Title
On an univalence criterion (New Extension of Historical Theorems for Univalent Function Theory)

Author(s)
Raducanu, Dorina; Owa, Shigeyoshi; Curt, Paula

Citation
数理解析研究所講究録 (2000), 1164: 125-132

Issue Date
2000-07

URL
http://hdl.handle.net/2433/64298

Right

Type
Departmental Bulletin Paper

Textversion
publisher
Kyoto University
On an univalence criterion

Dorina Răducanu, Shigeyoshi Owa and Paula Curt

Abstract

The method of subordination chains is used to establish an univalence criterion for holomorphic mappings in the open unit ball $B^n$ in $\mathbb{C}^n$. The authors consider an univalence criterion for holomorphic functions in $B^n$.

1 Introduction

Let $\mathbb{C}^n$ be the space of $n$-complex variables $z = (z_1, \ldots, z_n)$ with Euclidean inner product $\langle z, w \rangle = \sum_{k=1}^{n} z_k \overline{w}_k$ and the norm $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$.

Let $B^n$ be the open unit ball in $\mathbb{C}^n$, i.e $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$. We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from $B^n$ into $\mathbb{C}^n$, i.e the $n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm

$$\|A\| = \sup \{\|Az\| : \|z\| < 1\}, \ A \in \mathcal{L}(\mathbb{C}^n)$$

and

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbb{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings

$$f(z) = (f_1(z), \ldots, f_n(z)), \ z \in B^n$$

from $B^n$ into $\mathbb{C}^n$. We say that $f \in H(B^n)$ is \textit{locally biholomorphic} in $B^n$ if $f$ has local holomorphic inverse at each point in $B^n$ or equivalently, if the derivative
\[ Df(z) = \left( \frac{\partial f_k(z)}{\partial z_j} \right)_{1 \leq j, k \leq n} \]

is nonsingular at each point \( z \in B^n \).

A mapping \( v \in H(B^n) \) is called a Schwarz function if \( \|v(z)\| \leq \|z\| \), for all \( z \in B^n \).

Let \( f, g \in H(B^n) \). Then \( f \) is said to be subordinate to \( g \) (written by \( f \prec g \)) in \( B^n \) if there exists a Schwarz function \( v \) such that \( f(z) = g(v(z)) \), \( z \in B^n \). A function \( L : B^n \times [0, \infty) \to \mathbb{C}^n \) is said to be the subordination chain if \( L(\cdot, t) \) is holomorphic and univalent in \( B^n \), \( L(0, t) = 0 \) for all \( t \in [0, \infty) \), and \( L(z, s) \prec L(z, t) \) whenever \( 0 \leq s \leq t < \infty \).

The subordination chain \( L(z, t) \) is called a normalized subordination chain if \( DL(0, t) = e^t I \) for all \( t \geq 0 \).

The following result concerning subordination chains is due to J. A. Pfaltzgraff [5].

**Theorem 1** Let \( L(z, t) = e^t z + \ldots \) be a function from \( B^n \times [0, \infty) \) into \( \mathbb{C}^n \) such that:

(i) \( L(\cdot, t) \in H(B^n) \) for all \( t \in [0, \infty) \).

(ii) \( L(z, t) \) is a locally absolutely continuous function of \( t \), locally uniformly with respect to \( z \in B^n \).

Let \( h(z, t) \) be a function from \( B^n \times [0, \infty) \) into \( \mathbb{C}^n \) which satisfies the following condition:

(iii) \( h(\cdot, t) \in H(B^n) \), \( h(0, t) = 0 \), \( Dh(0, t) = I \) and \( \text{Re} \langle h(z, t), z \rangle \geq 0 \) for all \( t \geq 0 \) and \( z \in B^n \).

(iv) For each \( T > 0 \) and \( r \in (0, 1) \), there is a number \( K = K(r, T) \) such that \( \|h(z, t)\| \leq K(r, T) \) when \( \|z\| \leq r \) and \( t \in [0, T] \).

(v) For each \( z \in B^n \), \( h(\cdot, t) \) is a measurable function on \([0, \infty)\). Suppose \( h(z, t) \) satisfies

\[ \frac{\partial L(z, t)}{\partial t} = DL(z, t) h(z, t) \quad \text{a.e.} \ t \geq 0, \text{ for all } z \in B^n. \]
Further, suppose there is a sequence \( \{ t_m \}_{m \geq 0}, t_m > 0 \) which is increasing to \( \infty \) such that

\[
\lim_{m \to \infty} e^{-t_m} L(z, t_m) = F(z)
\]

locally uniformly in \( B^n \).

Then for each \( t \in [0, \infty), L(\cdot, t) \) is univalent in \( B^n \).

A version of Theorem 1 for subordination chains which are not normalized is due to P. Curt [2].

**Theorem 2** Let \( L(z, t) = a_1(t)z + \ldots, a_1(t) \neq 0 \) be a function from \( B^n \times [0, \infty) \) into \( \mathbb{C}^n \) such that:

(i) For each \( t \geq 0, L(\cdot, t) \in H(B^n) \).

(ii) \( L(z, t) \) is a locally absolutely continuous function of \( t \in [0, \infty) \), locally uniformly with respect to \( z \in B^n \).

(iii) \( a_1(t) \in C^1[0, \infty) \) and \( \lim_{t \to \infty} |a_1(t)| = \infty \).

(iv) \( h(\cdot, t) \in H(B^n) \) for all \( t \in [0, \infty) \).

(v) For each \( z \in B^n, h(z, \cdot) \) is a measurable function on \( [0, \infty) \).

(vi) \( h(0, t) = 0 \) and \( \text{Re}\langle h(z, t), z \rangle \geq 0 \) for all \( t \geq 0 \) and \( z \in B^n \).

(vii) For each \( T > 0 \) and \( r \in (0,1) \), there exists a number \( K = K(r, T) \) such that \( \| h(z, t) \| \leq K(r, T) \) when \( \| z \| \leq r \) and \( t \in [0, T] \).

Suppose that \( h(z, t) \) satisfies

\[
\frac{\partial L(z, t)}{\partial t} = DL(z, t) h(z, t) \quad \text{a.e. } t \geq 0, \text{ for all } z \in B^n. \tag{1}
\]

Further suppose that there exists a sequence \( \{ t_m \}_{m \geq 0}, t_m > 0 \) which is increasing to \( \infty \) such that
\[
\lim_{m \to \infty} \frac{L(z, t_m)}{a_1(t_m)} = F(z) \quad (2)
\]

locally uniformly in \( B^n \).

Then for each \( t \in [0, \infty) \), \( L(\cdot, t) \) is univalent in \( B^n \).

2 Main results

By using Theorem 2 we obtain an univalence criterion which generalize two univalence criteria for holomorphic mappings in \( B^n \).

**Theorem 3** Let \( f, g \) be holomorphic functions in \( B^n \) satisfying the conditions:

1° \( g \) is locally univalent in \( B^n \).

2° \( f(0) = g(0) = 0 \) and \( Df(0) = Dg(0) = I \).

Let \( \alpha \) be a real number with \( \alpha \geq 2 \). If

\[
\left\| (Dg(z))^{-1} Df(z) - \frac{\alpha}{2} I \right\| < \frac{\alpha}{2} \quad (3)
\]

and

\[
\left\| \|z\|^2 [Dg(z)^{-1} Df(z) - I] \right. \\
+ \left. (1 - \|z\|^{\alpha}) (Dg(z))^{-1} D^2 g(z)(z, \cdot) + \left(1 - \frac{\alpha}{2}\right) I \right\| \leq \frac{\alpha}{2} \quad (4)
\]

for all \( z \in B^n \), then \( f \) is an univalent function in \( B^n \).
Remark 1

The second derivative of function $g \in H(B^n)$ is a symmetric bilinear operator $D^2g(z)(\cdot, \cdot)$ on $\mathbb{C}^n \times \mathbb{C}^n$ and $D^2g(z)(w, \cdot)$ is the linear operator obtained by restricting $D^2g(z)$ to $\{w\} \times \mathbb{C}^n$. The linear operator $D^2g(z)(z, \cdot)$ has the matrix representation

$$D^2g(z)(z, \cdot) = \left( \sum_{m=1}^{n} \frac{\partial^2 g_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \leq j, k \leq n}.$$

Proof. We define the function $L(z, t)$ by

$$L(z, t) = f(e^{-t}z) + (e^{\alpha t} - 1) e^{-t}Dg(e^{-t}z)(z), \quad (z, t) \in B^n \times [0, \infty). \quad (5)$$

We have to show that $L(z, t)$ satisfies the conditions of Theorem 2 and hence $L(\cdot, t)$ is univalent in $B^n$, for all $t \in [0, \infty)$. It is easy to check that $a_1(t) = e^{(\alpha-1)t}$ and hence $a_1(t) \neq 0$, $\lim_{t \to \infty} |a_1(t)| = \infty$ and $a_1 \in C^1[0, \infty)$. We have

$$L(z, t) = a_1(t) z + (\text{holomorphic term}).$$

Thus $\lim_{t \to \infty} \frac{L(z, t)}{a_1(t)} = z$, locally uniform with respect to $B^n$ and hence (2) holds with $F(z) = z$. Obviously $L(z, t)$ satisfies the absolute continuity requirements of Theorem 2. From (5) we obtain

$$DL(z, t) = \frac{\alpha}{2} e^{(\alpha-1)t} Dg(e^{-t}z) [I - E(z, t)], \quad (6)$$

where, for all $(z, t) \in B^n \times [0, \infty)$, $E(z, t)$ is the linear operator defined by

$$E(z, t) = \left( 1 - \frac{2}{\alpha} \right) I - \frac{2}{\alpha} e^{-\alpha t} \left[ (Dg(e^{-t}z))^{-1} Df(e^{-t}z) - I \right] - \frac{2}{\alpha} (1 - e^{-\alpha t}) (Dg(e^{-t}z))^{-1} D^2g(e^{-t}z)(e^{-t}z, \cdot). \quad (7)$$
For $t = 0$, $E(z, 0) = -\frac{2}{\alpha} [(Dg(z))^{-1}Df(z) - \frac{\alpha}{2} I]$, it follows

$$\|E(z, 0)\| < 1, \text{for all } z \in B^n. \quad (8)$$

For $t > 0$, $E(\cdot, t) : \overline{B^n} \to \mathcal{L}(C^n)$ is holomorphic. By using the weak maximum modulus theorem[4], we obtain that $\|E(z, t)\|$ can have no maximum in $B^n$ unless $\|E(z, t)\|$ is of constant value throughout $\overline{B^n}$.

If $z = 0$ and $t > 0$, then we have

$$\|E(0, t)\| = \left\| \left(1 - \frac{2}{\alpha}\right)I \right\| = \left|1 - \frac{2}{\alpha}\right| < 1. \quad (9)$$

We also have

$$\|E(z, t)\| < \max_{\|w\|=1} \|E(w, t)\|.$$  

If we let $u = e^{-t}w$, with $\|w\| = 1$, then $\|u\| = e^{-t}$ and from (4) we obtain

$$\|E(w, t)\| = \frac{2}{\alpha} \left\| u \right\|^{\alpha} \cdot [(Dg(u))^{-1}Df(u) - I]
+ (1 - \|u\|^{\alpha}) (Dg(u))^{-1}D^2g(u)(u, \cdot) + \left(1 - \frac{\alpha}{2}\right) I \right\| \leq 1.$$  

Since $\|E(z, t)\| < 1$ for all $(z, t) \in B^n \times [0, \infty)$, it follows $I - E(z, t)$ is an invertible operator.

Further the calculation shows that

$$\frac{\partial L(z, t)}{\partial t} = \frac{\alpha}{2} e^{(\alpha-1)t}Dge^{-t}z[I + E(z, t)](z)
= DL(z, t)[I - E(z, t)]^{-1}[I + E(z, t)](z).$$
Hence $L(z, t)$ satisfies the differential equation (1) for all $t \geq 0$ and $z \in B^n$, where

$$h(z, t) = [I - E(z, t)]^{-1} [I + E(z, t)](z). \tag{10}$$

It remains to show that $h(z, t)$ satisfies the conditions $(iv)$ $(v)$ $(vi)$ and $(vii)$ of Theorem 2. Clearly, $h(z, t)$ satisfies the holomorphy and measurability requirements and $h(0, t) = 0$. The inequality

$$\|h(z, t) - z\| = \|E(z, t)(h(z, t) + z)\| \leq \|E(z, t)\| \|(h(z, t) + z)\| < \|(h(z, t) + z)\|$$

implies $\text{Re} \langle h(z, t), z \rangle \geq 0$, for all $z \in B^n$ and $t \geq 0$.

By using the inequality

$$\|[I - E(z, t)]^{-1}\| \leq [1 - \|E(z, t)\|]^{-1},$$

we obtain

$$\|h(z, t)\| \leq \frac{1 + \|E(z, t)\|}{1 - \|E(z, t)\|} \|z\|.$$

Since the conditions of Theorem 2 are satisfied it follows that the function $L[z, t), t \geq 0$ is univalent in $B^n$. In particular $f(z) = L(z, 0)$ is univalent in $B^n$.

**Remark 2**

1) If $g = f$ and $\alpha = 2$, then Theorem 3 becomes the $n$-dimensional version of Beker's univalence criterion [5].

2) For $\alpha = 2$ we obtain an univalence criterion due to P. Curt and D. Răducanu [3].
References


Dorina Răducanu:
Department of Mathematics
Transilvania University
Brașov 2200
România

Shigeyoshi Owa:
Department of Mathematics
Kinki University
Higashi - Osaka, Osaka 577-8502
Japan

Paula Curt:
Faculty of Mathematics
Kogălniceanu 1
Cluj-Napoca 3400
România