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LOCAL GEOMETRIC FINITENESS OF KLEINIAN GROUPS

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A Kleinian group is, by definition, a group of orientation preserving isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$ that acts freely and properly discontinuously. We try to extend a criterion for handy finitely generated Kleinian groups, geometric finiteness, to infinitely generated cases and come up with the following concept of local geometric finiteness: A Kleinian group $\Gamma$ is defined to be locally geometrically finite if every finitely generated subgroup of $\Gamma$ is geometrically finite.

In this note, we consider several conditions from which the local geometric finiteness follows. Especially we regard the following theorem due to Thurston (see [5, Th.3.11]) as a motivation for considering such conditions geometrically and clarify the relationship with analytic conditions given by the Hausdorff dimension of the limit set.

Theorem 1. Let $G$ be a geometrically finite Kleinian group with the non-empty region of discontinuity (i.e. of the second kind). Then every finitely generated subgroup of $G$ is geometrically finite. Namely, $G$ is locally geometrically finite.

First of all, we review geometric finiteness of Kleinian groups. The convex hull $\tilde{C}_G$ of the limit set $\Lambda(G)$ is the smallest, convex, closed subset in $\mathbb{H}^3$ that contains all geodesic lines with the end points in $\Lambda(G)$. The convex core $C_G$ is a convex, closed subset of the hyperbolic 3-manifold $N_G = \mathbb{H}^3/G$ that is the image of $\tilde{C}_G$ under the projection $\mathbb{H}^3 \rightarrow N_G$. Let $x \in \Lambda(G)$ be a parabolic fixed point of $G$. We say that a horoball $B_x$ in $\mathbb{H}^3$ tangent at $x$ is a cusp horoball if $B_x$ is equivariant under the stabilizer of $x$ in $G$. The image of a cusp horoball under the projection $\mathbb{H}^3 \rightarrow N_G$ is called a cusp neighborhood. Then one of mutually equivalent characterizations of geometric finiteness for $G$ is that the convex core $C_G$ is compact except for cusp neighborhoods (see [5, Th.3.7]). Another characterization is that $\Lambda(G)$ is coincident with the conical limit set $\Lambda_c(G)$ up to parabolic fixed points.

In this note, we define a Kleinian group $G$ to be analytically finite if the relative boundary $\partial C_G$ of the convex core in $N_G$ is compact except for cusp neighborhoods. It is obvious that if $G$ is geometrically finite then it is analytically finite. Moreover, the Ahlfors finiteness theorem (see [5, Th.4.1]) asserts that every finitely generated Kleinian group is analytically finite.
The assumption of Theorem 1 that $G$ has the non-empty region of discontinuity is essential; this is necessary for the proof and there exists a counterexample for the statement if we drop it. This is equivalent to saying that $\partial C_G$ is not empty. However, assuming for $G$ to be geometrically finite is too restricted; in order to prove Theorem 1, we only use a property of the convex core of a geometrically finite Kleinian group, boundedness of the hyperbolic distance from its boundary. We formulate this weaker condition precisely as follows: A Kleinian group $G$ is, by definition, geometrically bounded if $\partial C_G \neq \emptyset$ and if

$$\sup \{d(\partial C_G, q) \mid q \in C_G - P_G\} < \infty$$

is satisfied for the union $P_G$ of some cusp neighborhoods, where $d(\cdot, \cdot)$ means the hyperbolic distance.

By the definitions above, we can easily see the following fact:

**Proposition 1.** A Kleinian group $G$ is both geometrically bounded and analytically finite if and only if $G$ is geometrically finite with the non-empty region of discontinuity.

Now we state the extension of Theorem 1 by using the geometric boundedness and exhibit a proof for it.

**Theorem 2.** If a Kleinian group $G$ is geometrically bounded then $G$ is locally geometrically finite.

**Proof.** We denote $C_G - P_G$ by $(C_G)_0$ and $\tilde{C}_G - \tilde{P}_G$ by $(\tilde{C}_G)_0$ where $\tilde{P}_G$ is the union of cusp horoballs that is the inverse image of $P_G$. By assumption, $(\tilde{C}_G)_0$ is within a bounded distance of $\partial \tilde{C}_G$.

Let $\Gamma$ be a finitely generated subgroup of $G$. We define $(C_{\Gamma})_0 = C_{\Gamma} - P_{\Gamma}$ and $(\tilde{C}_{\Gamma})_0 = \tilde{C}_{\Gamma} - \tilde{P}_{\Gamma}$ similarly for $\Gamma$, where a cusp horoball $B_x \subset \tilde{P}_\Gamma$ for a parabolic fixed point $x$ of $\Gamma$ is chosen so that it is coincident with the cusp horoball for $G$. Then $(\tilde{C}_{\Gamma})_0 \cap (\tilde{C}_G)_0$ is within a bounded distance of $\partial \tilde{C}_{\Gamma}$ because $\tilde{C}_{\Gamma} \subset \tilde{C}_G$.

Since $\Gamma$ is analytically finite by the Ahlfors finiteness theorem, we see that

$$(\partial \tilde{C}_{\Gamma} \cap (\tilde{C}_{\Gamma})_0 \cap \tilde{P}_G)/\Gamma$$

is relatively compact. Thus, replacing $\tilde{P}_G$ with smaller cusp horoballs if necessary, we may assume that $(\tilde{C}_{\Gamma})_0 \cap \tilde{P}_G = \emptyset$ and hence $(\tilde{C}_{\Gamma})_0 \cap (\tilde{C}_G)_0$ is coincident with $(\tilde{C}_{\Gamma})_0$. This implies that $(\tilde{C}_{\Gamma})_0$ is within a bounded distance of $\partial \tilde{C}_{\Gamma}$, namely, $\Gamma$ is geometrically bounded. Hence, by Proposition 1, $\Gamma$ is geometrically finite. \qed

Next we move on the Hausdorff dimension of the limit set. The geometric boundedness has a connection with an analytic condition via the following result [4].
Proposition 2. If a Kleinian group $G$ is geometrically bounded then the Hausdorff
dimension $\dim \Lambda(G)$ of the limit set is strictly less than 2.

The conclusion of Proposition 2 is still a sufficient condition for local geometric
finiteness; it can be easily seen from a famous result due to Bishop and Jones [1].

Theorem 3. If a Kleinian group $G$ satisfies $\dim \Lambda(G) < 2$ then $G$ is locally geo-
metrically finite.

Proof. Let $\Gamma$ be a finitely generated subgroup of $G$. Then

$$\dim \Lambda(\Gamma) \leq \dim \Lambda(G) < 2.$$ 

By the theorem of Bishop and Jones, $\dim \Lambda(\Gamma) < 2$ implies that $\Gamma$ is geometrically
finite. $\square$

Actually, we can prove a slightly stronger result than Theorem 3.

Theorem 3'. If an infinitely generated Kleinian group $G$ satisfies $\dim \Lambda(G) < 2$
then every finitely generated subgroup $\Gamma$ of $G$ satisfies the strict inequality

$$\dim \Lambda(\Gamma) < \dim \Lambda(G).$$

Proof. By Theorem 3, $\Gamma$ is geometrically finite. Then the critical exponent of the
Poincaré series for $\Gamma$ is equal to $\dim \Lambda(\Gamma)$ and the Poincaré series diverges at this
critical exponent. As is shown in [3], if $\Lambda(\Gamma)$ is a proper subset of $\Lambda(G)$, which is
always the case for finitely generated $\Gamma$ and infinitely generated $G$, then the strict
inequality on the Hausdorff dimension follows. $\square$

Finally we weaken the assumption of Theorem 3 slightly and prove that local geo-
metric finiteness follows even from this weaker assumption. This is a consequence
of the theorem of Bishop and Jones again.

Theorem 4. If a Kleinian group $G$ satisfies both that the Hausdorff dimension
of the conical limit set $\Lambda_c(G)$ is strictly less than 2 and that the 2-dimensional
Hausdorff measure $\mu_2$ of $\Lambda(G)$ is zero, then $G$ is locally geometrically finite.

Proof. Any subgroup $\Gamma$ of $G$ satisfies $\dim \Lambda_c(\Gamma) < 2$ and $\mu_2(\Lambda(\Gamma)) = 0$, too. By
the theorem of Bishop and Jones, if $\Gamma$ is finitely generated but not geometrically
finite then either $\dim \Lambda_c(\Gamma) = 2$ or $\mu_2(\Lambda(\Gamma)) > 0$. Hence we can see that every
finitely generated subgroup $\Gamma$ is geometrically finite. $\square$

The assumption of Theorem 4 is by no means a sharp condition for local geo-
metric finiteness. In fact, we can construct the following examples:
Examples. Let $G$ be a Kleinian group of the second kind that is exhausted by a sequence of geometrically finite subgroups $\Gamma_n$ with $\dim \Lambda_c(\Gamma_n) \uparrow 2$. For instance, we can take such $G$ as a certain subgroup of a Kleinian group for an infinite cyclic cover of a closed hyperbolic manifold. Then $\dim \Lambda_c(G) = 2$, however $G$ is locally geometrically finite. On the other hand, we can construct an infinitely generated Schottky group $G$ of the second kind so that $\mu_2(\Lambda(G)) > 0$ (see [2, Chapter 8]). However, this $G$ is also locally geometrically finite. Moreover, combining these two examples, we can obtain a locally geometrically finite Kleinian group $G$ satisfying both $\dim \Lambda_c(G) = 2$ and $\mu_2(\Lambda(G)) > 0$.

Our next problem is to find an interesting necessary condition for local geometric finiteness.

References


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