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SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS OF SURFACES AND SECONDARY INVARIANTS

Ryoji Kasagawa

1. INTRODUCTION

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$ and $\mathcal{M}_g$ its mapping class group consisting of the isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$. We denote the 2-sphere with 3-holes by $P$. For any $a, b \in \mathcal{M}_g$, let $N_{a,b}$ be the $\Sigma_g$-bundle over $P$ with monodromies $a^{-1}$ and $b^{-1}$.

Meyer's signature 2-cocycle

$$sign_g : \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$$

is defined by $sign_g(a, b) = sign(N_{a,b})$, where $sign(N_{a,b})$ is the signature of 4-manifold $N_{a,b}$ (see [10, 1]). Novikov additivity for the signature of manifolds shows that $sign_g$ satisfies the cocycle condition. Meyer also defined a 2-cocycle $\tau_g$ on $Sp(2g, \mathbb{Z})$ over $\mathbb{Z}$, which is also called signature 2-cocycle. It is well-known that the equality $sign_g = \zeta_g^* \tau_g$ holds, where $\zeta_g$ is the standard representation of $\mathcal{M}_g$ to $Sp(2g, \mathbb{Z})$ induced from the obvious action of $\mathcal{M}_g$ on the first cohomology group of $\Sigma_g$.

Let $\iota$ be the hyperelliptic involution on $\Sigma_g$ depicted in Figure 1.

![Figure 1. The hyperelliptic involution $\iota$ on $\Sigma_g$.](image)

The hyperelliptic mapping class group $\mathcal{H}_g$ of $\Sigma_g$ is the subgroup of $\mathcal{M}_g$ consisting of elements which commute with the class of $\iota$. It is known that $\mathcal{M}_1 = \mathcal{H}_1 = SL(2, \mathbb{Z})$, $\mathcal{M}_2 = \mathcal{H}_2$ and that $\mathcal{H}_g(g \geq 3)$ is a subgroup of $\mathcal{M}_g$ of infinite index.
Meyer's signature cocycle $\text{sign}_g$ defines a nontrivial class of the second cohomology group of $\mathcal{M}_g$ with coefficients in $\mathbb{Z}$ and its restriction to $\mathcal{H}_g$ is also nontrivial. But it is trivial in the cohomology group of $\mathcal{H}_g$ with coefficients in $\mathbb{Q}$. Thus there exists a function or a 1-cochain

$$\phi_g: \mathcal{H}_g \to \mathbb{Q}$$

such that $\text{sign}_g = \delta \phi_g$, where $\delta$ denotes the coboundary operator defined by $\delta \phi_g(a, b) = \phi_g(b) - \phi_g(ab) + \phi_g(a)$ for $a, b \in \mathcal{H}_g$. It follows that $\phi_g$ is unique from the fact that the first cohomology group of $\mathcal{H}_g$ vanishes. This function $\phi_g$ is called Meyer function. It is known that it is conjugacy invariant. Its values are contained in $\frac{1}{2g+1}\mathbb{Z}$ and concrete values on Lickorish generators and BSCC maps are calculated by Endo [4], Matsumoto [9] and Morifuji [11].

In the case of $g = 1$, under the identification $\mathcal{M}_1 \cong \mathcal{H}_1 \cong SL(2, \mathbb{Z})$, Meyer [10] and Atiyah [1] gave the explicit expression of the Meyer function using the Dedekind sums (see also [7]). Thus we can compute the values of it. Moreover Atiyah [1] put various geometric interpretations on the values of $\phi_1$ on hyperbolic elements. Hereafter we regard $SL(2, \mathbb{Z})(=Sp(2, \mathbb{Z}))$ as the domain of $\phi_1$. Hence we have $\delta \phi_1 = \tau_1$.

In this paper we study some representations induced from the actions of subgroups of the mapping class groups of a surface on the first cohomology group of $\pi_1(\Sigma_g)$ with coefficients in the module obtained from the nontrivial representation of $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2 = \text{Aut}(\mathbb{Z})$. As an application of them, in the case of $g = 1, 2$ (see also [5, 6]) and 3, we define some functions on subgroups of $\mathcal{H}_g$ using Atiyah-Patodi-Singer $\rho$-invariants and state that the difference of our function from the Meyer function is a nontrivial homomorphism on the subgroup. Moreover we state that the Meyer function coincides with the average of our functions on a certain subgroup.

2. SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$ and $* \in \Sigma_g$ a base point. Let $\omega: \pi_1(\Sigma_g, *) \to \mathbb{Z}_2$ be a nontrivial homomorphism which is also regarded as an element of $H^1(\Sigma_g; \mathbb{Z}_2)$. If we regard $\mathbb{Z}_2$ as $\text{Aut}(\mathbb{Z})$, then using $\omega$, we can obtain
$\pi_1(\Sigma_g,*$)-module $\mathbb{Z}$, which is denoted by $\mathbb{Z}_\omega$. We consider the first cohomology group $H^1(\pi_1(\Sigma_g,*), \mathbb{Z}_\omega)$ which is isomorphic to $\mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2$. Moreover it has a natural pairing defined by the cup product, the pairing $\mathbb{Z}_\omega \otimes \mathbb{Z}_\omega \cong \mathbb{Z}$ and the evaluation on the fundamental class of $\Sigma_g$. It is found that this pairing induces a symplectic form on the quotient group $H^1(\pi_1(\Sigma_g,*), \mathbb{Z}_\omega)/\text{torsion}$ and that it is isomorphic to the standard one on $\mathbb{Z}^{2(g-1)}$.

Let $\mathcal{M}_{g*}$ be the mapping class group of $\Sigma_g$ with a base point and $\mathcal{M}_{g*}^\omega$ the subgroup of it consisting of elements which preserve $\omega$. This subgroup acts on the group $H^1(\pi_1(\Sigma_g,*), \mathbb{Z}_\omega)/\text{torsion}$ by pullback. Since this action preserves the symplectic form, if we take a symplectic basis for it, we have the representation

$$\zeta_{g*}^\omega : \mathcal{M}_{g*}^\omega \to \text{Sp}(2(g-1), \mathbb{Z}).$$

These representations are related to prym representations of Looijenga [8]. Some properties of $\zeta_{g*}^\omega$ were investigated in [5, 6].

In this section we study the restrictions of them to subgroups of the hyperelliptic mapping class group of genus $g \geq 3$.

The hyperelliptic mapping class group $\mathcal{H}_g$ of $\Sigma_g$ is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms which commute with $\iota$ under isotopy which also commutes with $\iota$ [3]. This description of $\mathcal{H}_g$ shows that it acts the set of the fixed points of $\iota$. Thus we have the representation $\sigma : \mathcal{H}_g \to \mathfrak{S}_{2g+2}$, where $\mathfrak{S}_{2g+2}$ denotes the symmetric group of degree $2g + 2$ which is the number of the fixed points of $\iota$. Let $\mathcal{H}_g^\sigma$ be the kernel of the representation of $\sigma$. Let $j : \mathcal{M}_{g*} \to \mathcal{M}_g$ be the natural homomorphism, then we have the short exact sequence $1 \to \pi_1(\Sigma_g,* \to \mathcal{M}_{g*} \xrightarrow{j} \mathcal{M}_g \to 1$. Put $\mathcal{H}_{g*} = j^{-1}(\mathcal{H}_g)$ and $\mathcal{H}_{g*}^\sigma = j^{-1}(\mathcal{H}_g^\sigma)$. The following lemma is known.

**Lemma 1.** For any $a \in \mathcal{H}_g^\sigma$, the induced homomorphism $a^*$ on $H^1(\Sigma_g; \mathbb{Z}_2)$ is the identity.

By this lemma, we have $\mathcal{H}_g^\sigma \subset \mathcal{M}_g^\omega$ and $\mathcal{H}_{g*}^\sigma \subset \mathcal{M}_{g*}^\omega$, for any $\omega \neq 0 \in H^1(\Sigma_g; \mathbb{Z}_2)$. We denote the class of $\iota$ in $\mathcal{H}_g^\sigma$ by the same letter $\iota$. 


Lemma 2. For any lift \( \tilde{i} \) of \( i \in \mathcal{H}^*_g \) to \( \mathcal{H}^*_g \), the image of \( \tilde{i} \) by \( \zeta^\omega \) commutes with those of all elements of \( \mathcal{H}^*_g \).

The fundamental group \( \pi_1(\Sigma_g, \ast) \) of \( \Sigma_g \) is presented by \( \langle \alpha_i, \beta_i \mid 1 \leq i \leq g \rangle \) where the generators are depicted in Figure 2.

![Figure 2. The generators of \( \pi_1(\Sigma_g, \ast) \).](image)

Let \( \alpha^*_i, \beta^*_i \) be the dual basis for \( H^1(\Sigma_g; \mathbb{Z}_2) \) to the one for \( H_1(\Sigma_g; \mathbb{Z}_2) \) which is given by the homology classes of \( \alpha_i, \beta_i \).

Lemma 3. For any nonzero class \( \omega \in H^1(\Sigma_g; \mathbb{Z}_2) \), there exists \( a \in \mathcal{H}_g \) such that \( a^* \omega = \alpha^*_k \) for some \( k \).

Direct computations show that the representation matrix of \( \zeta^\omega_{g*}(\tilde{i}) \) with respect to a symplectic basis for \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega)/\text{torsion} \) is given by \( \pm(I_{2(k-1)} \oplus (-I_{2(g-k)}) \oplus (-I_{2(\mathit{g-k})})) \), where \( I_{2(k-1)} \) and \( I_{2(g-k)} \) are the identity matrices of rank \( 2(k-1) \) and \( 2(g-k) \) respectively. And \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega)/\text{torsion} \) decomposes to the direct sum of two symplectic submodules over \( \mathbb{Z} \). This result and Lemma 2 imply the following lemma.

Lemma 4. For any nonzero \( \omega \in H^1(\Sigma_g; \mathbb{Z}_2) \), the representation matrix of \( \zeta^\omega_{g*}(\tilde{i}) \) with respect to some symplectic basis is \( \pm(I_{2(k-1)} \oplus (-I_{2(g-k)}) \oplus (-I_{2(\mathit{g-k})})) \) for some \( k \). Moreover \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega)/\text{torsion} \) decomposes to the direct sum of two symplectic submodules over \( \mathbb{Z} \) on which \( \zeta^\omega_{g*}(\tilde{i}) \) is \( \pm \) the identity.

If we take a fixed point \( e \) of \( i \) as a base point \( \ast \) of \( \Sigma_g \), we can consider the group \( \mathcal{H}^*_g \) as a subgroup of \( \mathcal{H}^*_g \).

Corollary 5. The representation \( \zeta^\omega_{g*} \) induces two representations of \( \mathcal{H}^*_g \) to \( Sp(2(k-1), \mathbb{Z}) \) and \( Sp(2(g-k), \mathbb{Z}) \), where \( k \) is the integer in Lemma 3.
3. SOME FUNCTIONS ON SUBGROUPS OF $\mathcal{H}_{g*}$ OF LOW GENUS.

In this section we consider the case of $g = 1, 2$ and $3$.

Let $H'$ be the set $H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}$ for $g = 1, 2$ and the set $\{\omega \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\} | k = 2 \text{ in Lemma } 4 \}$ for $g = 3$.

**Lemma 6.** The number $\sharp H'$ of the elements of $H'$ is 3, 15 and 35 for $g = 1, 2$ and $3$ respectively.

For each $\omega \in H'$, let $\Delta^\omega_{g*}$ denote $\mathcal{H}_{g*} \cap \mathcal{M}_{g*}^\omega$ for $g = 1, 2$ and $\mathcal{H}_{g*}^\omega$ for $g = 3$. For any $\omega \in H'$, the image of $\Delta^\omega_{g*}$ by $\zeta^\omega_{g*}$ is contained in $\{id\}, SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ for $g = 1, 2$ and $3$ respectively under an appropriate choice of a symplectic basis for the representation space. In the case of $g = 3$, let $\zeta^\omega_{g*+}$ and $\zeta^\omega_{g*-}$ be the composition of $\zeta^\omega_{g*}$ with the projection from $SL(2, \mathbb{Z})$ to the first and second factor $SL(2, \mathbb{Z})$ respectively.

For each $\omega \in H'$, the function

$$\Phi^\omega_{g*}: \Delta^\omega_{g*} \rightarrow \frac{1}{3} \mathbb{Z}$$

is defined by $0, (\zeta^\omega_{g*})^*\phi_1$ and $(\zeta^\omega_{g*+})^*\phi_1 + (\zeta^\omega_{g*-})^*\phi_1$ for $g = 1, 2$ and $3$ respectively. It is easy to see that these functions are well defined.

**Lemma 7.** The equality $\delta \Phi^\omega_{g*} = (\zeta^\omega_{g*})^*\tau_{g-1}$ holds on $\Delta^\omega_{g*}$ for each $\omega \in H'$.

4. THE MAIN THEOREM

In this section we define some functions on subgroups of the mapping class groups and state the main theorem.

Let $\omega$ be a nonzero class in $H^1(\Sigma_g; \mathbb{Z}_2)$. For any $a \in \mathcal{M}^\omega_{g*}$, put $M_a := \Sigma_g \times [0, 1]/(x, 0) \sim (a(x), 1)$. Then $M_a$ is a $\Sigma_g$-bundle over $S^1 = [0, 1]/0 \sim 1$ with the identification $i$ of $\Sigma_g$ with the fiber at $0 \in S^1$ and with the section $s: S^1 \rightarrow M_a$ defined by the base point $* \in \Sigma_g$. It is easily checked that there is a unique homomorphism $\omega_a: \pi_1(M_a, s(0)) \rightarrow \mathbb{Z}_2 = \{\pm 1\} \subset U(1)$ satisfying the equalities $i^*\omega_a = \omega$ and $s^*\omega_a = 1$. We define the function $\rho_\omega: \mathcal{M}^\omega_{g*} \rightarrow \mathbb{Q}$ by $\rho_\omega(a) := \rho_{\omega_a}(M_a)$ for each $a \in \mathcal{M}^\omega_{g*}$. Here $\rho_{\omega_a}(M_a)$ is the Atiyah-Patodi-Singer $\rho$-invariant for $(M_a, \omega_a)$. 

In general, the Atiyah-Patodi-Singer $\rho$-invariant is a diffeomorphism invariant for a pair of a closed oriented manifold of odd dimension and a unitary representation of the fundamental group of it to $U(n)$. If a metric on the manifold is given, then the invariant is defined by the difference of the $\eta$-invariant of the signature operator on the manifold and $n$ times that of signature operator with coefficients in the flat bundle obtained from the unitary representation. Thus $\rho$-invarinats take their values in $\mathbb{R}$. If a unitary representation factors through a finite group, then the value of the $\rho$-invariant belongs to $\mathbb{Q}$.

For each $\omega \in H'$, we define a rational valued function $\mu_{g*}^\omega$ on $\Delta_{g*}^\omega$ by

$$\mu_{g*}^\omega : \rho_{\omega} + \Phi_{g*}^\omega.$$ 

These functions have the following properties.

**Lemma 8.** For any $a \in \Delta_{g*}^\omega$ and $f \in \mathcal{H}_{g*}$, the following hold.

1. $\mu_{g*}^\omega(1) = 0$,
2. $\mu_{g*}^\omega(a^{-1}) = -\mu_{g*}^\omega(a)$,
3. $\mu_{g*}^\omega(f^{-1})^*\omega(faf^{-1}) = \mu_{g*}^\omega(a)$,
4. $\text{sign}_g = \delta \mu_{g*}^\omega$ on $\Delta_{g*}^\omega$.

The main property in this lemma is 4. In order to prove it, we need the following theorem proved by Atiyah, Patodi and Singer.

**Theorem 9** (Atiyah-Patodi-Singer [2]). Let $M$ be a closed oriented manifold of odd dimension and $\alpha: \pi_1(M) \to U(n)$ a unitary representation. If $M$ is the boundary of a compact oriented manifold $N$ with $\alpha$ extending to a unitary representation of $\pi_1(N)$ then $\rho_\alpha(M) = n \text{sign}(N) - \text{sign}_\alpha(N)$.

We consider the $\Sigma_g$-bundle $N_{a,b}$ over $P$, where $a, b \in M_{g*}^\omega$. There is a unique homomorphism $\omega_{a,b}: \pi_1(N_{a,b}) \to \mathbb{Z}_2 \subset U(1)$ satisfying the same condition as $\omega_a$.

We apply Atiyah-Patodi-Singer's theorem to the pair $(N_{a,b}, \omega_{a,b})$ and use the Leray-Serre spectral sequence of the fibration $N_{a,b} \to P$. Then we have the property 4 in Lemma 8. Using Lemma 8, it is easy to see that the function $\mu_{g*}^\omega$ descends to a function $\mu_{g}^\omega$ on $\Delta_{g}^\omega := j(\Delta_{g*}^\omega)$ for any $\omega \in H'$. 
Theorem 10. The difference $\phi_g - \mu_g^\omega$ is a nontrivial homomorphism from $\Delta_g^\omega$ to $\mathbb{Q}$ for any $\omega \in H'$ and the equality $\phi_g = \frac{1}{|H|} \sum_{\omega \in H} \mu_g^\omega$ holds on $\mathcal{H}_g^\sigma$ for $g = 1, 2$ and $3$.

Since the Meyer function $\phi_g$ has the same properties as those in Lemma 8, the former part of this theorem follows from Lemma 8 and nontrivial examples which can be given explicitly. The latter follows from Lemma 8 and $H^1(\mathcal{H}_g^\sigma, \mathbb{Q})^{\mathfrak{S}_{2g+2}} = \{0\}$ which is obtained from the fact of $H^1(\mathcal{H}_g, \mathbb{Q}) = \{0\}$ using the Hochschild-Leray-Serre spectral sequence of the short exact sequence $1 \to \mathcal{H}_g^\sigma \to \mathcal{H}_g \to \mathfrak{S}_{2g+2} \to 1$.

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