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Jørgensen groups of parabolic type

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ABSTRACT. This paper is a report without proofs on Jørgensen groups obtained recently. In this paper we consider Jørgensen groups of parabolic type. In particular, we consider four kinds of one-parameter families of Jørgensen groups. Here a Jørgensen group is a Kleinian group whose Jørgensen number is one.

0. Introduction.

It is an important problem to decide whether or not a non-elementary subgroup of the Möbius transformation group, which is denoted by Möb, is discrete. In 1976 Jørgensen [3] gave a necessary condition for a non-elementary Möbius transformation group $G=\langle A,B\rangle$ to be discrete : If $\langle A,B\rangle$ is a non-elementary discrete group, then

$$J(A,B):=|\mathrm{tr}^2(A)-4|+|\mathrm{tr}(ABA^{-1}B^{-1})-2|\geq 1.$$

The lower bound 1 is best possible.

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Let $\langle A, B\rangle$ be a marked two-generator subgroup of Möb. We call

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|$$

the *Jørgensen number* for $\langle A, B\rangle$. Let G be a two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for the group G is the infimum of $J(A, B)$ where A and B generate G. We call a non-elementary two-generator discrete subgroup G of Möb a *Jørgensen group* if $J(G) = 1$.

With respect to Jørgensen numbers it gives rise to the following problems:

(1) Problem 1 is to find all Jørgensen groups.

(2) Problem 2 is to find the infimum of Jørgensen numbers for some subspaces of the Kleinian space, for example for the Teichmüller space and for the Schottky space.

For Problem 2, Gilman [1] and Sato [7] gave the best lower bound of Jørgensen numbers for purely hyperbolic two-generator groups, and Sato [8], [9] gave the best lower bound of Jørgensen numbers for the classical Schottky space $R\mathbf{S}_2$ of real type of genus two. Namely,

$$\inf\{J(G) \mid G \in R\mathbf{S}_2\} = 4.$$

The family of groups, $P = \{G_\sigma = \langle A, B_{1/\sigma,\sigma}\rangle \mid G_\sigma \text{ is a discrete group}, \sigma \in \mathbf{C}\setminus\{0\}\}$ contains the Riley slice RS (see Keen and Series [5] for the definition of the Riley slice). If $\langle A, B_{1/\sigma,\sigma}\rangle$ is a group in P, then $J(A, B_{1/\sigma,\sigma}) = |\sigma|^2$. It is easily seen that $\inf\{J(G) \mid G \in P\} = 1$, since $J(A, B_{1/\sigma,\sigma}) = 1$ for $\sigma = 1$, that is, in this case the group is the classical modular group. Furthermore we easily see that

$$1 \leq \inf\{J(G) \mid G \in RS\} \leq 2.$$

As far as the author knows, the value of $\inf\{J(G) \mid G \in RS\}$.

For Problem 1, Jørgensen-Kiikka [3], Jørgensen-Lascurain-Pignataro [4], Sato [10] and Sato-Yamada [12] studied extreme discrete groups, that is, Jørgensen groups.

In particular Jørgensen-Kiikka [3] obtained the following theorem: Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A,B) = 1$, that is, a Jørgensen group. Then A is elliptic of order at least seven or A is parabolic.

In this paper we only consider the case where A is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$.

This paper contains six theorems. Theorems 2, 5 and 6 are new.In §1 we will state some definitions. In §2 theorems will be stated. The proofs of the theorems will appear elsewhere.

§1. Definitions

In this section we will state some definitons, for example a Jørgensen group. Let Möb denote the set of all Möbius transformations. In this paper we use a Kleinian group in the same meaning as a discrete group. A Kleinian group G is of the *first kind* if the limit set $\Lambda(G)$ of G is ll of the extended complex plane $\hat{\mathbf{C}}$ and it is of the *second kind* otherwise. A subgroup G of Möb is non-elementary grup if $\sharp\Lambda(G) \geq 3$.

In 1976 Jørgensen obtained the following important theorem called Jørgensen's inequality, which gives a necessary condition for a non-elementary Möbius transformtion group $G = \langle A, B\rangle$ to be discrete.

THEOREM A (Jørgensen [2]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

DEFINITION 1.1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ is

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 1.2. Let G be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for G is defined as follows:

$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}.$

DEFINITION 1.3. A non-elementary two-generator subgroup G of Möb is a *Jørgensen group* if G is a discrete group with $J(G) = 1$.

§2. Theorems.

In this section we will theorems without proofs. Jørgensen-Kiikka [3] obtained the following theorem for Jørgensen groups.

THEOREM B (Jørgensen-Kiikka [3]). *Let* $\langle A, B\rangle$ *be a non-elementary discrete group with* $J(A, B) = 1$, *that is, a Jørgensen group. Then* A *is elliptic of order at least seven or* A *is parabolic.*

In this paper we only consider the case where A is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$.

Let C_1 and C_2 be the following cylinders:

$C_1 = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$,

$C_2 = \{(\sigma, ik) \mid |\sigma| = 2,\ k \in \mathbf{R}\}$.

THEOREM 1 (Sato [10])

(i) *For each point inside the cylinder C_1, the corresponding group $G_{ik,\sigma}$ is not a Kleinian group.*

(ii) *Let (σ, ik) be a point outside of the cylinder C_2. If $|k| \geq 1$, then $G_{ik,\sigma}$ is a boundary group of the Schottky space of genus two.*

(iii) Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder C_1.

By Theorem 1 we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ with $\mu = ik$ ($k \in \mathbf{R}$) and $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$). For simplicity we set $B_{ik,\theta} := B_{ik,\sigma}$ and $G_{ik,\theta} = \langle A, B_{ik,\sigma}\rangle$ for $\sigma = -ie^{i\theta}$.

LEMMA 2.1. *Let $B_{ik,\theta}$ ($0 \leq \theta \leq \pi/2$) be as in the above, and let $\bar{B}_{ik,\theta}$ be the complex conjugate of $B_{ik,\theta}$. Then $B_{ik,\pi-\theta} = -\bar{B}_{ik,\theta}^{-1}$.*

We easily see the following by Lemma 2.1.

COROLLARY 2.2. *Let A and $B_{ik,\theta}$ be as in Lemma 3.1. Then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is discrete if and only if $G_{ik,\pi-\theta} = \langle A, B_{ik,\pi-\theta}\rangle$ is discrete for* $0 \leq \theta \leq \pi/2$.

LEMMA 2.3. *Let $B_{ik,\theta}$ and $G_{ik,\theta}$ be as in Lemma 3.1. Then $B_{ik,\pi+\theta} = B_{ik,\theta}$ and $G_{ik,\pi+\theta} = G_{ik,\theta}$.*

LEMMA 2.4. *A group $G_{ik,\theta}$ is a Kleinian group and so a Jørgensen group if and only if $G_{-ik,\theta}$ is a Kleinian group and so a Jørgensen group.*

This lemma follows from $B_{-ik,\theta} = B_{ik,\theta}^{-1}$. By Corollary 3.2, Lemmas 3.3 and 3.4, it suffices to consider the case of $(0 \leq \theta \leq \pi/2)$ and $k \geq 0$.

THEOREM 2 (Sato [11]). *Let $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ be the group generated by A and $B_{ik,\theta}$.*

(i) *If* $0 < \theta < \pi/6$ or $\pi/3 < \theta < \pi/2$, *then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is not a Kleinian group for every $k \in \mathbf{R}$.*

(ii) *If $|k| < 1/2$, then $G_{ik,\theta} = \langle A, B_{ik,\theta}\rangle$ is not a Kleinian group for every θ $(0 \leq \theta < 2\pi)$.*

THEOREM 3 (Sato-Yamada [12]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{ik,1} = \begin{pmatrix} ik & -(1+k^2) \\ 1 & ik \end{pmatrix}$$

and let $G_k = \langle A, B_k\rangle$ be the group generated by A and B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of $|k| > 1$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 2, 2, 3, 3)$ for each k, where $\Omega(G_k)$ denotes the region of discontinuity for G_k.*

(ii) *In the case of $|k| = 1$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 3, 3, \infty)$.*

(iii) *In the case of $\sqrt{3}/2 < |k| < 1$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a single Riemann surface with signature $(0; 3, 3, q)$ for k with $k^2 = \{1 + \cos(\pi/q)\}/2, q = 4, 5, 6, \cdots$.*

(iv) *In the case of $1/2 \leq |k| \leq \sqrt{3}/2$, G_k is a Kleinian group of the first kind and a Jørgensen group for $|k| = \sqrt{3}/2, \sqrt{2}/2$ or $1/2$. The volumes $V(G_{ik,1})$ of 3-orbifolds for $G_{ik,1}$ are as follows, where $L(\theta)$ is the Lobachevskiĭ function:*

$$L(\theta) = -\int_0^\theta \log|2\sin u| du.$$

(1) $V(G_{i\sqrt{3}/2,1}) = 5L(\pi/3)$.

(2) $V(G_{i\sqrt{2}/2,1}) = 2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}$.

(3) $V(G_{i/2,1}) = 7L(\pi/3)/2 - L(\varphi_0 + \pi/6) + L(\varphi_0 - \pi/6)$,

where $\varphi_0 = \sin^{-1}(1/2\sqrt{3})$.

(v) *In the case of* $0 < |k| < 1/2$, G_k *is not a Kleinian group for every* k.

(vi) *In the case of* $k = 0$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

REMARK. The group $G_{i/2,1}$ is conjugate to the Picard group in Möb and the group $G_{0,1}$ is the classical modular group.

THEOREM 4 (Sato [10]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_\theta := B_{\sqrt{3}i/2, -ie^{i\theta}} = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ *be the group generated by* A *and* B_θ $(0 \leq \theta \leq \pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta = \pi/6$, $G_{\pi/6}$ *has the following properties*:

(1) $G_{\pi/6}$ *is a Kleinian group of the first kind.*

(2) $G_{\pi/6}$ *is a Jørgensen group.*

(3) $V(G_{\pi/6}) = 6L(\pi/3)$, *where* $L(\theta)$ *is the Lobachevskiĭ function*:

$L(\theta) = -\int_0^\theta \log|2\sin u|du$.

(ii) *In the case of* $\theta = \pi/2$, $G_{\pi/2}$ *has the following properties*:

(1) $G_{\pi/2}$ *is a Kleinian group of the first kind.* (2) $G_{\pi/2}$ *is a Jørgensen group.*

(3) $V(G_{\pi/2}) = 2(L(\pi/6) + L(\pi/3))$.

(iii) *In the case of* $\theta = 0$, G_0 *has the following properties* :

(1) G_0 *is a Kleinian group of the second kind.*

(2) G_0 *is a Jørgensen group.*

(3) $\Omega(G_0)/G_0$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(iv) *If* $0 < \theta < \pi/6$ *or* $\pi/3 < \theta < \pi/2$, *then* G_θ *is not a Kleinian group.*

REMARKS. (1) enskip $G_{\pi/6}$ is congugate with the figure -eight knot group.

(2) Maskit [6] shows that the essentially same group as G_0 is discrete, that is, he shows that a group conjugate to G_π is discrete. Our proof is different from his.

THEOREM **5 (Sato [11]).** *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ -ie^{i\theta} & 0 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ *be the group generated by* A *and* B_θ $(0 \leq \theta \leq \pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta = 0, G_0$ *has the following properties:*

(1) G_0 *is a Kleinian group of the second kind.*

(2) $G_{\pi/2}$ *is a Jørgensen group.*

(3) $\Omega(G_0)/G_0$ *is a single Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of* $\theta = \pi/2$, $G_{\pi/2}$ *has the following properties:*

(1) $G_{\pi/2}$ *is a Kleinian group of the second kind.*

(2) G_0 *is a Jørgensen group.*

(3) $\Omega(G_{\pi/2})/G_{\pi/2}$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

(iii) *If* $0 < \theta < \pi/6, \pi/6 < \theta < \pi/4, \pi/4 < \theta < \pi/3$ *or* $\pi/3 < \theta < \pi/2$, *then* G_θ *is not a Kleinian group and so not a Jørgensen group.*

THEOREM **6 (Sato [11]).** *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{ik,-i} = \begin{pmatrix} k & i(k^2-1) \\ -i & k \end{pmatrix}$$

and let $G_k = \langle A, B_k \rangle$ *be the group generated by* A *and* B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $|k| > 1$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signatures* $(0; 2, 2, 2, 3)$ *and* $(0; , 2, 3, \infty)$ *for each* k, *where* $\Omega(G_k)$ *denotes the region of discontinuity for* G_k.

(ii) *In the case of* $|k| = 1$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

(iii) *In the case of* $\sqrt{3}/2 < |k| < 1$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is two Riemann surfaces with signatures* $(0; 2, 3, q)$ *and* $(0; 2, 3, \infty)$ *for* k *with* $k^2 = \{1 + \cos(\pi/q)\}/2, q = 4, 5, 6, \cdots$.

(iv) *In the case of* $1/2 \leq |k| \leq \sqrt{3}/2$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$ *for* $|k| = \sqrt{3}/2$.

(v) *In the case of* $0 < |k| < 1/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(vi) *In the case of* $k = 0$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

CORRECTION The part (from lines 23 through 31 on page 2 in the Introduction in the previous paper [10]) contains mistake, which gives no effect the paper. It should be changed as follows.

The family of groups, $P = \{G_\sigma = \langle A, B_{1/\sigma,\sigma} \rangle \mid G_\sigma$ is a discrete group, $\sigma \in \mathbf{C} \setminus \{0\}\}$ contains the Riley slice RS (see Keen and Series [14] for the definition of the Riley slice). If $\langle A, B_{1/\sigma,\sigma} \rangle$ is a group in P, then $J(A, B_{1/\sigma,\sigma}) = |\sigma|^2$. As far as the author knows, it is unknown whether or not $J(A, B_{1/\sigma,\sigma})$ achieves the infimum over the whole group P. It is easily seen that $\inf\{J(G) \mid G \in P\} = 1$, since

$J(A, B_{1/\sigma,\sigma}) = 1$ for $\sigma = 1$, that is, in this case the group is the classical modular group. Furthermore we easily see that

$$1 \leq \inf\{J(G) \mid G \in RS\} \leq 2.$$

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