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Kyoto University
On certain integral operators

SHIGEYOSHI OWA

Abstract. Let $A$ be the class of functions $f(z)$ which are analytic in the open unit disk $U$ with $f(0) = 0$ and $f'(0) = 1$. The object of the present paper is to consider the subordinations of certain integral operators for functions belonging to the class $A$.

1 Introduction

Let $A$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S^*$ be the subclass of $A$ consisting of functions $f(z)$ which are satisfy

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).$$

A function $f(z)$ in $S^*$ is said to be starlike in $U$. For functions $f(z)$ and $g(z)$ belonging to $A$, we say that $f(z)$ is subordinate to $g(z)$ if there exists the function $w(z)$ which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$. By virtue of the definition for subordinations, we know that: (i) The subordination $f(z) \prec g(z)$ implies that $f(0) = g(0)$ and $f(U) \subset g(U)$. (ii) If $g(z)$ is univalent in $U$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

For a function $f(z) \in A$, we consider the integral operator $I(f(z))$ given by

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\[ I_{\alpha,\beta}(f(z)) = \left\{ \frac{\alpha + \beta}{z^\beta} \int_0^z t^{\beta-1} f(t)^\alpha dt \right\}^{1/\alpha}, \]  \hspace{1cm} (1.2)\]

where \( \alpha \in \mathbb{C}, \alpha \neq 0 \), and \( \beta \in \mathbb{C} \).

**Remark 1.** (i) Libera [2] showed that if \( f(z) \in S^* \), then \( I_{1,1}(f(z)) \in S^* \).

(ii) Bernardi [1] showed that if \( f(z) \in S^* \), then \( I_{1,\beta}(f(z)) \in S^* \) when \( \beta = 1, 2, 3, \ldots \).

(iii) Miller, Mocanu and Reade [6] showed that if \( f(z) \in S^* \), then \( I_{\alpha,\beta}(f(z)) \in S^* \) when \( \alpha > 0 \) and \( \beta \geq 0 \).

To consider our integral operators for \( f(z) \in A \), we have to recall here the following lemmas.

**Lemma 1.** ([3]) Let \( f(z) \) and \( g(z) \) belong to \( A \) and \( g(z) \) be univalent in \( \overline{U} = U \cup \partial U \). If there exists points \( z_0 \in U \) and \( \zeta_0 \in \partial U \) such that

\[ f(|z| < |z_0|) \subset g(U) \quad \text{and} \quad f(z_0) = g(\zeta_0), \]

then \( z_0 f'(z_0) = m \zeta_0 g'(\zeta_0) \), where \( m \) is real and \( m \geq 1 \).

**Lemma 2.** ([4]) Let \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) be analytic in \( U \) with \( p(z) \not\equiv 1 \). Let \( \psi : \mathbb{C}^2 \rightarrow \mathbb{C} \) satisfies

(i) \( \psi \) is continuous in \( D \subset \mathbb{C}^2 \),

(ii) \( (1,0) \in D \) and \( \text{Re}(\psi(1,0)) > 0 \),

(iii) for all \( (iu_2,v_1) \in D \) such that \( v_1 \leq -(1 + u_2^2)/2, \text{Re}(\psi(iu_2,v_1)) \leq 0 \).

If \( (p(z),zp'(z)) \in D \) for all \( z \in D \) and \( \text{Re}(\psi(p(z),zp'(z))) > 0 \) for all \( z \in U \), then \( \text{Re}(p(z)) > 0 \) \( (z \in U) \).

Let a function \( L(z,t) \) be defined on \( U \times \{0,\infty\} \). Then \( L(z,t) \) is said to be the subordination chain (or Loewner chain) if it satisfies

(i) \( L(z,t) \) is analytic and univalent in \( U \) for all \( t \geq 0 \),

(ii) \( L(z,t) \) is continuously differentiable on \( t \geq 0 \) for all \( z \in U \),

(iii) \( L(z,t_1) \prec L(z,t_2) \) \( (0 \leq t_1 \leq t_2) \).
Lemma 3. ([7]) Let \( L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots \) \((a_1(t) \neq 0; t \geq 0)\) is a subordination chain if and only if it satisfies
\[
\text{Re} \left\{ z \frac{\partial L(z,t)}{\partial z} \frac{\partial L(z,t)}{\partial t} \right\} > 0 \quad (z \in U; t \geq 0).
\]
Further, we need

Lemma 4. ([5]) Let \( \alpha \in \mathbb{C}, \alpha \neq 0, \) and let \( \beta \in \mathbb{C}. \) Let a function
\[
h(z) = c \ast h_1 * h_2 z^2 + \ldots
\]
be analytic in \( U \) and \( \text{Re}(\alpha h(z) + \beta) > 0 \) \((z \in U)\). Then the solution of the Briot-Bouquet differential equation
\[
q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (h(0) = q(0))
\]
is analytic in \( U \) and \( \text{Re}(\alpha q(z) + \beta) > 0 \) \((z \in U)\).

2 Subordination theorems

Applying the above lemmas, we derive our main theorem in

Theorem 1. Let \( f(z) \in A \) and \( g(z) \in A. \) Let \( g(z) \) satisfy
(i) \( g(z)/z \neq 0 \) \((z \in U)\) and \( I_{\alpha, \beta}(g(\approx))/z \neq 0 \) \((z \in U)\) when \( \alpha \neq 1, \)
(ii) \( \phi(z) = (g(z)/z)^{\alpha} \) satisfies
\[
\text{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \quad (z \in U),
\]
where \( \delta < \text{Re}(\alpha + \beta), \)
\[
-1 < \delta \leq \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\text{Re}(\alpha + \beta))^2}}{4\text{Re}(\alpha + \beta)}
\]
if \( \text{Re}(\alpha + \beta) > 0, \) and
\[
-1 < \delta \leq \frac{1 + |\alpha + \beta|^2 + \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\text{Re}(\alpha + \beta))^2}}{4\text{Re}(\alpha + \beta)}
\]
if \( \text{Re}(\alpha + \beta) < 0 \).

If \( f(z) \) and \( g(z) \) satisfy the following subordination

\[
\frac{f(z)}{z} < \frac{g(z)}{z},
\]  

then

\[
\frac{I_{\alpha,\beta}(f(z))}{z} < \frac{I_{\alpha,\beta}(g(z))}{z}.
\]

Proof. Let us define \( F(z) \) and \( G(z) \) by

\[
F(z) = \left( \frac{I_{\alpha,\beta}(f(z))}{z} \right)^{\alpha} \quad \text{and} \quad G(z) = \left( \frac{I_{\alpha,\beta}(g(z))}{z} \right)^{\alpha},
\]

respectively. Without loss of generality, we can assume that \( G(z) \) is analytic and univalent in \( \overline{U} = U \cup \partial U \). Otherwise, we consider, for \( 0 < r < 1 \), \( F(rz)/r \) and \( G(rz)/r \) instead of \( F(z) \) and \( G(z) \), respectively.

First, we show that if the function \( q(z) \) is defined by

\[
q(z) = 1 + \frac{zG''(z)}{G'(z)},
\]

then \( \text{Re}(q(z) + \alpha + \beta) > 0 \) \((z \in U)\).

Since

\[
I_{\alpha,\beta}(g(z)) = \left\{ \frac{\alpha + \beta}{z^\beta} \int_0^z t^{\beta-1}g(t)^\alpha dt \right\}^{1/\alpha},
\]

we have

\[
\alpha \frac{z(I_{\alpha,\beta}(g(z)))'}{I_{\alpha,\beta}(g(z))} = -\beta + (\alpha + \beta)\frac{\phi(z)}{G(z)}.
\]

Also, we have

\[
\alpha \frac{z(I_{\alpha,\beta}(g(z)))'}{I_{\alpha,\beta}(g(z))} = \alpha + \frac{zG'(z)}{G(z)}.
\]

It follows from (2.6) and (2.7) that

\[
(\alpha + \beta)\phi(z) = (\alpha + \beta)G(z) + zG'(z).
\]
Differentiating both sides in (2.8), we obtain
\[ \beta z\phi'(z) = zG'(z) \left( \alpha + \beta + 1 + \frac{zG''(z)}{G'(z)} \right) \]
\[ = zG'(z)(q(z) + \alpha + \beta). \]
This gives us that
\[ 1 + \frac{z\phi''(z)}{\phi'(z)} = 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \alpha + \beta} \]
\[ = q(z) + \frac{zq'(z)}{q(z) + \alpha + \beta}. \]
If we define the function \( h(z) \) by
\[ h(z) = q(z) + \frac{zq'(z)}{q(z) + \alpha + \beta}, \]
then, \( q(0) = h(0) = 1 \) and
\[ \text{Re}(h(z) + \alpha + \beta) = \text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} + \alpha + \beta \right) \]
\[ > -\delta + \text{Re}(\alpha + \beta) > 0, \]
because \( \delta \) satisfies the conditions in our theorem and \( \delta \leq \text{Re}(\alpha + \beta) \).
Thus applying Lemma 4, we conclude that \( q(z) \) is analytic in \( U \) and \( \text{Re}(q(z) + \alpha + \beta) > 0 \) for all \( z \in U \).
Next, we show that \( \text{Re}(q(z) + \alpha + \beta) > 0 \) for all \( z \in U \).
Let us put
\[ \psi(u, v) = u + \frac{v}{u + \alpha + \beta} + \delta, \]
with \( u = u_1 + iu_2 \) and \( v = v_1 + iv_2 \). Then \( \psi(u, v) \) satisfies
(i) \( \psi(u, v) \) is continuous in \( D = (\mathbb{C} \setminus \{-\alpha - \beta\}) \times \mathbb{C} \),
(ii) \( (1, 0) \in D \) and \( \text{Re}\psi(1, 0) = 1 + \delta > 0 \),
(iii) for all \( (iu_2, v_1) \in D \) such that \( v_1 \leq -(1 + u_2^2)/2 \),
\[ \text{Re}\psi(iu_2, v_1) = \text{Re} \left( \frac{v_1}{iu_2 + \alpha + \beta} \right) + \delta \]
\[ = \delta - \frac{v_1 \text{Re}(\alpha + \beta)}{|\alpha + \beta|^2 + 2u_2 \text{Im}(\alpha + \beta) + u_2^2}. \]
We define
\[ E_\delta(u_2) = (2\delta - \text{Re}(\alpha + \beta))u_2^2 + 4\delta(\text{Im}(\alpha + \beta))u_2 + 2\delta |\alpha + \beta|^2 - \text{Re}(\alpha + \beta), \quad (2.9) \]
$k_1 = \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\text{Re}(\alpha + \beta))^2}}{4\text{Re}(\alpha + \beta)},$

and

$k_2 = \frac{1 + |\alpha + \beta|^2 + \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\text{Re}(\alpha + \beta))^2}}{4\text{Re}(\alpha + \beta)}.$

Then, the discrimination $\Delta$ of $E_\delta(u_2)$ given by (2.8) is represented by

$$\Delta = -4(\text{Re}(\alpha + \beta))^2 \delta^2 + 2\delta(1 + |\alpha + \beta|^2)\text{Re}(\alpha + \beta) - (\text{Re}(\alpha + \beta))^2.$$  

Therefore, if $\text{Re}(\alpha + \beta) > 0$, then $-1 < \delta \leq k_1$ implies $\Delta \leq 0$, and if $\text{Re}(\alpha + \beta) < 0$, then $-1 < \delta \leq k_2$ implies $\Delta \leq 0$.

This shows that $E_\delta(u_2) \leq 0$ for all real $u_2$, that is, that $\text{Re}(iu_2, v_1) \leq 0$ for all real $v_1$ and $u_2$ such that $v_1 \leq -(1 + u_2^2)/2$.

Further, we note that

$$\text{Re} \psi(q(z), zq'(z)) = \text{Re} \left( g(z) + \frac{zq'(z)}{q(z) + \alpha + \beta} + \delta \right)$$

$$= \text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} + \delta \right) > 0.$$

Thus, using Lemma 2, we conclude that $\text{Re}(q(z)) > 0$ for all $z \in YU$.

Finally, we prove that the subordination $f(z)/z \prec g(z)/z$ implies $F(z) \prec G(z)$.

Define the function $L(z, t)$ by

$$L(z, t) = G(z) + \frac{1 + t}{\alpha + \beta}zG'(z) \quad (t \geq 0).$$

Note that $G'(0) = 1$ and

$$\left. \frac{\partial L(z, t)}{\partial z} \right|_{z=0} = G'(0) \left( 1 + \frac{1 + t}{\alpha + \beta} \right) = 1 + \frac{1 + t}{\alpha + \beta} \neq 0.$$  

This shows that the function

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots$$

satisfies $a_1(t) \neq 0$ for all $t \geq 0$.

Furthermore, we have

$$\text{Re} \left\{ z \frac{\partial L(z, t)}{\partial t} \right\} = \text{Re} \left( \alpha + \beta + (1 + t) \left( 1 + \frac{zG''(z)}{G'(z)} \right) \right)$$

$$= \text{Re}(q(z) + \alpha + \beta) + t\text{Re}(z) > 0.$$
for all $z \in U$. Therefore, by virtue of Lemma 3, $L(z, t)$ is the subordination chain. Note that
\[
\phi(z) = G(z) + \frac{1}{\alpha + \beta}zG'(z) = L(z, 0)
\]
and
\[
L(z, 0) \prec L(z, t) \quad (t \geq 0)
\]
from the definition of the subordination chain. Next, we support that $F(z)$ is not subordinate to $G(z)$. Then, there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that
\[
F(|z| < |z_0|) \subset G(U) \quad \text{and} \quad F(z_0) = G(\zeta_0).
\]
This implies that $L(\zeta_0, t) \notin L(U, t)$. Since, by Lemma 1,
\[
z_0 F'(z_0) = (1 + t) \zeta_0 G'(\zeta_0) \quad (t \geq 0),
\]
we have
\[
L(\zeta_0, t) = G(\zeta_0) + \frac{1 + t}{\alpha + \beta} \zeta_0 G'(\zeta_0)
\]
\[
= F(z_0) + \frac{1}{\alpha + \beta} z_0 F'(z_0)
\]
\[
= \left( \frac{f(z_0)}{z_0} \right)^{\alpha} \in \psi(U),
\]
because $f(z)/z \prec g(z)/z$. This contradicts that $L(\zeta_0, t) \notin L(U, t)$. Therefore, the subordination $f(z)/z \prec g(z)/z$ has to imply $F(z) \prec G(z)$.

Now, since
\[
I_{\alpha, \beta}(g(z)) = 1 + c_1 z + c_2 z^2 + \ldots \neq 0 \quad (z \in U)
\]
when $\alpha \neq 1$, we conclude that
\[
F(z) = \left( \frac{I_{\alpha, \beta}(f(z))}{z} \right)^{\alpha} \prec G(z) = \left( \frac{I_{\alpha, \beta}(g(z))}{z} \right)^{\alpha}
\]
gives that
\[
I_{\alpha, \beta}(f(z)) \prec \frac{I_{\alpha, \beta}(g(z))}{z}.
\]
This completes the proof of our theorem.

\[
\square
\]

If we take $\alpha$ and $\beta$ such that $\alpha + \beta = 1$ in Theorem 1, then $-1 < \delta \leq 1/2$. Therefore, we have
Corollary 1. Let \( f(z) \in A \) and \( g(z) \in A \). Let \( g(z) \) satisfy
(i) \( g(z)/z \neq 0 \) (\( z \in U \)) and \( I_{\alpha,1-\alpha}(g(z))/z \neq 0 \) (\( z \in U \)) when \( \alpha \neq 1 \),
(ii) \( \phi(z) = (g(z)/z)^{\alpha} \) satisfies
\[
\text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} \right) > -\frac{1}{2}.
\]
Then
\[
\frac{f(z)}{z} < \frac{g(z)}{z} \quad \Rightarrow \quad \frac{I_{\alpha,1-\alpha}(f(z))}{z} < \frac{I_{\alpha,1-\alpha}(g(z))}{z},
\]
where
\[
I_{\alpha,1-\alpha}f(z) = \left\{ \frac{1}{z^{1-\alpha}} \int_{0}^{z} \left( \frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.
\]

If we make \( \alpha \) and \( \beta \) such that \( \alpha + \beta = -1 \) in Theorem 1, then \(-1 < \delta \leq -1/2\). Thus we have

Corollary 2. Let \( f(z) \in A \) and \( g(z) \in A \). Let \( g(z) \) satisfy
(i) \( g(z)/z \neq 0 \) (\( z \in U \)) and \( I_{\alpha,-1-\alpha}(g(z))/z \neq 0 \) (\( z \in U \)) when \( \alpha \neq 1 \),
(ii) \( \phi(z) = (g(z)/z)^{\alpha} \) satisfies
\[
\text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} \right) > \frac{1}{2}.
\]
Then
\[
\frac{f(z)}{z} < \frac{g(z)}{z} \quad \Rightarrow \quad \frac{I_{\alpha,-1-\alpha}(f(z))}{z} < \frac{I_{\alpha,-1-\alpha}(g(z))}{z},
\]
where
\[
I_{\alpha,-1-\alpha}f(z) = \left\{ \frac{1}{z^{1+\alpha}} \int_{0}^{z} \left( \frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.
\]

Taking \( \alpha \) and \( \beta \) such that \( \alpha + \beta = 1+i \) in Theorem 1, we have

Corollary 3. Let \( f(z) \in A \) and \( g(z) \in A \). Let \( g(z) \) satisfy
(i) \( g(z)/z \neq 0 \) (\( z \in U \)) and \( I_{\alpha,1+i-\alpha}(g(z))/z \neq 0 \) (\( z \in U \)) when \( \alpha \neq 1 \),
(ii) \( \phi(z) = (g(z)/z)^{\alpha} \) satisfies
\[
\text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} \right) > \frac{\sqrt{5} - 3}{4} \approx -0.190983.
\]
Then
\[
\frac{f(z)}{z} < \frac{g(z)}{z} \quad \Rightarrow \quad \frac{I_{\alpha,1+i-\alpha}(f(z))}{z} < \frac{I_{\alpha,1+i-\alpha}(g(z))}{z},
\]
where
\[
I_{\alpha,1+i-\alpha}f(z) = \left\{ \frac{1+i}{z^{1+i-\alpha}} \int_{0}^{z} t^{i} \left( \frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.
\]
Further, letting $\alpha + \beta = -1 + i/2$ in Theorem 1, we have

**Corollary 4.** Let $f(z) \in A$ and $g(z) \in A$. Let $g(z)$ satisfy
(i) $g(z)/z \neq 0$ ($z \in U$) and $I_{\alpha,-1+i/2-\alpha}(g(z))/z \neq 0$ ($z \in U$) when $\alpha \neq 1$,
(ii) $\phi(z) = (g(z)/z)^{\alpha}$ satisfies

$$\text{Re} \left( 1 + \frac{z\phi''(z)}{\phi'(z)} \right) > \frac{-9 + \sqrt{17}}{16} = -0.8201941.$$

Then

$$\frac{f(z)}{z} < \frac{g(z)}{z} \implies \frac{I_{\alpha,-1+i/2-\alpha}(f(z))}{z} < \frac{I_{\alpha,-1+i/2-\alpha}(g(z))}{z},$$

where

$$I_{\alpha,-1+i/2-\alpha}(f(z)) = \left\{ \frac{-2 + i}{2z-1+i/2-\alpha} \int_{0}^{z} t^{-1+i/2} \left( \frac{f(t)}{t} \right)^{\alpha} dt \right\}^{1/\alpha}.$$

### 3 An application of hypergeometric functions

For complex numbers $a, b,$ and $c$ with $c \neq 0, -1, -2, \ldots,$ the hypergeometric function $\tfrac{2}{1}F_1(a, b; c; z)$ is defined by

$$\tfrac{2}{1}F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of gamma functions, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & (n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \\ 1 & (n = 0) \end{cases}.$$

It is well-known that

$$\tfrac{2}{1}F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_{0}^{1} t^{a-1}(1-t)^{-a-1}(1-zt)^{-b} dt,$$

for $\text{Re}(c) > \text{Re}(a) > 0$.

If we consider

$$g(z) = \frac{z}{(1-z)^k} \quad (k \in \mathbb{C}),$$

then we have

$$I_{\alpha,\beta}(g(z)) = \left\{ \frac{\alpha + \beta}{z^{\beta}} \int_{0}^{z} t^{\alpha+\beta-1}(1-t)^{-\alpha k} dt \right\}^{1/\alpha}.$$
\[
\left\{(\alpha + \beta)z^\alpha \int_0^1 u^{\alpha + \beta - 1}(1 - zu)^{-\alpha k}du\right\}^{1/\alpha} = z_2 F_1(\alpha + \beta, \alpha k; \alpha + \beta + 1; z)^{1/\alpha}
\]

with Re(\(\alpha + \beta\)) > 0. Moreover, we note that
\[
g(z) = \frac{1}{(1 - z)^k} \neq 0 \quad (z \in U).
\]

Applying the above to Theorem 1, we obtain

**Theorem 2.** Let \(f(z) \in U\). Let \(\alpha > 0\) and \(\beta\) be a complex number such that Re(\(\alpha + \beta\)) > 0. If \(f(z)\) satisfies the subordination
\[
f(z) \prec \frac{1}{(1 - z)^k},
\]
then
\[
I_{\alpha,\beta}(f(z)) \prec 2F_1(\alpha + \beta, \alpha k; \alpha + \beta + 1; z)^{1/|\alpha|},
\]
where
\[
1 - \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + (\alpha + \beta)^2)^2 - 4(\text{Re}(\alpha + \beta))^2}}{2\text{Re}(\alpha + \beta)} \leq \alpha k < 3.
\]

Taking \(\alpha > 0\) and \(\beta = 1 - \alpha\) in Theorem 2, we have

**Example 1.** For \(f(z) \in A\) and \(0 \leq \alpha k < 3\),
\[
\frac{f(z)}{z} \prec \frac{1}{(1 - z)^k} \Rightarrow I_{\alpha,1-\alpha}(f(z)) \prec 2F_1(1, \alpha k; 2; z)^{1/\alpha},
\]
where
\[
I_{\alpha,1-\alpha}(f(z)) = \left\{\frac{1}{(1 - \alpha)} \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt\right\}^{1/\alpha}.
\]

Finally, if we make \(\alpha > 0\) and \(\beta = 1 + i - \alpha\) in Theorem 2, then we have

**Example 2.** For \(f(z) \in A\) and \((\sqrt{5} - 1)/2 \leq \alpha k < 3\),
\[
\frac{f(z)}{z} \prec \frac{1}{(1 - z)^k} \Rightarrow I_{\alpha,1+i-\alpha}(f(z)) \prec 2F_1(1 + i, \alpha k; 2 + i; z)^{1/\alpha},
\]
where
\[
I_{\alpha,1+i-\alpha}(f(z)) = \left\{\frac{1 + i}{(1 + i - \alpha)} \int_0^z i^t \left(\frac{f(t)}{t}\right)^\alpha dt\right\}^{1/\alpha}.
\]
References


Shigeyoshi Owa
Department of mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan