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<th>SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS OF SURFACES AND SECONDARY INVARIANTS (Hyperbolic Spaces and Related Topics II)</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1163: 78-84</td>
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<tr>
<td>Issue Date</td>
<td>2000-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64284">http://hdl.handle.net/2433/64284</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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1. INTRODUCTION

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$ and $\mathcal{M}_g$ its mapping class group consisting of the isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$. We denote the 2-sphere with 3-holes by $P$. For any $a, b \in \mathcal{M}_g$, let $N_{a,b}$ be the $\Sigma_g$-bundle over $P$ with monodromies $a^{-1}$ and $b^{-1}$.

Meyer's signature 2-cocycle

$$\text{sign}_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$$

is defined by $\text{sign}_g(a,b) = \text{sign}(N_{a,b})$, where $\text{sign}(N_{a,b})$ is the signature of 4-manifold $N_{a,b}$ (see [10, 1]). Novikov additivity for the signature of manifolds shows that $\text{sign}_g$ satisfies the cocycle condition. Meyer also defined a 2-cocycle $\tau_g$ on $Sp(2g, \mathbb{Z})$ over $\mathbb{Z}$, which is also called signature 2-cocycle. It is well-known that the equality $\text{sign}_g = \zeta_g^* \tau_g$ holds, where $\zeta_g$ is the standard representation of $\mathcal{M}_g$ to $Sp(2g, \mathbb{Z})$ induced from the obvious action of $\mathcal{M}_g$ on the first cohomology group of $\Sigma_g$.

Let $\iota$ be the hyperelliptic involution on $\Sigma_g$ depicted in Figure 1.

**FIGURE 1.** The hyperelliptic involution $\iota$ on $\Sigma_g$.

The hyperelliptic mapping class group $\mathcal{H}_g$ of $\Sigma_g$ is the subgroup of $\mathcal{M}_g$ consisting of elements which commute with the class of $\iota$. It is known that $\mathcal{M}_1 = \mathcal{H}_1 = SL(2, \mathbb{Z}), \mathcal{M}_2 = \mathcal{H}_2$ and that $\mathcal{H}_g(g \geq 3)$ is a subgroup of $\mathcal{M}_g$ of infinite index.
Meyer’s signature cocycle $\text{sign}_g$ defines a nontrivial class of the second cohomology group of $\mathcal{M}_g$ with coefficients in $\mathbb{Z}$ and its restriction to $\mathcal{H}_g$ is also nontrivial. But it is trivial in the cohomology group of $\mathcal{H}_g$ with coefficients in $\mathbb{Q}$. Thus there exists a function or a $1$-cochain

$$\phi_g: \mathcal{H}_g \to \mathbb{Q}$$

such that $\text{sign}_g = \delta \phi_g$, where $\delta$ denotes the coboundary operator defined by $\delta \phi_g(a, b) = \phi_g(b) - \phi_g(ab) + \phi_g(a)$ for $a, b \in \mathcal{H}_g$. It follows that $\phi_g$ is unique from the fact that the first cohomology group of $\mathcal{H}_g$ vanishes. This function $\phi_g$ is called Meyer function. It is known that it is conjugacy invariant. Its values are contained in $\frac{1}{2g+1}\mathbb{Z}$ and concrete values on Lickorish generators and BSCC maps are calculated by Endo [4], Matsumoto [9] and Morifuji [11].

In the case of $g = 1$, under the identification $\mathcal{M}_1 \cong \mathcal{H}_1 \cong SL(2, \mathbb{Z})$, Meyer [10] and Atiyah [1] gave the explicit expression of the Meyer function using the Dedekind sums (see also [7]). Thus we can compute the values of it. Moreover Atiyah [1] put various geometric interpretations on the values of $\phi_1$ on hyperbolic elements. Hereafter we regard $SL(2, \mathbb{Z})(=Sp(2, \mathbb{Z}))$ as the domain of $\phi_1$. Hence we have $\delta \phi_1 = \tau_1$.

In this paper we study some representations induced from the actions of subgroups of the mapping class groups of a surface on the first cohomology group of $\pi_1(\Sigma_g)$ with coefficients in the module obtained from the nontrivial representation of $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2 = Aut(\mathbb{Z})$. As an application of them, in the case of $g = 1, 2$ (see also [5, 6]) and 3, we define some functions on subgroups of $\mathcal{H}_g$ using Atiyah-Patodi-Singer $\rho$-invariants and state that the difference of our function from the Meyer function is a nontrivial homomorphism on the subgroup. Moreover we state that the Meyer function coincides with the average of our functions on a certain subgroup.

2. SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$ and $* \in \Sigma_g$ a base point. Let $\omega: \pi_1(\Sigma_g, *) \to \mathbb{Z}_2$ be a nontrivial homomorphism which is also regarded as an element of $H^1(\Sigma_g; \mathbb{Z}_2)$. If we regard $\mathbb{Z}_2$ as $Aut(\mathbb{Z})$, then using $\omega$, we can obtain
\[ \pi_1(\Sigma_g, \ast) \text{-module } \mathbb{Z}, \text{ which is denoted by } \mathbb{Z}_\omega. \] We consider the first cohomology group \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega) \) which is isomorphic to \( \mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2. \) Moreover it has a natural pairing defined by the cup product, the pairing \( \mathbb{Z}_\omega \otimes \mathbb{Z}_\omega \cong \mathbb{Z} \) and the evaluation on the fundamental class of \( \Sigma_g. \) It is found that this pairing induces a symplectic form on the quotient group \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega)/\text{torsion} \) and that it is isomorphic to the standard one on \( \mathbb{Z}^{2(g-1)}. \)

Let \( \mathcal{M}_{g*} \) be the mapping class group of \( \Sigma_g \) with a base point and \( \mathcal{M}_{g*}^\omega \) the subgroup of it consisting of elements which preserve \( \omega. \) This subgroup acts on the group \( H^1(\pi_1(\Sigma_g, \ast), \mathbb{Z}_\omega)/\text{torsion} \) by pullback. Since this action preserves the symplectic form, if we take a symplectic basis for it, we have the representation

\[ \zeta_{g*}^\omega : \mathcal{M}_{g*}^\omega \rightarrow \text{Sp}(2(g - 1), \mathbb{Z}). \]

These representations are related to prym representations of Looijenga [8]. Some properties of \( \zeta_{g*}^\omega \) were investigated in [5, 6].

In this section we study the restrictions of them to subgroups of the hyperelliptic mapping class group of genus \( g \geq 3. \)

The hyperelliptic mapping class group \( \mathcal{H}_g \) of \( \Sigma_g \) is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms which commute with \( \iota \) under isotopy which also commutes with \( \iota \) [3]. This description of \( \mathcal{H}_g \) shows that it acts the set of the fixed points of \( \iota. \) Thus we have the representation \( \sigma : \mathcal{H}_g \rightarrow \mathfrak{S}_{2g+2}, \) where \( \mathfrak{S}_{2g+2} \) denotes the symmetric group of degree \( 2g + 2 \) which is the number of the fixed points of \( \iota. \) Let \( \mathcal{H}_g^\sigma \) be the kernel of the representation of \( \sigma. \) Let \( j : \mathcal{M}_{g*} \rightarrow \mathcal{M}_g \) be the natural homomorphism, then we have the short exact sequence

\[ 1 \rightarrow \pi_1(\Sigma_g, \ast) \rightarrow \mathcal{M}_{g*} \overset{j}{\rightarrow} \mathcal{M}_g \rightarrow 1. \]

Put \( \mathcal{H}_{g*} = j^{-1}(\mathcal{H}_g) \) and \( \mathcal{H}_{g*}^\sigma = j^{-1}(\mathcal{H}_g^\sigma). \) The following lemma is known.

**Lemma 1.** For any \( a \in \mathcal{H}_g^\sigma, \) the induced homomorphism \( a^* \) on \( H^1(\Sigma_g; \mathbb{Z}_2) \) is the identity.

By this lemma, we have \( \mathcal{H}_g^\sigma \subset \mathcal{M}_g^\omega \) and \( \mathcal{H}_{g*}^\sigma \subset \mathcal{M}_{g*}^\omega, \) for any \( \omega \neq 0 \in H^1(\Sigma_g; \mathbb{Z}_2). \)

We denote the class of \( \iota \) in \( \mathcal{H}_g^\sigma \) by the same letter \( \iota. \)
Lemma 2. For any lift $i$ of $\iota \in H_{g}^{\sigma}$ to $H_{g*}^{\sigma}$, the image of $i$ by $\zeta_{g*}^{\omega}$ commutes with those of all elements of $H_{g*}^{\sigma}$.

The fundamental group $\pi_{1}(\Sigma_{g}, *)$ of $\Sigma_{g}$ is presented by $< \alpha_{i}, \beta_{i} \mid \prod_{i=1}^{g}[\alpha_{i}, \beta_{i}] = 1 >$, where the generators are depicted in Figure 2.

![Figure 2. The generators of $\pi_{1}(\Sigma_{g}, *)$.](image)

Let $\alpha_{i}^{*}, \beta_{i}^{*} (1 \leq i \leq g)$ be the dual basis for $H^{1}(\Sigma_{g}; \mathbb{Z}_{2})$ to the one for $H_{1}(\Sigma_{g}; \mathbb{Z}_{2})$ which is given by the homology classes of $\alpha_{i}, \beta_{i}$.

Lemma 3. For any nonzero class $\omega \in H^{1}(\Sigma_{g}; \mathbb{Z}_{2})$, there exists $a \in H_{g}$ such that $a^{*}\omega = \alpha_{k}^{*}$ for some $k$.

Direct computations show that the representation matrix of $\zeta_{g*}^{\alpha_{i}^{*}}(i)$ with respect to a symplectic basis for $H^{1}(\pi_{1}(\Sigma_{g}, *), \mathbb{Z}_{\omega})/torSion$ is given by $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)})$, where $I_{2(k-1)}$ and $I_{2(g-k)}$ are the identity matrices of rank $2(k-1)$ and $2(g-k)$ respectively. And $H^{1}(\pi_{1}(\Sigma_{g}, *), \mathbb{Z}_{\omega})/torSion$ decomposes to the direct sum of two symplectic submodules over $\mathbb{Z}$. This result and Lemma 2 imply the following lemma.

Lemma 4. For any nonzero $\omega \in H^{1}(\Sigma_{g}; \mathbb{Z}_{2})$, the representation matrix of $\zeta_{g*}^{\omega}(i)$ with respect to some symplectic basis is $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)})$ for some $k$. Moreover $H^{1}(\pi_{1}(\Sigma_{g}, *), \mathbb{Z}_{\omega})/torSion$ is decomposed to the direct sum of two symplectic submodules over $\mathbb{Z}$ on which $\zeta_{g*}^{\omega}(i)$ is $\pm$ the identity.

If we take a fixed point $e$ of $\iota$ as a base point $*$ of $\Sigma_{g}$, we can consider the group $H_{g}^{\sigma}$ as a subgroup of $H_{g*}^{\sigma}$.

Corollary 5. The representation $\zeta_{ge}^{\omega}$ induces two representations of $H_{g}^{\sigma}$ to $Sp(2(k-1), \mathbb{Z})$ and $Sp(2(g-k), \mathbb{Z})$, where $k$ is the integer in Lemma 3.
3. Some functions on subgroups of \( \mathcal{H}_{g*} \) of low genus.

In this section we consider the case of \( g = 1, 2 \) and 3.

Let \( H' \) be the set \( H^1(\Sigma_g; \mathbb{Z}_2) \) \( \setminus \{0\} \) for \( g = 1, 2 \) and the set \( \{\omega \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\} \mid k = 2 \text{ in Lemma 4} \} \) for \( g = 3 \).

**Lemma 6.** The number of the elements of \( H' \) is 3, 15 and 35 for \( g = 1, 2 \) and 3 respectively.

For each \( \omega \in H' \), let \( \Delta_{g*}^\omega \) denote \( \mathcal{H}_{g*} \cap \mathcal{M}_{g*}^\omega \) for \( g = 1, 2 \) and \( \mathcal{H}_{g*}^\omega \) for \( g = 3 \). For any \( \omega \in H' \), the image of \( \Delta_{g*}^\omega \) by \( \zeta_{g*}^\omega \) is contained in \( \{id\} \), \( SL(2, \mathbb{Z}) \) and \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) for \( g = 1, 2 \) and 3 respectively under an appropriate choice of a symplectic basis for the representation space. In the case of \( g = 3 \), let \( \zeta_{g*}^{\omega^+} \) and \( \zeta_{g*}^{\omega^-} \) be the composition of \( \zeta_{g*}^\omega \) with the projection from \( SL(2, \mathbb{Z}) \) to the first and second factor \( SL(2, \mathbb{Z}) \) respectively.

For each \( \omega \in H' \), the function

\[
\Phi_{g*}^\omega: \Delta_{g*}^\omega \to \frac{1}{3} \mathbb{Z}
\]

is defined by \( 0, (\zeta_{g*}^{\omega})^* \phi_1 \) and \( (\zeta_{g*}^{\omega^+})^* \phi_1 + (\zeta_{g*}^{\omega^-})^* \phi_1 \) for \( g = 1, 2 \) and 3 respectively. It is easy to see that these functions are well defined.

**Lemma 7.** The equality \( \delta \Phi_{g*}^\omega = (\zeta_{g*}^\omega)^* \tau_{g^{-1}} \) holds on \( \Delta_{g*}^\omega \) for each \( \omega \in H' \).

4. The main theorem

In this section we define some functions on subgroups of the mapping class groups and state the main theorem.

Let \( \omega \) be a nonzero class in \( H^1(\Sigma_g; \mathbb{Z}_2) \). For any \( a \in \mathcal{M}_{g*}^\omega \), put \( M_a := \Sigma_g \times [0, 1]/(x, 0) \sim (a(x), 1) \). Then \( M_a \) is a \( \Sigma_g \)-bundle over \( S^1 = [0, 1]/0 \sim 1 \) with the identification \( i \) of \( \Sigma_g \) with the fiber at \( 0 \in S^1 \) and with the section \( s: S^1 \to M_a \) defined by the base point \( * \) of \( \Sigma_g \). It is easily checked that there is a unique homomorphism \( \omega_a: \pi_1(M_a, s(0)) \to \mathbb{Z}_2 = \{ \pm 1 \} \subset U(1) \) satisfying the equalities \( i^* \omega_a = \omega \) and \( s^* \omega_a = 1 \). We define the function \( \rho_{\omega}: \mathcal{M}_{g*}^\omega \to \mathbb{Q} \) by \( \rho_{\omega}(a) := \rho_{\omega_a}(M_a) \) for each \( a \in \mathcal{M}_{g*}^\omega \). Here \( \rho_{\omega_a}(M_a) \) is the Atiyah-Patodi-Singer \( \rho \)-invariant for \((M_a, \omega_a)\).
In general, the Atiyah-Patodi-Singer \( \rho \)-invariant is a diffeomorphism invariant for a pair of a closed oriented manifold of odd dimension and a unitary representation of the fundamental group of it to \( U(n) \). If a metric on the manifold is given, then the invariant is defined by the difference of the \( \eta \)-invariant of the signature operator on the manifold and \( n \) times that of signature operator with coefficients in the flat bundle obtained from the unitary representation. Thus \( \rho \)-invarinats take their values in \( \mathbb{R} \). If a unitary representation factors through a finite group, then the value of the \( \rho \)-invariant belongs to \( \mathbb{Q} \).

For each \( \omega \in H' \), we define a rational valued function \( \mu_{g*}^\omega \) on \( \Delta_{g*}^\omega \) by

\[
\mu_{g*}^\omega := \rho_{\omega} + \Phi_{g*}^\omega.
\]

These functions have the following properties.

**Lemma 8.** For any \( a \in \Delta_{g*}^\omega \) and \( f \in \mathcal{H}_{g*} \), the following hold.

1. \( \mu_{g*}^\omega(1) = 0 \),
2. \( \mu_{g*}^\omega(a^{-1}) = \mu_{g*}^\omega(a) \),
3. \( \mu_{g*}^{(f^{-1})^*\omega}(fa^{-1}) = \mu_{g*}^\omega(a) \),
4. \( \text{sign}_g = \delta \mu_{g*}^\omega \) on \( \Delta_{g*}^\omega \).

The main property in this lemma is 4. In order to prove it, we need the following theorem proved by Atiyah, Patodi and Singer.

**Theorem 9** (Atiyah-Patodi-Singer [2]). *Let \( M \) be a closed oriented manifold of odd dimension and \( \alpha: \pi_1(M) \to U(n) \) a unitary representation. If \( M \) is the boundary of a compact oriented manifold \( N \) with \( \alpha \) extending to a unitary representation of \( \pi_1(N) \) then \( \rho_\alpha(M) = n \text{ sign}(N) - \text{sign}_\alpha(N) \).*

We consider the \( \Sigma_g \)-bundle \( N_{a,b} \) over \( P \), where \( a, b \in M_{g*}^\omega \). There is a unique homomorphism \( \omega_{a,b}: \pi_1(N_{a,b}) \to \mathbb{Z}_2 \subset U(1) \) satisfying the same condition as \( \omega_a \). We apply Atiyah-Patodi-Singer's theorem to the pair \( (N_{a,b}, \omega_{a,b}) \) and use the Leray-Serre spectral sequence of the fibration \( N_{a,b} \to P \). Then we have the property 4 in Lemma 8. Using Lemma 8, it is easy to see that the function \( \mu_{g*}^\omega \) descends to a function \( \mu_g^\omega \) on \( \Delta_g^\omega := j(\Delta_{g*}^\omega) \) for any \( \omega \in H' \).
Theorem 10. The difference $\phi_g - \mu_g^\omega$ is a nontrivial homomorphism from $\Delta_g^\omega$ to $\mathbb{Q}$ for any $\omega \in H'$ and the equality $\phi_g = \frac{1}{\# H} \sum_{\omega \in H'} \mu_g^\omega$ holds on $\mathcal{H}_g^\sigma$ for $g = 1, 2$ and 3.

Since the Meyer function $\phi_g$ has the same properties as those in Lemma 8, the former part of this theorem follows from Lemma 8 and nontrivial examples which can be given explicitly. The latter follows from Lemma 8 and $H^1(\mathcal{H}_g^\sigma, \mathbb{Q})^{S_{2g+2}} \rightarrow \{0\}$ which is obtained from the fact of $H^1(\mathcal{H}_g, \mathbb{Q}) = \{0\}$ using the Hochschild-Leray-Serre spectral sequence of the short exact sequence $1 \rightarrow \mathcal{H}_g^\sigma \rightarrow \mathcal{H}_g \rightarrow S_{2g+2} \rightarrow 1$.

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