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FORD DOMAINS OF PUNCTURED TORUS GROUPS AND TWO-BRIDGE KNOT GROUPS

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In [8], Jorgensen described the combinatorial structures of the Ford domains of discrete cyclic subgroups of Isom($\boldsymbol{H}^3$) by his so called "method of geometric continuity". By using the method, he also studied the combinatorial structures of the Ford domains of quasi-Fuchsian once-punctured torus groups (see [9]). The work is intimately related to the constructions of the complete hyperbolic structures of surface bundles over a circle given in [10] and [11] (cf. [18], [7]). But, unfortunately, the draft [9] has not been completed yet. Hopefully, it would be completed in the forthcoming book of Jorgensen and Marden [12]. For (attempts of) expositions of the results without proof, see [22], [3], [16] and [19].

This article is an announcement of our joint research which gives proofs to (parts of) the assertions in [9] and extends them to the results for the groups on the outside of the quasi-Fuchsian once-punctured torus space. To be explicit, we describe the Ford domains of (the fundamental groups or the holonomy groups) of hyperbolic manifolds (possibly with cone singularities) belonging to one of the following families (see Section 4 for the definitions of the terminologies):

- The quasi-Fuchsian once-punctured torus groups.
- The geometrically finite boundary groups of the quasi-Fuchsian once-punctured torus space, in particular, the groups in the Maskit embeddings of the Teichmüller space of once-punctured tori. (For geometrically infinite boundary groups, see the first author's work announced in [2], which relies on the result of Minsky [13].)
- The Koebe groups representing once-punctured tori.
- $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-extensions of the groups in the Riley slice of Schottky space.
- The hyperbolic cone-manifolds with underlying space a 2-bridge link complement having the upper and lower tunnels as cone axes.
- The hyperbolic 2-bridge link groups.

Roughly speaking, we have proved that, for any group in the list, there is a "chain" Σ (see Definition 4.9) of triangles in the modular diagram, such that the Ford domain is supported by the isometric hemi-spheres of the "(elliptic) generators" whose "slopes" are vertices of Σ, and that its combinatorial structure is recovered from Σ (cf. Theorems 5.4). In particular, we give a concrete and conceptual construction of the complete hyperbolic structures of the hyperbolic 2-bridge link complements, which leads to an affirmative answer to a conjecture proposed in the second author's joint work with J. Weeks [20] on the canonical decompositions of 2-bridge link complements. Actually, this joint work started aiming at this result. Then, why are the 2-bridge link groups related to the punctured torus groups? This is because the 2-bridge link groups are quotients of the fundamental group $\pi_1(S)$ of a four-times punctured sphere S, and $\pi_1(S)$ is "commensurable" with the fundamental group of a once-punctured torus T (see Section 1).

1. Fricke surfaces, modular diagram and 2-bridge links

Let T, S, $\mathcal{O}$, respectively, be a once-punctured torus, a 4-times punctured sphere, and a $(2,2,2,\infty)$-orbifold (i.e., the orbifold with underlying space a punctured sphere and with three cone points of cone angle π). They have $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ as the common covering space. To be precise, let Γ and $\tilde{\Gamma}$, respectively, be the groups of transformations on $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ generated by π-rotations about points in $\boldsymbol{Z}^2$ and $(\frac{1}{2}\boldsymbol{Z})^2$. Then $T = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\boldsymbol{Z}^2$, $S = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\Gamma$ and $\mathcal{O} = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\tilde{\Gamma}$. In particular, there is a $\boldsymbol{Z}_2$-covering $T \to \mathcal{O}$ and a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-covering $S \to \mathcal{O}$: the pair of these coverings is called the *Fricke diagram* and each of T, S, and $\mathcal{O}$ is called a *Fricke surface* (cf. [21]).

A simple loop in a Fricke surface is said to be *essential*, if it does not bound a disk, a disk with one puncture, or a disk with one cone point. Similarly, a simple arc in a Fricke surface joining punctures is said to be *essential*, if it does not cut off a subsurface homeomorphic to a surface obtained by deleting a point on the boundary of a disk, a disk with one puncture, or a disk with one cone point. Then the isotopy classes of essential simple loops [resp. essential simple arcs joining a given puncture to a puncture] in a Fricke surface are in one-to-one correspondence with $\hat{\boldsymbol{Q}} := \boldsymbol{Q} \cup \{1/0\}$: A representative of the isotopy class corresponding to $r \in \hat{\boldsymbol{Q}}$ is the projection of a line in $\boldsymbol{R}^2$ (the line being disjoint from $\boldsymbol{Z}^2$ for the loop case, and intersecting $\boldsymbol{Z}^2$ for the arc case). The element $r \in \hat{\boldsymbol{Q}}$ associated to a circle or an arc is called its *slope*. An essential loop of slope r in T or $\mathcal{O}$ [resp. S] is denoted by α_r [resp. $\tilde{\alpha}_r$]. Note that the projection from α $(\subset T)$ to α $(\subset \mathcal{O})$ is a homeomorphism, while the projection from $\tilde{\alpha}_r$ $(\subset S)$ to α $(\subset \mathcal{O})$ is a 2-fold covering.

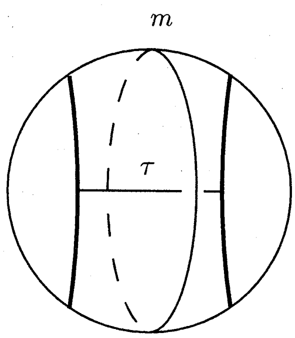
Consider the ideal triangle in the hyperbolic plane $\boldsymbol{H}^2 = \{z \in \boldsymbol{C} \mid \Im(z) > 0\}$ spanned by the ideal vertices $\{0/1, 1/1, 1/0\}$. Then the translates of this ideal triangle by the action of $SL(2,\boldsymbol{Z})$ form a tessellation of $\boldsymbol{H}^2$. This is called the *modular diagram* and is denoted by $\mathcal{D}$. The set of ideal vertices of $\mathcal{D}$ is equal to $\hat{\boldsymbol{Q}}$, and a typical ideal triangle σ of $\mathcal{D}$ is spanned by $\{\frac{p_1}{q_1}, \frac{p_1+p_2}{q_1+q_2}, \frac{p_2}{q_2}\}$ where $\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \in SL(2,\boldsymbol{Z})$. For each ideal triangle σ in $\mathcal{D}$, the union of the lines in $\boldsymbol{R}^2$ intersecting $\boldsymbol{Z}^2$ with slopes the ideal vertices of σ determines a Γ-invariant ideal triangulation of $\boldsymbol{R}^2 - \boldsymbol{Z}^2$ which projects to a *maximal arc system* of each of T, S, and $\mathcal{O}$.

The abstract simplicial complex having the combinatorial structure of $\mathcal{D}$ is also denoted by the same symbol $\mathcal{D}$. Then $\boldsymbol{H}^2$ is identified with $|\mathcal{D}| - |\mathcal{D}^{(0)}|$, where $\mathcal{D}^{(0)}$ denotes the 0-skeleton of $\mathcal{D}$ and $|\cdot|$ denotes the underlying topological space of a simplicial complex. The *distance* $d(r_1, r_2)$ between two elements r_1 and r_2 of $\hat{\boldsymbol{Q}} = \mathcal{D}^{(0)}$ is defined to be the minimal number of edges in a simplicial path in $\mathcal{D}$ joining r_1 to r_2.

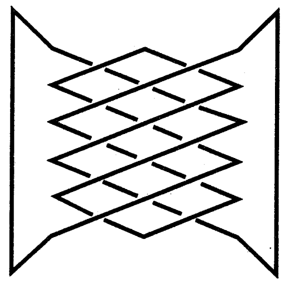
In the remainder of this section, we recall basic facts concerning the 2-bridge links. First, we recall the definitions of a trivial tangle and a rational tangle. A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. A *meridian* m of (B^3, t) is a simple loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. The arc τ illustrated in Figure 1.1 (1) is called the *core* of (B^3, t). A *rational tangle* is a trivial tangle (B^3, t) endowed with a homeomorphism from $\partial(B^3, t)$ to the *Conway sphere* $(\boldsymbol{R}^2, \boldsymbol{Z}^2)/\Gamma$. Then the meridian m of a rational tangle is regarded as a loop in the Fricke surface $S = (\boldsymbol{R}^2 - \boldsymbol{Z}^2)/\Gamma$ and hence it has a well-defined slope. The *slope* of a rational tangle is defined to be the slope of its meridian, and a rational tangle of slope r is denoted by $(B^3, t(r))$.

The *two bridge link* $S(r)$ *of slope* r is defined as the "sum" of the rational tangles of slopes ∞ and r; i.e., $(S^3, S(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map (see Figure 1.1 (2)). (Note that each of the boundaries of the rational tangles are identified with the Conway sphere $(\boldsymbol{R}^2, \boldsymbol{Z}^2)/\Gamma$, so the term "identity map" has a well-defined meaning.) The cores of $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, are called the *upper tunnel* and the *lower tunnel* of $S(r)$.

(1)



(2)



$S(2/5)$

Fig. 1.1

The following is a reformulation of a well-known consequence (cf. [18]) of Thurston's uniformization theorem of Haken manifolds ([14],[15]):

Theorem 1.1. *According as the distance* $d(\infty, r)$ *is* 0, 1, 2, *or* ≥ 3, *the 2-bridge link* $S(r)$ *is the 2-component trivial link, the trivial knot, a torus link, or a hyperbolic link.*

More generally, for any pair (r_1, r_2) of elements of $\hat{\boldsymbol{Q}}$, $S(r_1, r_2)$ denotes the link defined by $(S^3, S(r_1, r_2)) = (B^3, t(r_1)) \cup (B^3, t(r_2))$. This link is homeomorphic to $(S^3, S(r))$, where r is the image of r_2 by an element $A \in SL(2, \boldsymbol{Z})$ such that $A(r_1) = \infty$. Hence, $S(r_1, r_2)$ is hyperbolic if and only if $d(r_1, r_2) \geq 3$.

2. A way from punctured torus groups to 2-bridge link groups

In this section, we explain our strategy for the construction of the complete hyperbolic structures of the 2-bridge link complements. Let M be the space obtained from the link exterior $\mathrm{cl}(S^3 - N(S(r)))$ by deleting open regular neighborhoods of the upper and lower tunnels. Then M is homeomorphic to $S \times [0, 1]$, and the link complement is recovered from M by attaching 2-handles along the loops $\tilde{\alpha}_\infty$ on $S \times 0$ and $\tilde{\alpha}_r$ on $S \times 1$ (see Figure 2.1). We will give "geometric realization" of this procedure as follows. We start from a very simple Fuchsian representation of $\pi_1(S)$, deform the representation in the quasi-Fuchsian space, and obtain as the limit the "double cusp group" in which $\tilde{\alpha}_\infty$ and $\tilde{\alpha}_r$ correspond to accidental parabolic transformations. The quotient of the hyperbolic space by the image of each representation in the above procedure is homeomorphic to $\mathrm{int}(M)$. Each representation in the above procedure extends to a representation of $\pi_1(\mathcal{O})$, and Jorgensen's analysis of punctured torus groups [9] describes how the Ford domain of the image of the representation of $\pi_1(\mathcal{O})$ changes during the procedure.

Next, we get out of the closure of the quasi-Fuchsian space, and consider deformations of the representation such that $\tilde{\alpha}_\infty$ and $\tilde{\alpha}_r$ become elliptic transformations. Though the representations are not discrete anymore except for special cases, each of them can be regarded as the holonomy representation of the *hyperbolic cone-manifold* $S(r; 2\theta_1, 2\theta_2)$, illustrated in Figure 2.1, for some θ_1 and θ_2 with $0 \le \theta_i \le \pi$ $(i = 1, 2)$. Note that the $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-symmetry of S extends to those of $(S^3, S(r))$ and $S(r; 2\theta_1, 2\theta_2)$. We denote the quotient orbifold $(S^3 - S(r))/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)$ by $\mathcal{O}(r)$ and the quotient cone-manifold $S(r; 2\theta_1, 2\theta_2)/(\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2)$ by $\mathcal{O}(r; \theta_1, \theta_2)$ (see Figure 2.2). Then we can construct a fundamental domain of the cone-manifold $\mathcal{O}(r; \theta_1, \theta_2)$, whose combinatorial structure is essentially equal to that of the Ford domain of the $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-extension of the double cusp group. Furthermore, we can see the combinatorial structure of the fundamental domain of $\mathcal{O}(r; \theta_1, \theta_2)$ does not change as long as $0 \le \theta_i < \pi$ $(i = 1, 2)$. When θ_1 or θ_2 becomes π, the fundamental domain changes drastically. However, it is possible to describe the drastic change, and we can understand the combinatorial structure of the Ford domain of $\mathcal{O}(r; 2\pi, 2\pi) = \mathcal{O}(r)$. Moreover the (extended) Ford domain of $S^3 - S(r)$ is equal to that of $\mathcal{O}(r)$, and it is dual to the topological ideal triangulation given by [20]. This proves that the topological triangulation is isotopic to the canonical decomposition.

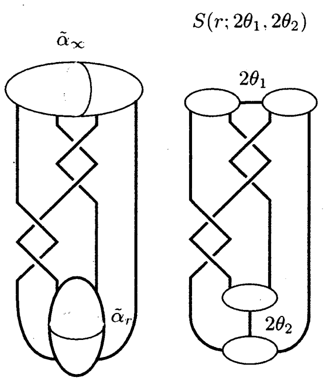


Fig. 2.1

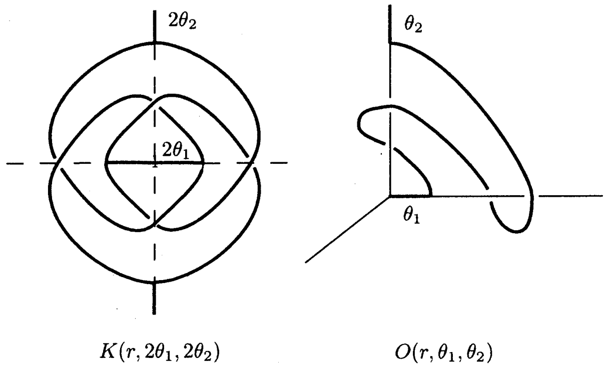


Fig. 2.2

When the link $S(r)$ is a torus link, i.e., when $d(\infty, r) = 2$, the holonomy representation of $\mathcal{O}(r; \theta_1, \theta_2)$ degenerates into a real representation when (θ_1, θ_2) becomes (π, π); the image of the limit representation is isomorphic to the orbifold fundamental group of the (2-dimensional) base orbifold of the Seifert fibered structure of the link complement. In particular, we have the following result.

Theorem 2.1. *The topological cone-manifold* $S(r; 2\theta_1, 2\theta_2)$ *is a hyperbolic cone-manifold if and only if one of the following conditions holds:*
(1) $d(\infty, r) \ge 3$.
(2) $d(\infty, r) = 2$ *and* $(\theta_1, \theta_2) \ne (\pi, \pi)$.
(3) $d(\infty, r) = 1$.

Remark 2.2. (1) By the argument of Parkkonen [17], we can see that the holonomy representation of the cone-manifold $S(r; 2\theta_1, 2\theta_2)$ is discrete if and only if $\theta_i = 2\pi/n_i$ for some integer n_i $(i = 1, 2)$.

(2) In (3) of the above theorem, the cone-manifold structure projects to that of $\mathcal{O}(r; 2\theta_1, 2\theta_2)$ if and only if $(\theta_1, \theta_2) \neq (\pi, \pi)$.

3. Fundamental groups of Fricke surfaces

Since T and S are finite regular coverings of the orbifold $\mathcal{O}$, the fundamental groups of T and S are regarded as normal subgroups of the orbifold fundamental group of $\mathcal{O}$ of finite index. These groups have the following group presentations:

$$\pi_1(T) = \langle A_0, B_0 \rangle, \tag{1}$$

$$\pi_1(S) = \langle K_0, K_1, K_2, K_3 \mid K_0K_1K_2K_3 = 1 \rangle, \tag{2}$$

$$\pi_1(\mathcal{O}) = \langle P_0, Q_0, R_0 \mid P_0^2 = Q_0^2 = R_0^2 = 1 \rangle, \tag{3}$$

Here the generators satisfy the following conditions: Put $K = (P_0Q_0R_0)^{-1}$, then K is represented by the puncture of $\mathcal{O}$ and satisfies the relation $K^2 = [A_0, B_0]$, $A_0 = KP_0 = R_0Q_0$, $B = K^{-1}R_0 = P_0Q_0$, $K_0 = K$, $K_1 = K^P$, $K_2 = K^Q$, $K_3 = K^R$, where X^Y denotes YXY^{-1}. (Warning: Note that this convention may be different from the usual one and that $(X^Y)^Z \neq X^{YZ} = (X^Z)^Y$.) *Throughout this paper, we reserve the symbol K to denote the element of $\pi_1(\mathcal{O})$ defined in the above.*

Definition 3.1. (1) An ordered pair (A, B) of elements in $\pi_1(T)$ is a *generator pair* of $\pi_1(T)$ if they generate $\pi_1(T)$ and satisfies $[A, B] = K^2$. In this case, A and B, respectively are called the *left* and *right* generators, and (A, AB, B) is called a *generator triple.* The slope of an essential loop in T realizing A [resp. B] is called the *slope* of A [resp. B] and is denoted by $s(A)$ [resp. $s(B)$].

(2) An ordered triple (P, Q, R) of elements of $\pi_1(\mathcal{O})$ is called an *elliptic generator triple* if they generate $\pi_1(\mathcal{O})$ and satisfies $P^2 = Q^2 = R^2 = 1$ and $(PQR)^{-1} = K$. A member of an elliptic generator triple is called an *elliptic generator.* $\mathcal{EG}$ denotes the set of all elliptic generators.

Proposition 3.2. *(1) For any elliptic generator triple (P, Q, R), the following holds:*

(1.1) The triple of any three consecutive elements in the following bi-infinite sequence is also an elliptic generator triple.

$$\cdots, P^{K^{-1}}, Q^{K^{-1}}, R^{K^{-1}}, P, Q, R, P^K, Q^K, R^K, \cdots$$

(1.2) (P, R, Q^R) is also an elliptic generator triple.

(2) Conversely, any elliptic generator triple is obtained from (P, Q, R) by successively applying the operations in (1).

(3) If (P, Q, R) is an elliptic generator triple of $\pi_1(\mathcal{O})$, then $(KP, KQ, K^{-1}R)$ is a generator triple of $\pi_1(T)$. Conversely, every generator triple of $\pi_1(T)$ is so obtained.

For each elliptic generator P of $\pi_1(\mathcal{O})$, KP and $K^{-1}P$, respectively, are left and right generators of $\pi_1(T)$ by Proposition 3.2. Further, we see $s(KP) = s(K^{-1}P)$. We define the *slope* $s(P)$ of P by $s(P) := s(KP) = s(K^{-1}P)$. *Throughout this paper, we assume that the slopes of A_0 and B_0 in the group presentation (1) are* $0/1$ *and* $1/0$, *respectively and that the slopes of P_0, Q_0 and R_0 in the group presentation (3) are* $0/1$, $1/1$ *and* $1/0$, *respectively.*

Proposition 3.3. *(1) For two elliptic generators P and P', $s(P) = s(P')$ if and only if $P' = P^{K^n}$ for some integer n.*

(2) For any elliptic generator triple (P, Q, R), the oriented triangle $\langle s(P), s(Q), s(R)\rangle$ of $\mathcal{D}$ is a coherent with the triangle $\langle 0/1, 1/1, 1/0\rangle$.

(3) The slopes of two elliptic generator triples span the same triangle of $\mathcal{D}$ if and only if they are related by the operation (1.1) of Proposition 3.2.

(4) For any elliptic generator triple (P, Q, R), we have $s(Q^R) = s(Q^P)$, and this slope is the image of $s(Q)$ by the reflection in the edge $\langle s(P), s(R)\rangle$.

(5) Let (A, AB, B) be a generator triple of $\pi_1(T)$. Then (AB^{-1}, A, B) is also a generator triple, and both $\langle s(A), s(AB), s(B)\rangle$ and $\langle s(AB^{-1}), s(A), s(B)\rangle$ are coherent with the triangle $\langle 0/1, 1/1, 1/0\rangle$. In particular, $s(AB^{-1})$ is the image of $s(AB)$ by the reflection in the edge $\langle s(A), s(B)\rangle$.

Convention 3.4. When we mention to a triangle $\langle s_0, s_1, s_2\rangle$ of $\mathcal{D}$, we always assume that the orientation of the triangle by this order of the vertices is coherent with the orientation of $\langle 0/1, 1/1, 1/0\rangle$.

By Propositions 3.2 and 3.3, we see that for each triangle $\sigma = \langle s_0, s_1, s_2\rangle$ of $\mathcal{D}$, there is a bi-infinite sequence $\{P_n\}_{n\in \boldsymbol{Z}}$ of elliptic generators satisfying the following conditions:

1. For each $n \in \boldsymbol{Z}$, we have $s(P_n) = s_{[n]}$, where $[n]$ denotes the integer in $\{0, 1, 2\}$ such that $[n] \equiv n \pmod 3$.
2. The triple of any three consecutive elements P_{n-1}, P_n, P_{n+1} is an elliptic generator triple.
3. $P_n^{K^m} = P_{n+3m}$.

Further, such a sequence is unique modulo sifts of suffix by multiples of 3.

Definition 3.5. (1) The above sequence $\{P_n\}_{n\in \boldsymbol{Z}}$ is called the *sequence of elliptic generators associated with* σ, and it is denoted by $\mathcal{EG}(\sigma)$.

(2) More generally, for a subcomplex Σ of $\mathcal{D}$, $\mathcal{EG}(\Sigma)$ denotes the set of elliptic generators, $\{P \in \mathcal{EG} \mid s(P) \in \Sigma^{(0)}\}$.

4. Marked representations

First, we introduce the family of $PSL(2, \boldsymbol{C})$-representations of the fundamental groups of the Fricke surfaces which are studied in this paper.

Definition 4.1. (Type-preserving representation) (1) A $PSL(2, \boldsymbol{C})$-representation of $\pi_1(\mathcal{O})$ is *type-preserving* if it is not reducible (i.e., it does not have a common fixed point in the closure of hyperbolic space) and sends K to a parabolic transformation.

(2) $\mathcal{X}$ denotes the space of the type-preserving $PSL(2, \boldsymbol{C})$-representations of $\pi_1(\mathcal{O})$ modulo conjugacy.

The following lemma can be proved by using the arguments in [6, Proof of Proposition 1.1].

Lemma 4.2. *(1) Let ρ be a type-preserving $PSL(2, \boldsymbol{C})$-representation of $\pi_1(\mathcal{O})$. Then the restriction of ρ to $\pi_1(T)$ lifts to an $SL(2, \boldsymbol{C})$-representation $\tilde{\rho}$ such that* $\mathrm{tr}(\tilde{\rho}(K^2)) = -2$.

(2) Conversely, every $PSL(2, \boldsymbol{C})$-representation of $\pi_1(T)$ obtained from an $SL(2, \boldsymbol{C})$-representation $\tilde{\rho}$ of $\pi_1(T)$ [resp. $\pi_1(S)$] such that $\mathrm{tr}(\tilde{\rho}(K^2)) = -2$, *extends to a type-preserving $PSL(2, \boldsymbol{C})$-representation of $\pi_1(\mathcal{O})$.*

Definition 4.3. (1) An $SL(2, \boldsymbol{C})$-representation $\tilde{\rho}$ of $\pi_1(T)$ is *type-preserving* if $\mathrm{tr}(\tilde{\rho}(K^2)) = -2$.

(2) $\tilde{\mathcal{X}}$ denotes the space of the type-preserving $SL(2, \boldsymbol{C})$-representations of $\pi_1(T)$ modulo conjugacy with the albegraic topology.

Definition 4.4. (Markoff map) For a type-preserving $SL(2, \boldsymbol{C})$-representation $\tilde{\rho} \in \tilde{\mathcal{X}}$, let ϕ be the map from $\mathcal{D}^{(0)} = \hat{\boldsymbol{Q}}$ to $\boldsymbol{C}$ define by $\phi(r) = \mathrm{tr}(\tilde{\rho}(\alpha_r))$, where α_r is an element of $\pi_1(T)$ represented by a simple loop of slope r. We call it the *Markoff map* associated with $\tilde{\rho}$.

Then it is known by [5] and [9] that $\tilde{\rho}$ is recovered from the Markoff map ϕ. Throughout this paper, we employ the following convention:

Convention 4.5. (1) When we choose a representative ρ of an element of $\mathcal{X}$, we always assume

$$\rho(K) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(2) We do not distinguish between an element of $\mathcal{X}$ and its representative: they are denoted by the same symbol as long as there is no fear of confusion.
(3) When we mention to ρ, the symbols $\tilde{\rho}$ and ϕ, respectively, denote a lift of ρ and the Markoff map associated with $\tilde{\rho}$.

We now give the definitions of the Maskit embeddings, Koebe groups and the Riley slice of Schottky space, which appeared in the introduction.

Definition 4.6. (Maskit slice) We call ρ a *Maskit representation* of slope s and sign ϵ if it satisfies the following conditions.

1. $\rho(\alpha_s)$ is a parabolic transformation or equivalently $\phi(s) = \pm 2$.
2. The connected components of $\Omega(\rho)$ are of two kinds:
   - A simply connected $\mathrm{Im}(\rho)$-invariant component Ω_0 for which the orbit space $\Omega_0/\mathrm{Im}(\rho)$ is homeomorphic to the orbifold $\mathcal{O}$ or equivalently $\Omega_0/\rho(\pi_1(T))$ is homeomorphic to T.
   - Non-invariant component Ω_i, $i \geq 1$, that are conjugate to one another under $\mathrm{Im}(\rho)$ and for which each orbit space $\Omega_i/\mathrm{Stab}_{\mathrm{Im}(\rho)}(\Omega_i)$ is conformally the $(2, \infty, \infty)$-orbifold, i.e., the hyperbolic orbifold with underlying space a twice-punctured sphere with a cone point of cone angle π. The latter condition is equivalent to the condition that $\Omega_i/\mathrm{Stab}_{\rho(\pi_1(T))}(\Omega_i)$ is homeomorphic to a trice-punctured sphere.
3. The region $\{z \in \boldsymbol{C} \mid \epsilon\Im(z) > M\}$ for sufficiently large positive real number M is contained in the component Ω_0.

The subspace of $\mathcal{X}$ consisting of the conjugacy classes of the Maskit representations of slope s and sign ϵ is called the *Maskit slice* (or the *Maskit embedding* of the Teichmuller space of punctured tori) of slope s and of sign ϵ and is denoted by $\mathcal{M}(s, \epsilon)$.

Definition 4.7. (Koebe slice) We call ρ a *Koebe representation* of order n (≥ 3), slope s and sign ϵ if it satisfies the following conditions.

1. $\rho(\alpha_s)$ is an elliptic transformation with rotation angle $2\pi/n$ or equivalently $\phi(s) = \pm 2\cos(\pi/n)$.
2. The connected components of $\Omega(\rho)$ are of two kinds:

- A simply connected $\mathrm{Im}(\rho)$-invariant component Ω_0 for which the orbit space $\Omega_0/\mathrm{Im}(\rho)$ is homeomorphic to the orbifold $\mathcal{O}$ or equivalently $\Omega_0/\rho(\pi_1(T))$ is homeomorphic to T.
- Non-invariant component Ω_i, $i \geq 1$, that are conjugate to one another under $\mathrm{Im}(\rho)$ and for which each orbit space $\Omega_i/\mathrm{Stab}_{\mathrm{Im}(\rho)}(\Omega_i)$ is conformally the $(2, n, \infty)$-orbifold, i.e., the hyperbolic orbifold with underlying space a once-punctured sphere with a cone point of cone angle $2\pi/n$ and a cone point of cone angle π. The latter condition is equivalent to the condition that $\Omega_i/\mathrm{Stab}_{\rho(\pi_1(T))}(\Omega_i)$ is homeomorphic to the (n, n, ∞)-orbifold.

3. The region $\{z \in \boldsymbol{C} \mid \epsilon\Im(z) > M\}$ for sufficiently large positive real number M is contained in the component Ω_0.

The subspace of $\mathcal{X}$ consisting of the conjugacy classes of the Koebe representations of slope s and of sign ϵ is called the *Koebe slice* of order n (≥ 3), slope s and sign ϵ, and it is denoted by $\mathcal{K}(n, s, \epsilon)$.

Definition 4.8. (Riley slice) We call ρ a *Riley representation* of slope s if it satisfies the following conditions.

1. $\rho(\alpha_s)$ is an elliptic transformation with rotation angle π or equivalently $\phi(s) = 0$.
2. $\Omega(\rho)$ is connected and the orbit space $\Omega(\rho)/\mathrm{Im}(\rho)$ is homeomorphic to the orbifold $\mathcal{O}$ or equivalently $\Omega(\rho)/\rho(\pi_1(S))$ is homeomorphic to S.

The subspace of $\mathcal{X}$ consisting of the conjugacy classes of the Riley representations of slope s is called the *Riley slice* (of Schottky space) of slope s, and it is denoted by $\mathcal{R}(s)$. Each $\mathcal{R}(s)$ is equivalent to $\mathcal{R}$ introduced at the beginning of this section.

Next, we introduce some concepts and notations which are needed later.

Definition 4.9. (Chain) (1) A *chain* is a non-empty ordered set $\Sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_m\}$, such that $\sigma_1, \sigma_2, \cdots, \sigma_m$ are triangles of $\mathcal{D}$ intersecting an oriented open geodesic segment of $\boldsymbol{H}^2$ in this order. The number m is called the *length* of the chain.

(2) When a chain $\Sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_m\}$ is given, the symbol σ^- [resp. σ^+] denotes σ_1 [resp. σ_m]: we call it the $(-)$-*terminal triangle* [resp. $(+)$-*terminal triangle*] of Σ. If the length m is greater than 1, then the symbols s^- [resp. s^+] denotes the vertex of σ^- [resp. σ^+] which is not a vertex of σ_2 [resp. σ_{m-1}]: we call it the $(-)$-*terminal vertex* [resp. $(+)$-*terminal vertex*] of Σ.

Remark 4.10. If Σ has length 1, then we regard $\sigma^- = \sigma^+ = \sigma_1$; however, s^ϵ ($\epsilon = \pm$) are undefined. If Σ has length 0, then σ^ϵ ($\epsilon = \pm$) are undefined.

Definition 4.11. A *marked representation* of $\pi_1(\mathcal{O})$ (a marked representation, in brief) is a pair $(\rho; \Sigma)$ of a type-preserving representation ρ of $\pi_1(\mathcal{O})$ and a chain Σ. Σ is called the *marking* of $(\rho; \Sigma)$. When Σ consists of a single triangle σ [resp. a single edge τ] $(\rho; \Sigma)$ is denoted by $(\rho; \sigma)$ [resp. $(\rho; \tau)$] and is called a *singly marked representation* [resp. *thin marked representation*].

5. Ford domains

In this section, we recall the definition of the Ford domain and give a rough exposition of the main theorem. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of $PSL(2, \boldsymbol{C})$. Then it acts on the

Riemann sphere $\hat{\boldsymbol{C}}$ by $A(z) = (az+b)/(cz+d)$. Suppose $A(\infty) \neq \infty$, then the *isometric circle* $I(A)$ of A is defined by

$$I(A) = \{z \in \boldsymbol{C} \mid |A'(z)| = 1\} = \{z \in \boldsymbol{C} \mid |cz+d| = 1\}.$$

$I(A)$ is the circle in $\boldsymbol{C}$ whose center is $-d/c = A^{-1}(\infty) = \text{pole}(A)$ and has radius $1/|c|$. The *isometric hemisphere* $Ih(A)$ is the hyper-plane of $\boldsymbol{H}^3$ bounded by $I(A)$. We use the following notation:

$c(A) =$ the center of $I(A)$,
$D(A) =$ the disk in $\boldsymbol{C}$ bounded by $I(A)$,
$E(A) = \text{cl}(\boldsymbol{C} - D(A))$,
$Dh(A) =$ the closed half-space in $\boldsymbol{H}^3$ bounded by $Ih(A)$ whose closure contains $c(A)$.
$Eh(A) = \text{cl}(\boldsymbol{H}^3 - Dh(A))$.

The symbols $\bar{I}h(A)$, $\bar{D}h(A)$ and $\bar{E}h(A)$, respectively, denote the closure of $Ih(A)$, $Dh(A)$ and $Eh(A)$ in the closure $\bar{\boldsymbol{H}}^3 = \boldsymbol{H}^3 \cup \boldsymbol{C}$ of the hyperbolic space $\boldsymbol{H}^3$.

Definition 5.1. (Extended Ford domain I) Let ρ be an element of $\mathcal{X}$, such that $\text{Im}(\rho)$ is discrete. The *extended Ford domain* of ρ, denoted by $Ph(\rho)$, is defined to be the common exterior of the isometric hemi-spheres of the elements of $\text{Im}(\rho)$ which do not fix ∞, that is,

$$Ph(\rho) = \bigcap\{Eh(\rho(X)) \mid X \in \pi_1(\mathcal{O}) - \text{Stab}_\rho(\infty)\},$$

where $\text{Stab}_\rho(\infty) = \{X \in \pi_1(\mathcal{O}) \mid \rho(X)(\infty) \neq \infty\}$.

Note that the intersection of $Ph(\rho)$ with a vertical fundamental polyhedron of the discrete subgroup $\text{Stab}_\rho(\infty)$ is a fundamental polyhedron of $\text{Im}(\rho)$. Here, a *vertical polyhedron* is a polyhedron of the form $F \times \boldsymbol{R}_+ \subset \boldsymbol{C} \times \boldsymbol{R}_+ = \boldsymbol{H}^3$ for some polygon F in $\boldsymbol{C}$.

Even if ρ is not discrete, we can define an analogue of the Ford domain provided that ρ is the holonomy representation of a hyperbolic cone-manifold. (This means that the there is a "morphism" f from the topological orbifold $\mathcal{O}$ into a hyperbolic cone-manifold (M, Σ), such that $f_* : \pi_1(\mathcal{O} - \{\text{the three cone points}\}) \to \pi_1(M - \Sigma)$ is an epimorphism and that $h \circ f = \rho \circ j$, where $h : \pi_1(M - \Sigma) \to PSL(2, \boldsymbol{C})$ is the holonomy representation and j is the natural homomorphism $\pi_1(\mathcal{O} - \{\text{the three cone points}\}) \to \pi_1(\mathcal{O})$.)

Definition 5.2. (Extended Ford domain II) Let ρ be an element of $\mathcal{X}$. *An extended Ford domain* $Ph(\rho)$ of ρ means a $\langle\rho(K)\rangle$-invariant polyhedron in $\boldsymbol{H}^3$ bounded by the isometric hemispheres of a family of elements of $\text{Im}(\rho)$, such that the intersection of which with a vertical polyhedron is a fundamental polyhedron of the hyperbolic cone-manifold.

To describe the Ford domains, we introduce the following notations.

Definition 5.3. Let $(\rho; \Sigma)$ be a marked representation, such that $\phi^{-1}(0) \cap \Sigma^{(0)} = \emptyset$. Then $Eh(\rho; \Sigma)$ denotes the region in $\boldsymbol{H}^3$ and $\boldsymbol{C}$ defined by the following formula:

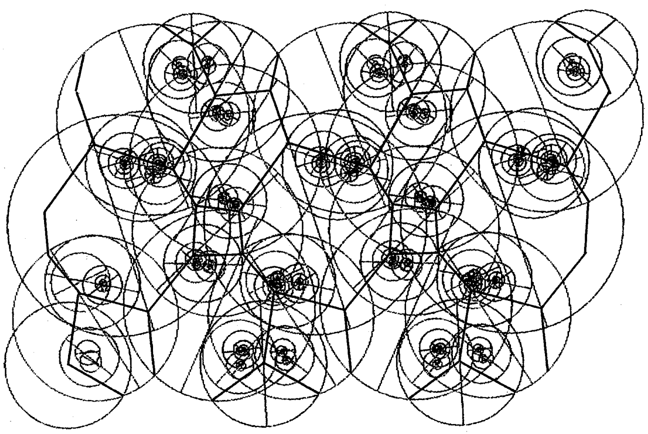
$$Eh(\rho; \Sigma) := \bigcap\{Eh(\rho(P)) \mid P \in \mathcal{EG}(\Sigma)\}$$

The definition of $Eh(\rho;\Sigma)$ is generalized as follows. Let $(\rho;\Sigma)$ be a marked representation, such that $\phi^{-1}(0) \cap (\Sigma^{(0)} - \{s^-, s^+\}) = \emptyset$. Let P be an elliptic generator of slope s^ϵ. Then $\rho(P)$ is the π-rotation about a vertical geodesic. The *isometric circle* $I(\rho(P))$ is defined to be the $\rho(K)$-invariant line in $\boldsymbol{C}$ passing through the (unique) fixed point of $\rho(P)$ in $\boldsymbol{C}$. The *isometric hemisphere* $Ih(\rho(P))$ is defined to be the vertical hyper-plane of $\boldsymbol{H}^3$ above $I(\rho(P))$. $Eh(\rho(P))$ is defined to be the closed half-space of $\boldsymbol{H}^3$ bounded by $Ih(\rho(P))$ which lies on the $-(\epsilon)$-side of $Ih(\rho(P))$ with respect to the imaginal coordinate. Then $Eh(\rho;\Sigma)$ is defined by the formula in Definition 5.3.

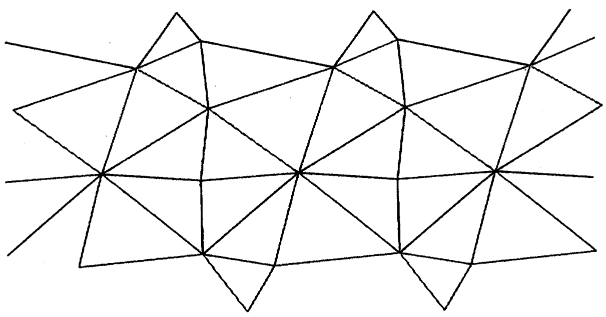
The following theorem gives a rough exposition of the main result of our joint work:

Theorem 5.4. *For any representation ρ belonging to one of the families listed in the introduction, there is a chain Σ, such that $Ph(\rho) = Eh(\rho;\Sigma)$. Furthermore, the combinatorial structure of $Ph(\rho)$ is determined by Σ.*

See Figure 5.1 for a typical example of the Ford domain of a quasi-Fuchsian representation, where the second figure illustrates the "geometric dual" to the Ford domain. This figure is created by the program Opti [23], made by the third author. We strongly recommend the readers to play with the program Opti [23], made by the third author: it is the best way to understand the contents of this article. For the explicit meaning of the "dual" and the full statement of the result, please see [4].



$Ph(\rho)$



$\mathcal{L}(\rho;\Sigma)$

Fig.5.1

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