INNER RADIUS OF UNIVALENCE FOR A STRONGLY STARLIKE DOMAIN (New Extension of Historical Theorems for Univalent Function Theory)

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Citation
数理解析研究所講究録 2000, 1164: 144-150

Issue Date
2000-07

URL
http://hdl.handle.net/2433/64296

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
INNER RADIUS OF UNIVALENCE FOR A STRONGLY STARLIKE DOMAIN

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Abstract. The inner radius of univalence of a domain $D$ with Poincaré density $\rho_D$ is the possible smallest number $\sigma$ such that the condition $\|S_f\|_D = \sup_{w \in D} \rho_D(w)^{-2} |S_f(w)| \leq \sigma$ implies the univalence of $f$ for a nonconstant meromorphic function $f$ on $D$, where $S_f$ is the Schwarzian derivative of $f$. In this note, we will give a lower estimate of the inner radius of univalence for strongly starlike domains of order $\alpha$ with a concrete bound in terms of the order $\alpha$.

1. Main result

For a constant $0 \leq \alpha \leq 1$, a holomorphic function $f$ on the unit disk is called strongly starlike of order $\alpha$ if $f$ satisfies the condition

$$(1) \quad \left| \arg \frac{zf'(z)}{f(z) - f(0)} \right| \leq \frac{\pi \alpha}{2} \quad (z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}).$$

Note that a strongly starlike function is starlike in the usual sense. Every strongly starlike function $f$ of order $\alpha < 1$ is bounded. In fact, Brannan and Kirwan [1] showed that

$$(2) \quad |f(z) - f(0)| \leq |zf'(0)|M(\alpha) \quad (z \in \mathbb{D}).$$

Here $M(\alpha)$ is defined by

$$(3) \quad M(\alpha) = \exp \left[ \int_0^1 \left\{ \left( \frac{1 + t}{1 - t} \right)^\alpha - 1 \right\} \frac{dt}{t} \right],$$

where $\Gamma$ is the Gamma function and $\gamma$ is the Euler constant.

A proper subdomain $D$ of the complex plane $\mathbb{C}$ is said to be strongly starlike of order $\alpha$ with respect to a point $w_0 \in D$ if $D$ is simply connected and if the Riemann mapping function $f : \mathbb{D} \to D$ with $f(0) = w_0$ is strongly starlike of order $\alpha$. A strongly starlike domain of order 1 is nothing but a usual starlike domain. In what follows, without any pain, we always assume that $w_0 = 0$.

1991 Mathematics Subject Classification. Primary 30C45; Secondary 30C62.
Key words and phrases. strongly starlike, logarithmic spiral.

The author was partially supported by the Ministry of Education, Grant-in-Aid for Encouragement of Young Scientists, 11740088.

This article will, hopefully, appear in some journal.
We now introduce a standard domain adapted to the strong starlikeness. For a constant \( \alpha \) with \( 0 < \alpha < 1 \), we denote by \( V_\alpha \) the bounded domain enclosed by the logarithmic spirals
\[
\gamma_\alpha = \{ \exp((\tan(\pi \alpha/2) + i)\theta); 0 \leq \theta \leq \pi \} \quad \text{and} \quad \bar{\gamma}_\alpha = \{ w; \bar{w} \in \gamma_\alpha \}.
\]
Let \( D \) be a proper subdomain of \( \mathbb{C} \) containing the origin. It will be convenient to consider the periodic function \( R = R_D : \mathbb{R} \to (0, +\infty) \) of period \( 2\pi \) defined by
\[
R(\theta) = \sup\{ r > 0; [0, re^{i\theta}] \subset D \},
\]
where \([a, b]\) denotes the closed line segment joining points \( a \) and \( b \) in \( \mathbb{C} \). Note that \( R \) is lower semi-continuous.

In the sequel, we will use the convention \( a \cdot D = \{ aw; w \in D \} \) for \( a \in \mathbb{C} \) and a domain \( D \). Also, set \( D'^{\prime} = I(\text{Ext} \, D) \), where \( \text{Ext} \, D = \hat{\mathbb{C}} \backslash \overline{D} \) and \( I(z) = 1/z \).

The next result will be fundamental for our aim here, whose proof can be found in [6].

**Theorem A.** Let \( D \) be a proper subdomain of \( \mathbb{C} \) with \( 0 \in D \) satisfying the condition \( \text{Int} \, \overline{D} = D \) and let \( \alpha \) be a constant with \( 0 < \alpha < 1 \). Then the following conditions are equivalent.

(a) \( D \) is strongly starlike of order \( \alpha \) with respect to the origin.
(b) \( D'^{\prime} \) is strongly starlike of order \( \alpha \) with respect to the origin.
(c) For each point \( w \in D \) we have \( w \cdot V_\alpha \subset D \).
(d) The radius function \( R = R_D \) is absolutely continuous and satisfies \( |R'/R| \leq \tan(\pi \alpha/2) \) a.e. in \( \mathbb{R} \).

**Remark.** The implication (a)\( \Rightarrow \) (d) is essentially due to Fait, Krzyż and Zygmunt [2]. Actually, we will employ their idea which was used to show the quasiconformal extendability of strongly starlike functions.

Let \( D \) be a subdomain of \( \mathbb{C} \) with the hyperbolic metric \( \rho_D(z)|dz| \) of constant curvature \(-4\). The inner radius of univalence of \( D \), which will be denoted by \( \sigma(D) \), is the possible maximal number \( \sigma \) for which the condition \( \|S_f\|_D \leq \sigma \) implies the univalence of the nonconstant meromorphic function \( f \) on \( D \), where \( S_f \) denotes the Schwarzian derivative \( (f''/f')' - (f''/f')^2/2 \) of \( f \) and \( \| \varphi \|_D = \sup_{w \in D} \rho_D(w)^{-2} |\varphi(w)| \). Note that \( \sigma(D) \) is Möbius invariant in the sense that \( \sigma(L(D)) = \sigma(D) \) for a Möbius transformation \( L \). In particular, \( \sigma(D'^{\prime}) = \sigma(\text{Ext} \, D) \). The reader may consult the textbook [4] by Lehto as a general reference for the inner radius of univalence and related notions. When \( D \) is simply connected, theorems of Ahlfors and Gehring imply that \( \sigma(D) > 0 \) if and only if \( D \) is a quasidisk, furthermore, \( \sigma(D) \) is estimated from below by a positive constant \( c(K) \) depending only on \( K \) for a \( K \)-quasidisk \( D \). However, it is hard to give an explicit lower estimate of \( \sigma(D) \) for a concrete quasidisk \( D \) in general. Our second result concerns the inner radius of univalence of strongly starlike domains.

**Theorem 1.** A strongly starlike domain \( D \) of order \( \alpha \) satisfies
\[
\sigma(D) \geq \frac{2}{M(\alpha)^2} \cdot \frac{\cos(\pi \alpha/2)}{1 + \sin(\pi \alpha/2)},
\]
where \( M(\alpha) \) is defined by (3).
Remarks. 1. When \( \alpha \) tends to 0, the right-hand side above tends to 2. On the other hand, it is known that \( \sigma(D) = 2 \). See also the final section.

2. By a result of Fait, Krzyż and Zygmunt [2], we know that a strongly starlike domain \( D \) of order \( \alpha \) is a \( K(\alpha) \)-quasidisk, where \( (K(\alpha) - 1)/(K(\alpha) + 1) = \sin(\pi \alpha/2) \). Hence, as a corollary, we have \( \sigma(D) \geq c(K(\alpha)) \). So, the novelty of this theorem lies in the explicitness of the estimate.

3. From Theorem 1 we observe that \( D^v \) is strongly starlike of order \( \alpha \) under the assumption of Theorem 1. Hence, we obtain \( \sigma(\text{Ext} D) \geq 2 \cos(\pi \alpha/2)/(1 + \sin(\pi \alpha/2))M(\alpha)^2 \) simultaneously. We also note that the standard domain \( V_\alpha \) has the property \( \sigma(V_\alpha) = \sigma(\text{Ext} V_\alpha) \).

2. Mapping function of \( V_\alpha \)

Let \( S \) denote the set of holomorphic univalent functions on the unit disk \( \mathbb{D} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \). For \( 0 \leq \alpha \leq 1 \), we define the function \( k_\alpha \) in the class \( S \) by the relation

\[
zk_\alpha'(z)/k_\alpha(z) = \left(\frac{1+z}{1-z}\right)^\alpha
\]

on \( \mathbb{D} \). More explicitly, \( k_\alpha \) can be expressed by

\[
k_\alpha(z) = z \exp \left[ \int_0^z \left( \frac{1+\zeta}{1-\zeta} \right)^\alpha - 1 \right] \frac{d\zeta}{\zeta}
\]

This function is known to play a role of the usual Koebe function in the class of normalized strongly starlike functions of order \( \alpha \) in many cases. Actually \( k_1 \) is nothing but the Koebe function.

Noting \( k_\alpha(1) = M(\alpha) \), we consider the function

\[
g_\alpha(z) = k_\alpha(z)/M(\alpha) = \exp \left[ \int_1^z \left( \frac{1+\zeta}{1-\zeta} \right)^\alpha \frac{d\zeta}{\zeta} \right]
\]

The following fact is useful to note. Although this result was stated in [6], we give a direct proof here.

Lemma 1. \( g_\alpha(\mathbb{D}) = V_\alpha \) for \( 0 < \alpha < 1 \).

Proof. If we set \( g_\alpha(e^{it}) = r(t)e^{i\Theta(t)} = R(\theta)e^{i\theta}, \) then we have \( e^{it}g_\alpha'(e^{it})/g_\alpha(e^{it}) = \Theta'(t) - ir'(t)/r(t). \) Since \( \arg(zg_\alpha'(z)/g_\alpha(z)) = \pi \alpha/2 \) for \( z = e^{it} \) with \( t \in (0, \pi) \), we obtain

\[
\frac{R'(\theta)}{R(\theta)} = \frac{r'(t)}{r(t)\Theta'(t)} = -\tan \frac{\pi \alpha}{2},
\]

which yields \( \log R(\theta) = -\theta \tan (\pi \alpha/2) \) for \( \theta = \Theta(t) \in (0, \pi) \). In the same way, we have \( \log R(-\theta) = -\theta \tan (\pi \alpha/2) \) for \( \theta \in (0, \pi) \). These imply that the radius function \( R \) of \( g_\alpha(\mathbb{D}) \) agrees with that of \( V_\alpha \), and hence \( g_\alpha(\mathbb{D}) = V_\alpha \). \( \square \)
3. Proof of Main Theorem

First, we recall the construction of a quasiconformal reflection in the boundary of a strongly starlike domain given by [2]. Let $D$ be a strongly starlike domain of order $\alpha \in (0, 1)$ with respect to the origin and $R$ be its radius function. Then we can take the quasiconformal reflection $\lambda$ in $\partial D$ defined by

$$\lambda(re^{i\theta}) = \frac{R(\theta)^2}{r}e^{i\theta}$$

for all $r > 0$ and $\theta \in \mathbb{R}$ following [2]. We then calculate

$$\partial \lambda = \frac{RR'}{r^2} \quad \text{and} \quad \overline{\partial} \lambda = \frac{e^{2i\theta}}{r^2}(iRR' - R^2)$$

at $w = re^{i\theta}$.

Now we use the following estimate to prove our main result. This estimate is originally due to Lehto [3], however, the following more general form can be found in [5].

**Theorem B.** Let $D$ be a quasidisk with quasiconformal reflection $\lambda$ in $\partial D$. Then the following inequality holds:

$$(7) \quad \sigma(D) \geq \epsilon(\lambda, D) := 2\inf_{w \in D} \frac{\left|\partial \lambda(w)\right| - \left|\overline{\partial} \lambda(w)\right|}{\left|\lambda(w) - w\right|^2 \rho_D(w)^2}.$$

Let us return to our case. By (6) and Theorem 1 (d), we obtain the estimates

$$\left|\overline{\partial} \lambda(w)\right| - \left|\partial \lambda(w)\right| = \frac{R^2}{r^2} \left(1 + \frac{\tan^2(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}\right)$$

and

$$\left|\lambda(w) - w\right| = \frac{R^2}{r} - r = \frac{R^2 - r^2}{r}$$

for almost all $w = re^{i\theta} \in D$.

Secondly, we estimate $\rho_D$ from above. Fix $w = re^{i\theta}$ and set $R = R(\theta)$. If we think of the domain $W = w_0 \cdot V_\alpha$, where $w_0 = Re^{i\theta} \in \partial D$, from Theorem 1 (c), we have $W \subset D$. The monotonicity property of the Poincaré metric then implies $\rho_D(w) \leq \rho_W(w)$. Now we write $\tau_\alpha = \rho_\alpha$. Then $\rho_W(w) = \tau_\alpha(w/w_0)/|w_0| = \tau_\alpha(r/R)/R$. Consequently, we have $\rho_D(w) \leq \tau_\alpha(r/R)/R$.

Summarizing the above, we have the estimate

$$\frac{\left|\overline{\partial} \lambda(w)\right| - \left|\partial \lambda(w)\right|}{\left|\lambda(w) - w\right|^2 \rho_D(w)^2} \geq \frac{R^2}{r^2} \cdot \frac{1}{(1 - (r/R)^2)^2 \tau_\alpha(r/R)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}.$$
Hence,
\[ \epsilon(\lambda, D) \geq \frac{2}{\sup_{0<u<1}(1-u^2)\tau_\alpha(u)^2} \cdot \frac{\cos(\pi\alpha/2)}{1+\sin(\pi\alpha/2)}. \]

So, if we can show the following lemma, the proof of our main theorem will be finished.

**Lemma 2.** The Poincaré density \( \tau_\alpha \) of \( V_\alpha \) satisfies
\[ \sup_{0<u<1}(1-u^2)\tau_\alpha(u) = M(\alpha). \]

**Proof.** Since \( g_\alpha : \mathbb{D} \to V_\alpha \) is biholomorphic by Lemma 1, we have \((1-|z|^2)^{-1} = \tau_\alpha(g_\alpha(z))|g_\alpha'(z)|\) for \( z \in \mathbb{D} \). Note here that \( u = u(x) = g_\alpha(x) > 0 \) and \( g_\alpha'(x) > 0 \) for positive \( x \). If we set
\[ Q(x) = (1-u(x)^2)\tau_\alpha(u(x)) = \frac{1-u(x)^2}{(1-x^2)u'(x)} \]
for \( x \in (0,1) \), we have only to show that \( Q \) is non-increasing in the interval \((0,1)\) because
\[ \lim_{x \to 0} Q(x) = \tau_\alpha(0) = 1/|g_\alpha'(0)| = M(\alpha). \]

Since \( xu'/u = \{(1+x)/(1-x)\}^\alpha \), we have the expression
\[ Q = \frac{1-u^2}{1-x^2} \cdot \frac{x}{u} \cdot \frac{(1-x)}{(1+x)^\alpha} = \frac{1-u^2}{1-x^2} \cdot \frac{x}{u} \cdot \frac{1-x}{(1+x)^{1+\alpha}(1-x)^{1-\alpha}}. \]

Taking the logarithmic derivative, we obtain
\[ x \frac{Q'}{Q} = 1 - \frac{xu'}{u} \cdot \frac{1+u^2}{1-u^2} + \frac{2x(x-\alpha)}{1-x^2} = \frac{1-2\alpha x + x^2}{1-x^2} - \frac{1-u^2}{1-u^2} \left( \frac{1+x}{1-x} \right)^\alpha. \]

Therefore, \( Q' \geq 0 \) if and only if
\[ \frac{1-u^2}{1+u^2} \leq \left( \frac{1+x}{1-x} \right)^\alpha \cdot \frac{1-x^2}{1-2\alpha x + x^2} = \frac{(1+x)^{1+\alpha}(1-x)^{1-\alpha}}{1-2\alpha x + x^2} =: P(x). \]

By representation (5), we see
\[ \frac{1-u^2}{1+u^2} = \tanh \left[ - \int_1^x \left( \frac{1+t}{1-t} \right)^\alpha \frac{dt}{t} \right]. \]

Hence, the assertion \( Q' \geq 0 \) on the interval \((0,1)\) is further equivalent to the validity of the statement that
\[ \int_x^1 \left( \frac{1+t}{1-t} \right)^\alpha \frac{dt}{t} \leq \arctanh P(x) \]
holds whenever \( P(x) < 1 \).

We now investigate the behaviour of the function \( P \) on \((0,1)\). Since
\[ \frac{P'(x)}{P(x)} = \frac{4(x-\alpha)(\alpha x - 1)}{(1-x^2)(1-2\alpha x + x^2)}, \]
\( P \) is increasing in \((0,\alpha)\) and decreasing in \((\alpha,1)\). Noting \( P(0) = 1 \) and \( P(1) = 0 \), we observe that \( P(x) > 1 \) for \( x \in (0,\beta) \) and that \( 0 < P(x) < 1 \) for \( x \in (\beta,1) \) for some number \( \beta \) between \( \alpha \) and 1. Here, we use the following elementary fact.
Lemma 3. Let $S$ and $T$ be continuous functions on the interval $(\beta, 1]$ that are positive, have continuous integrable derivatives on $(\beta, 1)$ and satisfy $S(1) = T(1) = 0$ and $S'(x) \leq T'(x)$ for $x \in (\beta, 1)$. Then $S(x) \geq T(x)$ for $x \in (\beta, 1]$.

Thus it is enough to show the inequality

$$-\frac{1}{x} \left( \frac{1+x}{1-x} \right)^{a} \geq \frac{d}{dx} \arctanh P(x) = \frac{P'(x)}{1 - P(x)^{2}} = \frac{-\frac{4(x-a)(1-\alpha)}{(1-2ax+X^{2})^{2} - (1+x)^{2+2\alpha}(1-x)^{2-2\alpha}}}{(1+x)^{2\alpha}}$$

for $x \in (\beta, 1)$. This inequality is equivalent to

$$(1-2\alpha x + x^{2})^{2} - (1+x)^{2+2\alpha}(1-x)^{2-2\alpha} \leq 4(x-a)(1-\alpha)(1-ax)$$

$$\Leftrightarrow (1+x)^{2+2\alpha}(1-x)^{2-2\alpha} \geq (1-2ax+x^{2})^{2} - 4X(X-\alpha)(1-\alpha X) = (1-x^{2})^{2} \Leftrightarrow (\frac{1+x}{1-x})^{2\alpha} \geq 1.$$ 

The last inequality is certainly valid for $x \in (0, 1)$. So, now the proof is complete.

Remark. We can see from the proof that $\varepsilon(\lambda, V_{\alpha}) = 2\cos(\pi\alpha/2)/(1+\sin(\pi\alpha/2))M(\alpha)^{2}$ holds, where $\lambda$ is the quasiconformal reflection constructed for $V_{\alpha}$ as above.

4. Upper estimate of $\sigma(V_{\alpha})$

In this section, we give a rough upper estimate of $\sigma(V_{\alpha})$ in order to examine how good our estimate (4) is.

Theorem 2. For $0 < \alpha < 1$, we have $\sigma(V_{\alpha}) \leq 2(1-\alpha)^{2}$.

Proof. We consider the holomorphic function $f(w) = \log(1-w)$ on the domain $\mathbb{C}\setminus[1, +\infty)$. Although $f$ is univalent, $f(V_{\alpha})$ has an outward pointing cusp. So, $f(V_{\alpha})$ is not a quasidisk.

On the other hand, for a quasidisk $D$, if $\|S_{f}\|_{D} < \sigma(D)$, we know that $f(D)$ is also a quasidisk (see [4, p. 120]). Hence, we conclude $\sigma(V_{\alpha}) \leq \|S_{f}\|_{V_{\alpha}}$ for the above $f$.

Now we estimate $\|S_{f}\|_{V_{\alpha}}$. First note that $V_{\alpha} \subset W := \{w; |\arg(1-w)| < (1-\alpha)\pi/2\}$. By the monotonicity of the Poincaré metric, we have $\|S_{f}\|_{V_{\alpha}} \leq \|S_{f}\|_{W}$. Since $(1-w)^{1/(1-\alpha)}$ maps $W$ conformally onto the right half plane, we compute

$$\rho_{W}(w) = \frac{|1-w|^{a/(1-\alpha)}}{2(1-\alpha)\text{Re}[(1-w)^{1/(1-\alpha)}]}.$$ 

On the other hand, $S_{f}(w) = 1/2(1-w)^{2}$. Thus, we calculate

$$\|S_{f}\|_{W} = \sup_{w \in W} 2(1-\alpha)^{2}\text{Re} \left[ \frac{(1-w)^{1/(1-\alpha)}}{|1-w|^{1/(1-\alpha)}} \right] = 2(1-\alpha)^{2}.$$ 

Now the proof is completed. \hfill $\Box$

Finally, we exhibit the graphs of the function $2\cos(\pi\alpha/2)/(1+\sin(\pi\alpha/2))M(\alpha)^{2}$ and $2(1-\alpha)^{2}$ below.
REFERENCES


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