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Alpha-spiral mappings of a Banach space into the complex plane

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Abstract

Let $E$ be a complex Banach space and let $B$ be the unit ball in $E$, i.e. $B = \{x \in E : \|x\| < 1\}$. In this paper we define the class of $\alpha$-spiral mappings of the unit ball $B$ into the complex plane $\mathbb{C}$.

1 Introduction

Let $E^*$ be the dual space of $E$. For any $A \in E^*$ we consider $\chi(A) = \{x \in E : A(x) \neq 0\}$ and $\gamma(A) = E \setminus \chi(A)$. If $A \neq 0$ then $\chi(A)$ is dense in $E$ and $\chi(A) \cap \hat{B}$ is dense in $\hat{B}$, where $\hat{B} = \{x \in E : \|x\| = 1\}$.

Let $H(B)$ be the family of all functions $f : B \to \mathbb{C}, f(0) = 0$, which are holomorphic in $B$, i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B$. If $f \in H(B)$, then, in some neighbourhoods $V$ of the origin, $f(x) = \sum_{m=1}^{\infty} P_{m,f}(x)$, where the series is uniformly convergent on $V$ and $P_{m,f} : E \to \mathbb{C}$ are continuous and homogeneous polynomials of degree $m$.

Let $\alpha \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}$ and let $z_0 \in \mathbb{C} \setminus \{0\}$. The condition

$$z(t) = z_0 e^{-(\cos \alpha + i \sin \alpha)t}, \quad t \in \mathbb{R}$$

defines an $\alpha$-spiral curve in the complex plane.

Let $D$ be a domain in $\mathbb{C}$, such that $0 \in D$. If for any $z_0 \in D \setminus \{0\}$ the arc of $\alpha$-spiral curve between the points $z_0$ and the origin is contained in $D$, then $D$ is an $\alpha$-spiral domain with respect to the origin.
Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$. We denote by $SP(\alpha)$ the family of all univalent functions $f : U \to \mathbb{C},$

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are $\alpha$-spiral in $U$, i.e. $f(U)$ is an $\alpha$-spiral domain with respect to the origin.

**Theorem 1** ([3]) Let $f$ be an holomorphic function from $U$ into $\mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$, $f(z) \neq 0$, for all $z \in U \setminus \{0\}$ and let $\alpha \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}$. Then $f \in SP(\alpha)$ if and only if

\[
\Re \left[ e^{i\alpha} z f'(z) \right] / f(z) > 0 \quad \text{for all } z \in U.
\]

2 The class of alpha-spiral mappings on a Banach space

Let $A \in E^*$, $A \neq 0$ and $\alpha \in \mathbb{R}$, $|\alpha| < \frac{\pi}{2}$. We denote by $SP_A(\alpha)$ the family of all functions $f \in H(B)$ which have the form

\[
f(x) = A(x) + \sum_{n=2}^{\infty} P_{nf}(x)
\]

such that, for any $a \in \chi(A) \cap \hat{B}$, $f$ is univalent on $B_a = \{ za : z \in U \}$ and $f(B_a)$ is an $\alpha$-spiral domain with respect to the origin.

For any function $f$ of the form (1) and $a \in \chi(A) \cap \hat{B}$ we consider the function $f_a : U \to \mathbb{C}$.

\[
f_a(z) = \frac{f(za)}{A(a)}, \quad z \in U.
\]

Obviously
\[ f_a(z) = z + \sum_{n=2}^{\infty} \frac{P_{nf}(a)}{A(a)} z^n, \quad z \in U. \]  

(2)

Moreover, it is easy to check that

\[ f_a^{(n)}(z) = \frac{f^{(n)}(z a)(a,\ldots,a)}{A(a)}, \quad n \in \mathbb{N}, z \in U. \]

By using the properties of \( \alpha \)-spiral functions in the unit disk, we obtain some estimations of \( |P_{n,f}(a)| \) and \( \|P_{n,f}\| \) in the class \( \text{SP}_A(\alpha) \).

**Theorem 2** If \( f \in \text{SP}_A(\alpha) \) and \( a \in \hat{B} \), then

\[ |P_{n,f}(a)| \leq \frac{|A(a)|}{(n-1)!} \prod_{k=1}^{n-1} [(k-1)^2 + 4k \cos^2 \alpha]^\frac{1}{2}, \quad n \geq 2 \]

(3)

This inequality is sharp and the equality holds for the function

\[ f(x) = \frac{A(x)}{(1-H(x))^{2s}}, \quad x \in B \]

where \( s = e^{-i\alpha} \cos \alpha, H \in E^*, H(a) = 1 \) and \( \|H\| = 1 \).

Proof. Suppose that \( f \in \text{SP}_A(\alpha) \) and \( n \geq 2 \). If \( a \in \chi(A) \cap \hat{B} \), then \( f_a \in \text{SP}(\alpha) \) and hence we get (3). If \( a \in \gamma(A) \cap \hat{B} \), then evidently \( a = \lim_{m \to \infty} a_m \), where \( a_m \in \chi(A), m \in \mathbb{N} \). There exists \( r_m \in /R_+ \) such that \( a_m/r_m \in \hat{B} \). Clearly \( (r_m)_{m \geq 0} \) is bounded for 0 is an interior point of \( B \).

Since \( a_m/r_m \in \chi(A) \cap \hat{B}, m \in \mathcal{N} \), by the first part of the proof we have
\[ |P_{n,f} \left( \frac{a_m}{r_m} \right) | \leq |A \left( \frac{a_m}{r_m} \right) | \frac{1}{(n-1)!} \prod_{k=1}^{n-1} \left[ (k-1)^2 + 4k \cos^2 \alpha \right]^\frac{1}{2}, \quad m \in \mathbb{N}. \]

Hence

\[ |P_{n,f} (a_m) | \leq r_m^{n-1} \frac{|A(a_m)|}{(n-1)!} \prod_{k=1}^{n-1} \left[ (k-1)^2 + 4k \cos^2 \alpha \right]^\frac{1}{2}, \quad m \in \mathbb{N}. \]

By taking the limit with \( m \to \infty \), we obtain \( P_{n,f} (a) = 0 \).

**Corollary 1**

Any \( f \in SP_A (\alpha) \) vanishes on \( \gamma (A) \cap B \).

Proof. Let \( f \in SP_A (\alpha) \). Since \( P_{n,f} (a) = 0 \) for all \( a \in \gamma (A) \cap \hat{B} \), \( f \) vanishes on \( B_a \). Let \( x \in \gamma (A) \cap B, x \neq 0 \). Then \( a = \frac{x}{||x||} \in \gamma (A) \cap \hat{B} \) and \( f (za) = 0 \) for all \( z \in U \). Putting \( z = ||x|| \), we get \( f (x) = 0 \).

**Corollary 2** If \( f \in SP_A (\alpha) \) and \( n \geq 2 \), then

\[ \|P_{n,f}\| \leq \frac{\|A\|}{(n-1)!} \prod_{k=1}^{n-1} \left[ (k-1)^2 + 4k \cos^2 \alpha \right]^\frac{1}{2} \quad (4) \]

The inequality is sharp, being attained by

\[ f (x) = \frac{A (x)}{(1 - H (x))^{2s}}, \quad x \in B. \]

We shall give some necessary and sufficient conditions for holomorphic functions to belong to the class \( SP_A (\alpha) \).
Theorem 3 If \( f \in SP_A(\alpha) \), then

\[
\text{Re} \left[ e^{\iota \alpha} \frac{f'(x)(x)}{f(x)} \right] > 0, \quad \text{for any } x \in \chi(A) \cap B
\]  

(5)

Proof. Let \( x \in \chi(A) \cap B, x \neq 0 \). Then \( a = \frac{x}{\|x\|} \in \chi(A) \cap \hat{B} \) and hence the function \( f_a \) belongs to the class \( SP(\alpha) \). We have

\[
\text{Re} \left[ e^{\iota \alpha} \frac{zf_a'(z)}{f_a(z)} \right] > 0, \quad z \in U.
\]

From the equality

\[
\frac{f'(za)(za)}{f(za)} = \frac{zf_a'(z)}{f_a(z)}, \quad z \in U,
\]

we obtain

\[
\text{Re} \left[ e^{\iota \alpha} \frac{f'(za)(za)}{f(za)} \right] > 0, \quad z \in U.
\]

Putting \( z = \|x\| \), we get (5).

Theorem 4 Let \( f \in H(B), f'(0) = A \) and \( f(x) \neq 0 \), for all \( x \in B \setminus \{0\} \). If

\[
\text{Re} \left[ e^{\iota \alpha} \frac{f'(x)(x)}{f(x)} \right] > 0, \quad x \in B
\]

then \( f \in SP_A(\alpha) \).
Proof. Let \( a \in \chi(A) \cap \hat{B} \). Since \( f_a(0) = 0, f'_a(0) = 1, f_a(z) \neq 0 \), for all \( z \in U \setminus \{0\} \) and

\[
\text{Re} \left[ e^{i\alpha} \frac{zf_a'(z)}{f_a(z)} \right] = \text{Re} \left[ e^{i\alpha} \frac{f'(za)(za)}{f(za)} \right] > 0, \quad z \in U,
\]

we obtain that \( f_a \) is an \( \alpha \)-spiral function in \( U \). Then \( f \) is univalent in \( B_a \) and \( f(B_a) \) is an \( \alpha \)-spiral domain with respect to the origin. Hence \( f \in SP_A(\alpha) \).

**Remark**

The above results can be generalized by replacing the unit ball \( B \) with a bounded and open set \( D \subset E, D \neq \emptyset \) such that \( zD \subset D \), for \( z \in \overline{U} = \{z \in \mathbb{C}, |z| \leq 1\} \). In this case, for \( \alpha = 0 \) some of the results due to E.Janiec [4] are obtained.
References


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