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Integral Means of the Fractional Derivative for Certain Starlike and Convex Functions of order $\alpha$ (New Extension of Historical Theorems for Univalent Function Theory)

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Integral Means of the Fractional Derivative for Certain Starlike and Convex Functions of order $\alpha$

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Abstract

In this paper we study a subclass of analytic functions consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (\theta \text{ real, } a_k \geq 0; \, n \in N).$$

We show the integral means of the fractional derivative for starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$ belonging to the subclass.

1 Introduction

Denote by $A$ the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $U = \{z : z \in C, \, |z| < 1\}$, and by $A(n)$ the subclass of $A$ consisting of all functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0; \, n \in N = \{1, 2, 3, \cdots\}).$$

We denote by $T(n)$ the subclass of $A(n)$ of univalent functions in $U$, further by $T_{\alpha}(n)$ and $C_{\alpha}(n)$ the subclasses of $T(n)$ consisting of functions which are starlike of order $\alpha (0 \leq \alpha < 1)$ and convex of order $\alpha (0 \leq \alpha < 1)$, respectively. These subclasses $T(n)$, $T_{\alpha}(n)$ and $C_{\alpha}(n)$ were introduced by Chatterjea[1]. When $n = 1$ these notations are

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usually used as $T(1) = T, \quad T_{\alpha}(1) = T^{*}(\alpha)$ and $C_{\alpha}(1) = C(\alpha)$, which were introduced earlier by Silverman[7]. Chatterjea[1] showed that a function $f(z)$ of the form (1.1) is in $T_{\alpha}(n)$ if and only if $\sum_{k=n+1}^{\infty}(k - \alpha)a_{k} \leq 1 - \alpha$, and that a function $f(z)$ of the form (1.1) is in $C_{\alpha}(n)$ if and only if $\sum_{k=n+1}^{\infty}k(k - \alpha)a_{k} \leq 1 - \alpha$. In the case of $n = 1$ these results coincide with Theorem 2 and Corollary 2 of Silverman[7], respectively.

Denote by $A(n, \theta)$ the subclass of $A$ consisting of all functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_{k}z^{k} \quad (\theta \text{ real}, \quad a_{k} \geq 0; \quad n \in N)$$

(see, Sekine and Owa[6]).

We note that $A(n, 0) = A(n)$. We define the subclasses $T(n, \theta)$, $T_{\alpha}(n, \theta)$ and $C_{\alpha}(n, \theta)$ of $A(n, \theta)$ by the same way as those for the subclasses $T(n)$, $T_{\alpha}(n)$ and $C_{\alpha}(n)$ of $A(n)$, respectively. Then it is clear that $T(n, 0) = T(n), \quad T_{\alpha}(n, 0) = T_{\alpha}(n)$ and $C_{\alpha}(n, 0) = C_{\alpha}(n)$.

Sekine and Owa[6] proved that a function $f(z)$ in $A(n, \theta)$ is in $T_{\alpha}(n, \theta)$ if and only if

$$\sum_{k=n+1}^{\infty}(k - \alpha)a_{k} \leq 1 - \alpha \quad \text{(1.2)}$$

and that a function $f(z)$ in $A(n, \theta)$ is in $C_{\alpha}(n, \theta)$ if and only if

$$\sum_{k=n+1}^{\infty}k(k - \alpha)a_{k} \leq 1 - \alpha. \quad \text{(1.3)}$$

We note that the coefficient inequalities (1.2) and (1.3) do not contain $\theta$ and coincide with the coefficient inequalities for $T_{\alpha}(n)$ and $C_{\alpha}(n)$ of Chatterjea[1], respectively.

We have the following results needed later. Since the proofs are similar to those in [5], we omit the proofs(see, [5]).

**Theorem 1.1** The extremal points of $T_{\alpha}(n, \theta)$ are functions

$$f_{1}(z) = z \quad \text{and} \quad f_{k}(z) = z - e^{i(k-1)\theta} \frac{1 - \alpha}{k - \alpha}z^{k} \quad (k \geq n + 1). \quad \text{(1.4)}$$

**Theorem 1.2** The extremal points of $C_{\alpha}(n, \theta)$ are functions

$$f_{1}(z) = z \quad \text{and} \quad f_{k}(z) = z - e^{i(k-1)\theta} \frac{1 - \alpha}{k(k - \alpha)}z^{k} \quad (k \geq n + 1). \quad \text{(1.5)}$$

### 2 Fractional derivative and Subordination

In this section we recall the concepts of fractional derivative and subordination. Further we give several known results needed later.
Definition 2.1 ([4]) The fractional derivative of order \( \lambda \) is defined by
\[
D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad (0 \leq \lambda < 1),
\]
where \( f(z) \) is an analytic function in a simple connected region of the \( z \)-plane containing the origin and the many-values of \( (z-\xi)^{-\lambda} \) is removed by requiring \( \log(z-\xi) \) to be real when \( z-\xi > 0 \).

Remark 2.1
\[
(2.1) \quad D_z^\lambda z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\lambda)} z^{m-\lambda} \quad (m \in \mathbb{N}),
\]
where \( 0 \leq \lambda < 1 \).

For analytic functions \( g(z) \) and \( h(z) \) in \( U \) with \( g(0) = h(0) \), \( g(z) \) is said to be subordinate to \( h(z) \) if exists an analytic function \( w(z) \) so that \( w(0) = 0 \), \( |w(z)| < 1 \) (\( z \in U \)) and \( g(z) = h(w(z)) \), we denote this subordination by \( g(z) \prec h(z) \).

In 1925, Littlewood[3] proved the following subordination theorem.

Theorem 2.1 ([3]) If \( g \) and \( f \) are analytic in \( U \) with \( g \prec f \), then for \( \lambda > 0 \) and \( 0 < r < 1 \),
\[
\int_0^{2\pi} |g(re^{i\theta})|^{\lambda} d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^{\lambda} d\theta.
\]

Making use of Theorem 2.1, Silverman[8] proved the following integral means for univalent function with negative coefficients.

Theorem 2.2 ([8]) Suppose \( f(z) \in T, \lambda > 0 \), and \( f_2(z) = z - z^2/2 \). Then for \( z = re^{i\theta}, 0 < r < 1 \),
\[
\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta.
\]

Further, Kim and Choi[2] showed the integral means of the fractional derivative for \( T \), \( C \), \( T^*(\alpha) \) and \( C(\alpha) \). In this paper, we show the integral means of the fractional derivative of order \( \lambda \) for the functions belonging to \( T^*_{\alpha}(n; \vartheta) \) and \( C_{\alpha}(n; \vartheta) \).

3 Results

Theorem 3.1 Suppose \( f(z) \in T^*_{\alpha}(n; \vartheta), \beta > 0 \), and \( f_{n+1}(z) \) is defined by (1.4). Then for \( z = re^{i\theta} \) and \( 0 < r < 1 \),
\[
\int_0^{2\pi} |D_z^\lambda f(z)|^\beta d\theta \leq \int_0^{2\pi} |D_z^\lambda f_{n+1}(z)|^\beta d\theta \quad (0 \leq \lambda < 1).
\]
Proof. If \( f(z) \in T_{a}^{*}(n; \theta) \), then we have \( f(z) = \sum_{k=0}^{\infty} e^{i(k-1)\theta} a_k z^k \) \((a_k \geq 0)\). By Remark 2.1 for the function \( f(z) \), we have

\[
D_{z}^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left( 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k \Phi(k) z^{k-1} \right),
\]

where

\[
\Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq n+1).
\]

Since \( \Phi(k) \) is a non-increasing function of \( k \), it follows that

\[
0 < \Phi(k) \leq \Phi(n+1) = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}.
\]

On the other hand, for the function

\[
f_{n+1}(z) = z - e^{in\theta} \frac{1-\alpha}{n+1-\alpha} z^{n+1},
\]

we have

\[
D_{z}^\lambda f_{n+1}(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left( 1 - \frac{e^{in\theta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n \right).
\]

To prove this theorem we must show that

\[
\int_{0}^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k \Phi(k) z^{k-1} \right|^\beta d\theta \leq \int_{0}^{2\pi} \left| 1 - \frac{e^{in\theta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n \right|^\beta d\theta.
\]

Since

\[
\int_{0}^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k \Phi(k) z^{k-1} \right|^\beta d\theta \leq \int_{0}^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} (k-\alpha) a_k \Phi(k) z^{k-1} \right|^\beta d\theta,
\]

by virtue of Theorem 2.1, it suffices to show that

\[
(3.1) \quad 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} (k-\alpha) a_k \Phi(k) z^{k-1} < 1 - \frac{e^{in\theta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n.
\]

If we put

\[
1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} (k-\alpha) a_k \Phi(k) z^{k-1} = 1 - \frac{e^{in\theta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} (w(z))^n,
\]

then we have

\[
(w(z))^n = \frac{(n+1-\alpha)\Gamma(n+2-\lambda)}{e^{in\theta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)} \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} (k-\alpha) a_k \Phi(k) z^{k-1}.
\]
Therefore we have

\[
|w(z)|^n \leq \frac{(n + 1 - \alpha)\Gamma(n + 2 - \lambda)}{(1 - \alpha)\Gamma(2 - \lambda)\Gamma(n + 2)} \frac{\Phi(n + 1)}{(n + 2)} \sum_{k=n+1}^{\infty} (k - \alpha)a_k |z|^{k-1}
\]

\[
|w(z)|^n \leq \frac{(n + 1 - \alpha)\Gamma(n + 2 - \lambda)}{(1 - \alpha)\Gamma(2 - \lambda)\Gamma(n + 2)} \sum_{k=n+1}^{\infty} (k - \alpha)a_k |z|^{k-1}
\]

By applying the coefficient inequality (1.2) to the inequality above we have

\[
|w(z)|^n \leq |z| < 1,
\]

that is, \(|w(z)| < 1\). Therefore we have the subordination (3.1).

**Theorem 3.2** Suppose \(f(z) \in C_\alpha(n; \theta), \beta > 0\), and \(f_{n+1}(z)\) is defined by (1.5). Then for \(z = re^{i\theta}\) and \(0 < r < 1\),

\[
\int_0^{2\pi} |D_z^\lambda f(z)|^\beta d\theta \leq \int_0^{2\pi} |D_z^\lambda f_{n+1}(z)|^\beta d\theta \quad (0 \leq \lambda < 1).
\]

**Proof.** By the assumption, we note

\[
f_{n+1}(z) = z - e^{in\theta \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)}}.
\]

Also we note that

\[
(n + 1) \sum_{k=n+1}^{\infty} (k - \alpha)a_k \leq \sum_{k=n+1}^{\infty} k(k - \alpha)a_k \leq 1 - \alpha,
\]

that is,

\[
\sum_{k=n+1}^{\infty} \frac{k - \alpha}{1 - \alpha} \leq \frac{1}{n + 1}.
\]

By means of two notes above, we can prove this theorem by an argument similar to that in Theorem 3.1.

**References**


