it is easy to show that

$$\begin{aligned}
&\|\langle D\rangle^{-\sigma}\int_0^T U(-s)F(s)ds\|_{L^2_x}^2\\
=&\left|\int_0^T (F(t),\langle D\rangle^{-2\sigma}\int_0^T U(t-s)F(s)ds)dt\right|\\
\le&\ C\|F\|^2_{L^2_{x_j}(L^1_{T,\widehat{x_j}})}.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

To prove Theorem 2.1 (12), we use the smoothing properties of $U(t)$ and $G(t)$. One can see the one space dimensional version of the smoothing properties in [1]. The n space dimensional version is

Lemma 2.4 *Let $2\le p<\infty$. Then, we have*

$$\|\partial_j^{1/2-1/p}U(\cdot)\phi\|_{L^p_{x_j}(L^2_{T,\widehat{x_j}})}\ \le\ CT^{1/p}\|\phi\|_{L^2_x},\tag{17}$$

$$\|\partial_j^{1-1/p}G(\cdot)F\|_{L^p_{x_j}(L^2_{T,\widehat{x_j}})}\ \le\ CT^{1/p}\|F\|_{L^1_{x_j}(L^2_{T,\widehat{x_j}})}.\tag{18}$$

proof of Lemma 2.4. The results follow from Stein's interpolation theorem and L^p-boundedness of the Hilbert transform. □

Now we start to show the outline of the proof for Theorem 2.1.

Proof of Theorem 2.1. We first prove (11) by the duality argument. Applying Lemma 2.3 (16), we have

$$\begin{aligned}
&\int_0^T (F(s),\langle D\rangle^{-\sigma}U(s)\phi)ds\\
=&\left(\langle D\rangle^{-\sigma}\int_0^T U(-s)F(s)ds,\phi\right)\\
\le&\ \|\langle D\rangle^{-\sigma}\int_0^T U(-s)F(s)ds\|_{L^2_x}\|\phi\|_{L^2_x}\\
\le&\ C\|F\|_{L^2_{x_j}(L^1_{T,\widehat{x_j}})}\|\phi\|_{L^2_x}.
\end{aligned}$$

Hence, we obtain (11). We next prove (12). Since

$$\langle x\rangle^\tau U(t)\phi = U(t)\langle x\rangle^\tau\phi + iG(t)[\langle x\rangle^\tau, -\Delta]U(\cdot)\phi,$$

it follows from (11) that

$$\begin{aligned}
&\|\langle x\rangle^\tau U(\cdot)\phi\|_{L^2_{x_j}(L^\infty_{T,\widehat{x_j}})} \\
\leq\ & C\|\phi\|_{\sigma',\tau} + C\sup_k \int_0^T \|\langle x\rangle^{-(1-\tau)}\partial_k^{\sigma'+1}U(s)\phi\|_{L^2_x}ds \\
\leq\ & C\|\phi\|_{\sigma',\tau} + CT^{1/2}\sup_k \|\langle x\rangle^{-(1-\tau)}\partial_k^{\sigma'}U(\cdot)\phi\|_{L^2_{T,x}} \\
\leq\ & C\|\phi\|_{\sigma',\tau} + CT^{1/2}\sup_k \|\partial_k^{\sigma'+1}U(\cdot)\phi\|_{L^p_{x_k}(L^2_{T,\widehat{x_k}})},
\end{aligned} \tag{19}$$

where $1/2 = 1/p + (1-\tau-\epsilon)$ for some $\epsilon > 0$. Applying Lemma 2.4 (17) to the second term in RHS of (19), we obtain Theorem 2.1. □

References

[1] D. Bekiranov, T. Ogawa and G. Ponce, *On the well-posedness of Benny's interaction equation of short and long waves*, Advances in Diff. Eq., (1996), pp. 919–937.

[2] T. Cazenave and F.B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. T.M.A., **100** (1990), pp. 807–836.

[3] H. Chihara, *Gain of regularity for semilinear Schrödinger equations*, preprint.

[4] H. Chihara, *Local existence for semilinear Schrödinger equations*, Math. Japon., **42** (1995), 35–52.

[5] F. M. Christ and M. Weinstein, *Dispersive small amplitude solution to the generalized Korteweg-de Vries equation*, J. Funct. Anal., **100**(1991), pp. 87–109.

[6] P. Constantin and J.-C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc., **1** (1989), pp. 413–446.

[7] N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equations*, Comm. in P.D.E., **18** (1993), pp. 823–833.

[8] N. Hayashi and H. Hirata, *Local existence in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system in the usual Sobolev space*, to appear in Proceedings of Royal Society of Edinburgh A.

[9] N. Hayashi and H. Hirata, *Global existence of small solutions to nonlinear Schrödinger equations*, preprint.

[10] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Math. Supp. Studies, Studies in Applied Math., **8** (1983), pp. 93–128.

[11] C. E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, **10** (1993), pp. 255–288.

[12] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for generalized Korteweg-de Vries equation via the contraction mapping principle*, Comm. pure Appl. Math., **46** (1993), pp. 527–620.

[13] C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., **40** (1991), pp. 33–69.

[14] C. E. Kenig and A. Ruiz, *A strong type* $(2,2)$ *estimate for the maximal function associated to the Schrödinger equation*, Trans. Amer. Math. Soc., **280** (1983), pp. 239–246.

[15] P. Sjölin, *Regularity of solutions to the Schrödinger equations*, Duke Math., **55** (1987), pp. 699–715.

[16] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis in Euclidean Spaces*, Princeton University Press, (1971).

[17] R. S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equation*, Duke Math. J., **44** (1977), pp. 705–714.

[18] L. Vega, *The Schrödinger equation: pointwise convergence to the initial data*, Proc. Amer. Math. Soc., **102** (1988), pp. 874–878.

[19] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys., **110** (1987), 415–426.

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