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On some inverse properties for univalent functions

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Abstract. The object of the present paper is to investigate some inverse properties for univalent functions in the open unit disk $U$. Starlikeness and convexity for functions in $U$ are shown.

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S$ be the subclass of $A$ consisting of functions $f(z)$ which are univalent in $U$. It is very famous as Bieberbach conjecture that if $f(z) \in S$, then

$$|a_n| \leq n \quad (n = 2, 3, 4, \ldots).$$

(1.2)

The equality holds true for the Koebe function $k(z)$ which given by

$$k(z) = \frac{z}{(1 - e^{i\theta}z)^2} \quad (\theta \in \mathbb{R}).$$

(1.3)

This Bieberbach conjecture was proved by de Branges [1].

In the present paper, we investigate some inverse properties for functions $f(z)$ belonging to the class $S$.

Let $B$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the coefficient inequalities (1.2). Recently, Kim and Nunokawa [2, Theorem 1] proved that if $f(z) \in B$, then $f(z)$ is univalent in $|z| < r_0$, where $r_0$ is the unique solution of the equation

$$2r^3 - 6r^2 + 7r - 1 = 0.$$ 

(1.4)

This result is sharp.

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2 Inverse properties

For the functions $f(z)$ belonging to the class $B$, we derive

**Theorem 1.** If $f(z) \in B$, then

$$\frac{2r^2 - 4r + 1}{(1 - r)^2} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1 - r)^2}$$

(2.1)

for $|z| = r < 1$. The result is sharp for $f(z) = z/(1 - e^{i\theta}z)^{2}$.

**Proof.** Since $f(z) \in B$ satisfies (1.2), we have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n$$

$$\leq |z| + \sum_{n=2}^{\infty} n |z|^n = \frac{r}{(1 - r)^2}$$

(2.2)

for $|z| = r < 1$.

Therefore, $f(z)$ absolutely converges in $U$, and so, $f(z)$ is analytic in $U$. On the other hand, we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n$$

$$\geq r - \sum_{n=2}^{\infty} nr^n \geq \frac{(2r^2 - 4r + 1)r}{(1 - r)^2}$$

(2.3)

for $|z| = r < 1$.

\[\square\]

**Remark 1.** Theorem 1 shows that $|f(z)/z| > 0$ for $|z| < r_1 = \frac{2 - \sqrt{2}}{2} \approx 0.29289$. Thus Theorem 1 is sharp.

Next we show

**Theorem 2.** If $f(z) \in B$, then $f(z)$ is univalent and starlike in $|z| < r_2$, where

$$r_2 = \frac{1}{1 + \sqrt{2}} \left( 1 - \sqrt{\frac{e}{2e - 1}} \right) \approx 0.08998.$$  

(2.4)
Proof. By means of Theorem 1, we have $|f(z)/z| > 0$ in $|z| < r_1 = (2 - \sqrt{2})/2$, and therefore, $\log(f(z)/z)$ is harmonic in $|z| < r_1$.

From the harmonic function theory, we know that

$$\log \frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \left( \log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{\zeta + z}{\zeta - z} d\varphi,$$

where $\zeta = re^{i\varphi}(0 \leq \varphi \leq 2\pi), z = re^{i\theta}(0 \leq \theta \leq 2\pi)$, and $0 \leq r < \rho \leq r_1 = (2 - \sqrt{2})/2$.

By using the logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi} \int_0^{2\pi} \left( \log \left| \frac{f(\zeta)}{\zeta} \right| \right) \frac{2\zeta z}{(\zeta - z)^2} d\varphi.\quad (2.6)$$

Because, we have

$$\frac{1}{(1-r)^2} < \frac{(1-r)^2}{2r^2 - 4r + 1}\quad (2.7)$$

for $|z| = r < 1$, then, from Theorem 1 and (2.7), we derive

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 - \frac{1}{2\pi} \int_0^{2\pi} \left( \max_{|\zeta| = \rho} \left| \log \left| \frac{f(\zeta)}{\zeta} \right| \right| \right) \frac{2\rho r}{\rho^2 - 2\rho \cos(\varphi - \theta) + r^2} d\varphi\quad (2.8)$$

where $0 \leq r < \rho < r_1 = (2 - \sqrt{2})/2$.

Putting $\rho = (1 + \sqrt{2})r$, we have

$$\frac{2\rho r}{\rho^2 - r^2} \log \left( \frac{\rho^2 - 2\rho + 1}{2\rho^2 - 4\rho + 1} \right) = \log \left( \frac{1}{2} + \frac{1}{4 \left\{ (1 + \sqrt{2})r - 1 \right\}^2 - 2} \right) = 1.\quad (2.9)$$

Consequently, we see that (2.8) and (2.9) imply

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0\quad (2.10)$$

in $|z| < r_2$, where $r_2$ is the smallest positive root of the equation

$$\frac{1}{2} + \frac{1}{4 \left\{ (1 + \sqrt{2})r - 1 \right\}^2 - 2} = e\quad (2.11)$$

or

$$r_2 = \frac{1}{1 + \sqrt{2}} \left( 1 - \sqrt{\frac{e}{2e - 1}} \right) \approx 0.08998.\quad (2.12)$$

This completes the proof of Theorem 2. \qed
Remark 2. In the proof of Theorem 2, we put $\rho = (1 + \sqrt{2})r$. But we don’t prove that this is best or not. Therefore, Theorem 2 is not sharp.

From Theorem 2, we make

Corollary 1. If a function $f(z)$ of the form (1.1) satisfies

$$|a_n| \leq 1 \quad (n = 2, 3, 4, \ldots),$$

then $f(z)$ is univalent and convex in $|z| < r_2$.

Applying the same method as the proof of Theorem 2, we can obtain some routh results on the other cases, but we expect that someone get exact results.

References
