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An example of Kleinian groups with indiscrete horoball heights

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1 Introduction

Let $M$ be a hyperbolic 3-manifold with a cusp, i.e., $M$ is the quotient of the hyperbolic 3-space $\mathbb{H}^3$ by a torsion-free Kleinian group $\Gamma$ with a parabolic element. For simplicity, we suppose that $M$ contains precisely one cusp, i.e., all parabolic fixed points of $\Gamma$ are equivalent with respect to the action of $\Gamma$. We shall identify $\mathbb{H}^3$ with the upper half of the Euclidean 3-space $\mathbb{E}^3$ so that $\infty$ becomes a parabolic fixed point of $\Gamma$. Let $C$ be the maximal cusp of $M$. The horoball pattern, $\mathcal{H}(M)$, of $M$ is the set of horoballs in $\mathbb{H}^3$ which project onto $C$ and the centers are distinct from $\infty$. Let $h : \mathbb{H}^3 \rightarrow \mathbb{R}_+$ be the height function defined by using the coordinate of $\mathbb{E}^3$. Then the discreteness of $h(\mathcal{H}(M)) \subset \mathbb{R}_+$ is an invariant of $M$.

Theorem 1.1. Suppose that $\Gamma$ is geometrically finite. Then $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$.

It is natural to expect that there exists a manifold $M$ such that $h(\mathcal{H}(M))$ is indiscrete in $\mathbb{R}_+$. The main result in this paper is the following theorem. For any quasi-Fuchsian group of the once-punctured torus, $\Gamma$, we can define the end invariant $\lambda(\Gamma) = (\lambda^{-}(\Gamma), \lambda^{+}(\Gamma)) \in \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$, where $\Delta$ is the diagonal of $\partial \mathbb{H}^2 \times \partial \mathbb{H}^2$. It is proved in [7] that $\lambda$ is a bijective map from the closure of the quasi-Fuchsian space of the once-punctured torus to $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ and that $\lambda^{-1}$ is continuous.

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Theorem 1.2. Let $\lambda_\infty$ be the real number which has the expansion into the continued fraction

$$\lambda_\infty = [2, 3, 4, \ldots] = \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \ddots}}}.$$ 

Put $\Gamma_\zeta = \lambda^{-1}(\lambda_\infty, \zeta)$ and $M_\zeta = \mathbb{H}^3/\Gamma_\zeta$ for any $\zeta \in \mathbb{H}^2$. Then $h(\mathcal{H}(M_\zeta))$ is indiscrete in $\mathbb{R}_+$. 

2 Horoball pattern

Let $M$ be a hyperbolic 3-manifold with a single cusp. Let $\Pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ be the universal covering. The maximal cusp of $M$ is defined as follows: Let $v$ be a parabolic fixed point of $\Gamma$ and $\Gamma_v$ the stabilizer of $v$ in $\Gamma$. Then $\Gamma_v$ consists of the parabolic elements in $\Gamma$ which stabilizes $v$. There exists a horoball $H$ centered at $v$ such that the quotient $H/\Gamma_v$ is embedded in $M$. The set $H/\Gamma_v \subset M$ is called a cusp of $M$. If we gradually expand $H$ then $H/\Gamma_v$ eventually has a self-intersection in $M$. The maximal cusp is the subset $H/\Gamma_v$ of $M$ with this maximal size. Let $\mathcal{H}(M)$ be the set of horoballs in $\mathbb{H}^3$ which project onto the maximal cusp and the centers are distinct from $v$.

We shall identify $\mathbb{H}^3$ with the upper half of $\mathbb{E}^3$, i.e., $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{E}^3 \mid z > 0\}$, so that $v$ is identified with $\infty$. (Note that $\partial \mathbb{H}^3$ is identified with $\mathbb{C} \cup \{\infty\}$.) For a point $(x, y, z) \in \mathbb{H}^3$, we define $h(x, y, z) = z$.

Definition 2.1. For a set $X \subset \mathbb{H}^3$, the Euclidean height $h(X)$ of $X$ is defined by

$$h(X) = \sup\{h(x) \mid x \in X\}.$$ 

We remark that the discreteness of $h(\mathcal{H}(M)) \subset M$ is independent of the choice of a parabolic fixed point $v$ and an identification of $\mathbb{H}^3$ with the upper half space.

In the following, we prove a stronger version of Theorem 1.1 (Theorem 2.3).

Definition 2.2. (1) The rank of a parabolic fixed point $v$ of $\Gamma$ is the rank of an abelian group $\Gamma_v$.

(2) Suppose that the rank of $v$ is one. We say that $v$ is doubly cusped if there exist two open round disks in $\Omega(\Gamma)$ which are disjoint and stabilized by $\Gamma_v$, where $\Omega(\Gamma)$ denotes the domain of discontinuity of $\Gamma$. 

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(3) A parabolic fixed point of $\Gamma$ is said to be bounded if (i) it is of rank $2$ or (ii) it is of rank $1$ and doubly cusped.

**Theorem 2.3.** Suppose that $\infty$ is a bounded parabolic fixed point of $\Gamma$. Then $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$.

We remark that Theorem 1.1 follows immediately from Theorem 2.3 and Proposition 2.4 below. (See [6, Chapter VI, Proposition A.10] for example.)

**Proposition 2.4.** Suppose that $\Gamma$ is geometrically finite. Then any parabolic fixed point of $\Gamma$ is bounded.

**Proof of Theorem 2.3.** Since $\infty$ is bounded, there exists a compact subset $K$ of $\mathbb{C}$ with the following property: For any $w \in \Lambda(\Gamma) - \{\infty\}$, there exists $\gamma \in \Gamma_\infty$ such that $\gamma w \in K$, where $\Lambda(\Gamma)$ denotes the limit set of $\Gamma$. Suppose that $h(\mathcal{H}(M))$ is indiscrete in $\mathbb{R}_+$. Note that $\gamma H \in \mathcal{H}(M)$ for any $\gamma \in \Gamma$ and $H \in \mathcal{H}(M)$ and that each element of $\Gamma_\infty$ keeps the Euclidean heights of horoballs as it is a Euclidean parallel translation of the upper half space. Thus there exists a sequence of horoballs $\{H_n\} \subset \mathcal{H}(M)$ such that the sequence $\{h(H_n)\}$ converges to some point $h_\infty \in \mathbb{R}_+$, $h(H_n) \neq h_\infty$ for any $n \in \mathbb{N}$ and that the centers of $H_n$ ($n \in \mathbb{N}$) are contained in $K$. By taking a subsequence, which we denote by the same symbol, we may assume that the horoballs $H_n$ ($n \in \mathbb{N}$) are distinct from one another. Then, from the definition, they are mutually disjoint in the interior. Since $\{h(H_n)\}$ converges to $h_\infty \in \mathbb{R}_+$, there exist two positive numbers $h_+$ and $h_-$ such that $h_- \leq h(H_n) \leq h_+$ for any $n \in \mathbb{N}$. Thus the Euclidean volume of each $H_n$ ($n \in \mathbb{N}$) is bounded below by a positive number. On the other hand, each $H_n$ is contained in the set $\{(x, y, z) \mid (x, y) \in B(K, h_+/2), z \leq h_+\}$ whose Euclidean volume is equal to $\text{Area}(B(K, h_+/2))h_+ < \infty$, where $B(K, h_+/2)$ denotes the $(h_+/2)$-neighborhood of $K$. This is a contradiction. \hfill $\square$

### 3 Punctured torus groups

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{C})$ with $c \neq 0$, the isometric hemisphere $Ih(\gamma)$ of $\gamma$ is the Euclidean hemisphere with equator $\{z \in \mathbb{C} \mid |cz + d| = 1\}$. For a Kleinian group $\Gamma$, let $\mathcal{I}(\Gamma)$ be the set of isometric hemispheres defined by

$$\mathcal{I}(\Gamma) = \{Ih(\gamma) \mid \gamma \in \Gamma, \gamma(\infty) \neq \infty\}.$$ 

In this section, we study the Euclidean heights of isometric hemispheres which support faces of the Ford domain of a once-punctured torus group. This can be used to prove Theorem 1.2 by the following lemma.
**Lemma 3.1.** For a hyperbolic 3-manifold with a single cusp $M = \mathbb{H}^3/\Gamma$, $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$ if and only if $h(\mathcal{I}(\Gamma))$ is discrete in $\mathbb{R}_+$.

*Proof.* Let $H$ be a horoball in $\mathbb{H}^3$ which projects onto the maximal cusp. We can see that $h(\gamma H) = 1/(|c|^2 h(\partial H))$ for any $\gamma \in \Gamma$ with $\gamma(\infty) \neq \infty$. Thus we have

$$h(\mathcal{H}(M)) = \{1/(|c|^2 h(\partial H)) | \gamma \in \Gamma, \gamma(\infty) \neq \infty\}.$$  

On the other hand, from the definition, we have

$$h(\mathcal{I}(\Gamma)) = \{1/|c| | \gamma \in \Gamma, \gamma(\infty) \neq \infty\}.$$  

Thus it is obvious that $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$ if and only if $h(\mathcal{I}(\Gamma))$ is discrete in $\mathbb{R}_+$. \qed

Let $T$ be the once-punctured torus and $\rho_0 : \pi_1(T) \to PSL(2, \mathbb{R}) \subset PSL(2, \mathbb{C})$ its Fuchsian representation. The *quasi-Fuchsian space* $QF$ of the once-punctured torus is the set of quasi-conformal deformations of $\rho_0$ quotiented by the conjugation in $PSL(2, \mathbb{C})$ and equipped with the algebraic topology. We denote the closure of $QF$ in the representation space of $\pi_1(T)$ by $\overline{QF}$. In this paper, we loosely identify an element of $\overline{QF}$ and its image in $PSL(2, \mathbb{C})$.

For any $\Gamma \in QF$, $\mathbb{H}^3/\Gamma$ is homeomorphic to $T \times (-1,1)$ and hence has two ends $\mathcal{E}^\pm$. We can associate an end invariant $\lambda(\Gamma) = (\lambda^- (\Gamma), \lambda^+ (\Gamma))$ with $\Gamma$ as follows:

1. If the end $\mathcal{E}^\epsilon$ is geometrically finite, then $\lambda^\epsilon (\Gamma)$ is the marked conformal structure of the Riemann surface at infinity.

2. If the end $\mathcal{E}^\epsilon$ is geometrically infinite, then $\lambda^\epsilon (\Gamma)$ is the ending lamination of the end.

Then each $\lambda^\pm (\Gamma)$ is defined as the point in the closure of the Teichmüller space of $T$, which is isomorphic to $\overline{\mathbb{H}^2}$.

**Theorem 3.2 ([7]).** $\lambda : \overline{QF} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ is bijective and $\lambda^{-1}$ is continuous.

To prove Theorem 1.2, it is convenient to study the representations of $\pi_1(T)$. (See [4] and [2, 3] for detail.) The once-punctured torus $T$ has the symmetry $\tau$ depicted in Figure 1. Let $\mathcal{O}$ be the quotient of $T$ by $\langle \tau \rangle$, which is the orbifold $S^2(\infty, 2, 2, 2)$. Let $p : T \to T/\langle \tau \rangle = \mathcal{O}$ be the covering projection.

By the following proposition, we can study the elements of $\overline{QF}$ by using a representation of $\pi_1^{orb}(\mathcal{O})$. In the rest of this paper, we regard $QF$ as a set of representations of $\pi_1^{orb}(\mathcal{O})$. 


Figure 1: Covering \( p : T \to \mathcal{O} \)

**Proposition 3.3.** For any \( \rho \in \overline{\mathcal{O}} \), there exists a unique representation \( \overline{\rho} : \pi_1^{orb}(\mathcal{O}) \to PSL(2, \mathbb{C}) \) such that \( \overline{\rho} \circ p_* = \rho \).

We can see that the fundamental group of \( \mathcal{O} \) has the following presentation:

\[
\pi_1^{orb}(\mathcal{O}) = \langle P_0, Q_0, R_0 | P_0^2 = Q_0^2 = R_0^2 = 1 \rangle,
\]
where each \( P_0, Q_0 \) and \( R_0 \) is represented by a loop which goes around a branch point. (See Figure 2.) Put \( K = R_0Q_0P_0 \). Then \( K \) is represented by a loop which goes around the puncture.

**Definition 3.4 (Elliptic generators).** (1) A triple \( (P, Q, R) \) of elements of \( \pi_1^{orb}(\mathcal{O}) \) is called an elliptic generator triple if the following conditions are satisfied:

(i) \( \pi_1^{orb}(\mathcal{O}) = \langle P, Q, R \rangle \).

(ii) \( P^2 = Q^2 = R^2 = 1 \) and \( RQP = K \).

(2) An element \( P \) of \( \pi_1^{orb}(\mathcal{O}) \) is said to be an elliptic generator if there exist \( Q, R \in \pi_1^{orb}(\mathcal{O}) \) such that \( (P, Q, R) \) is an elliptic generator triple.

**Remark 3.5.** For an elliptic generator triple \( (P, Q, R) \), put \( A = KP \) and \( B = K^{-1}R \). Then \( p_*(\pi_1(T)) = \langle A, B \rangle \) and \( ABA^{-1}B^{-1} = K^2 \).
Figure 2: Generators of $\pi_1^{orb}(O)$

Let $D^{(0)}$ be the isotopy classes of essential simple closed curves in $T$. Then $D^{(0)}$ can be identified with $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial \mathbb{H}^2$. Let $\sigma_0$ be the geodesic triangle in $\overline{\mathbb{H}^2}$ spanned by $\infty, 0, 1$, which we denote by $\langle \infty, 0, 1 \rangle$.

**Definition 3.6 (Modular diagram).** The modular diagram $\mathcal{D}$ is the simplicial complex defined by the triangulation $\{\gamma \sigma_0 | \gamma \in SL(2, \mathbb{Z})\}$ of $\mathbb{H}^2 \cup D^{(0)}$.

By the definition, the element $KP \in \pi_1(T)$ is represented by an essential simple closed curve $C$ in $T$ for any elliptic generator $P$. We denote the isotopy class of $C$ by $s(P)$, and call it the slope of $P$.

**Lemma 3.7.** (1) For elliptic generators $P$ and $P'$, $s(P) = s(P')$ if and only if $P' = K^nPK^{-n}$ for some $n \in \mathbb{Z}$.

(2) For any elliptic generator triple $(P, Q, R)$, the three points $s(P)$, $s(Q)$ and $s(R)$ span a triangle in $\mathcal{D}$.

(3) For any triangle $\sigma$ in $\mathcal{D}$, there exists an elliptic generator triple $(P, Q, R)$ such that $\sigma = \langle s(P), s(Q), s(R) \rangle$. 
Let \( \rho : \pi_1(T) \to PSL(2, \mathbb{C}) \) be a representation in \( \overline{QF} \). Then \( \rho \) lifts to a representation \( \hat{\rho} : \pi_1(T) \to SL(2, \mathbb{C}) \). We define the Markoff map \( \phi : D^{(0)} \to \mathbb{C} \) by \( \phi(s(P)) = \text{tr} \hat{\rho}(KP) \).

**Lemma 3.8.** (1) For any triangle \( \langle s_0, s_1, s_2 \rangle \) in \( D \),
\[
\phi(s_0)^2 + \phi(s_1)^2 + \phi(s_2)^2 = \phi(s_0)\phi(s_1)\phi(s_2).
\]

(2) For any different triangles \( \langle s_0, s_1, s_2 \rangle \) and \( \langle s_0, s_1, s'_2 \rangle \) in \( D \),
\[
\phi(s_2) + \phi(s'_2) = \phi(s_0)\phi(s_1).
\]

**Remark 3.9.** (1) By Lemma 3.8(2), a Markoff map is determined from the values at the vertices of a single triangle in \( D \).

(2) We can see that any Markoff map induces a unique representation of \( \pi_{1}^{orb}(\mathcal{O}) \) to \( PSL(2, \mathbb{C}) \).

In [5], Jorgensen studies the Ford domains of quasi-Fuchsian groups of the once-punctured torus. We can apply the argument to the boundary groups of quasi-Fuchsian space of once-punctured torus. (See [1] for an outline.) We can use several results obtained by this study. For the rest of this paper, we suppose that \( \rho(K) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) for any \( \rho \in \overline{QF} \).

**Lemma 3.10.** Let \( \rho \in \overline{QF} \). For any elliptic generator \( P \) with \( \rho(P)(\infty) \neq \infty \), \( h(Ih(\rho(P))) \) is equal to \( 1/|\phi(s(P))| \), where \( \phi \) is a Markoff map which induces \( \rho \).

**Definition 3.11.** Let \( \lambda_\infty \) be the real number which has the expansion into the continued fraction \( \lambda_\infty = [2, 3, \ldots] \). For \( \zeta \in \mathbb{H}^2 \), let \( \Gamma_\zeta = \lambda^{-1}(\lambda_\infty, \zeta) \) and \( \phi_\zeta \) be a Markoff map which induces \( \rho_\zeta \in \overline{QF} \) with \( \text{Im} \rho_\zeta = \Gamma_\zeta \). (See Figure 3\(^1\).)

Let \( s_n \) be the rational number which has the expansion into the continued fraction \( s_n = [2, 3, \ldots, n] \). Since any parabolic element in \( \Gamma_\zeta \) is the image of an element which is conjugate in \( \pi_{1}^{orb}(\mathcal{O}) \) into the cyclic group \( \langle K \rangle \), the following lemma holds.

**Lemma 3.12.** No \( \phi_\zeta(s_n) \ (n \in \mathbb{N}) \) is equal to \( \pm 2 \).

As a corollary to the characterization of the Ford domains of once-punctured torus groups, we have the following lemma (cf. [5, Lemma 5]).

\(^1\)This figure is drawn by using OPTI [8].
Lemma 3.13. There exists a subsequence of \( \{s_n\} \), which we denote by the same symbol, such that the sequence \( \{\phi_\zeta(s_n)\} \) converges to one of \( \pm 2 \).

Proof of Theorem 1.2. Let \( s'_n \) and \( s''_n \) (\( n \in \mathbb{N} \)) be the points in \( D^{(0)} \) depicted in Figure 4. Let \( x_n \) (resp. \( y_n, z_n \) and \( w_n \)) be the value of \( \phi_\zeta \) at \( s_n \) (resp. \( s_{n-1}, s'_n \) and \( s''_n \)). Then, by Lemma 3.13, there exists a subsequence of \( \{s_n\} \), which we denote by the same symbol, such that each \( \{x_n\} \) and \( \{y_n\} \) converges to one of \( \pm 2 \). We may suppose that both \( \{x_n\} \) and \( \{y_n\} \) converge to 2, if necessary, by changing \( \phi_\zeta \) to another Markoff map which induces \( \rho_\zeta \). Then, by Lemma 3.8,

\[
x_n^2 + y_n^2 + z_n^2 = x_n y_n z_n, \tag{3.1}
\]
\[
y_n + w_n = x_n z_n. \tag{3.2}
\]
Figure 4: Slopes $s_n$, $s'_n$ and $s''_n$

By (3.1), we have

$$z_n = \frac{x_n y_n \pm \sqrt{x_n^2 y_n^2 - 4(x_n^2 + y_n^2)}}{2}.$$  \hspace{1cm} (3.3)

Thus, by taking a subsequence, $\{z_n\}$ converges to $2(1 + \epsilon\sqrt{-1})$ ($\epsilon \in \{\pm 1\}$). Then, by (3.2), $\{w_n\}$ converges to $2(1 + 2\epsilon\sqrt{-1})$.

Suppose that $h(I(\Gamma_\zeta))$ is discrete in $\mathbb{R}_+$. Then, by Lemma 3.10, each $\{|x_n||n \in \mathbb{N}\}$, $\{|y_n||n \in \mathbb{N}\}$, $\{|z_n||n \in \mathbb{N}\}$ and $\{|w_n||n \in \mathbb{N}\}$ is a finite set. Hence both $|x_n|$ and $|y_n|$ are equal to 2, $|z_n|$ is equal to $|2(1 + \epsilon\sqrt{-1})| = 2\sqrt{2}$ and $|w_n|$ is equal to $|2(1 + 2\epsilon\sqrt{-1})| = 2\sqrt{5}$ for sufficiently large $n$. Put $x_n = 2e^{\theta_n \sqrt{-1}}$, $y_n = 2e^{\varphi_n \sqrt{-1}}$ and $z_n = 2\sqrt{2}e^{\psi_n \sqrt{-1}}$ for such $n$. Since both $\{x_n\}$ and $\{y_n\}$ converge to 2 and $\{z_n\}$ converges to $2(1 + \epsilon\sqrt{-1})$, both $\{\theta_n\}$ and $\{\varphi_n\}$ converge to 0 and $\{\psi_n\}$ converges to $\epsilon\pi/2$. By (3.2), we have $|x_n z_n - y_n| = |w_n|$. Thus

$$|4\sqrt{2}e^{(\theta_n + \varphi_n)\sqrt{-1}} - 2e^{\varphi_n \sqrt{-1}}| = 2\sqrt{5},$$

and hence $\varphi_n - \theta_n - \psi_n = \epsilon\pi/4$. Then, by (3.3),

$$2\sqrt{2}e^{(\varphi_n - \theta_n - \epsilon\pi/4)\sqrt{-1}} = 2e^{(\theta_n + \varphi_n)\sqrt{-1}} \left( 1 + \epsilon \sqrt{1 - (e^{-2\theta_n \sqrt{-1}} + e^{-2\varphi_n \sqrt{-1}})} \right),$$

and hence

$$e^{2\varphi_n \sqrt{-1}} = 2\epsilon \sqrt{-1} e^{-4\theta_n \sqrt{-1}} + (1 - 2\epsilon \sqrt{-1}) e^{-2\varphi_n \sqrt{-1}}. \hspace{1cm} (3.4)$$

Note that the absolute value of the left hand side of (3.4) is equal to 1. Put $f(\theta) = |2\sqrt{-1}e^{-4\theta \sqrt{-1}} + (1 - 2\sqrt{-1})e^{-2\varphi \sqrt{-1}}|$. Then $f'(0)$ is not equal to 0. Therefore $\theta_n$ is equal to 0 for sufficiently large $n$. This contradicts Lemma 3.12. \qed
References


