Title: RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS (New Extension of Historical Theorems for Univalent Function Theory)

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Citation: 数理解析研究所講究録 (2000), 1164: 45-52

Issue Date: 2000-07

Type: Departmental Bulletin Paper

URL: http://hdl.handle.net/2433/64310

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RADIUS OF STRONGLY STARLIKENESS
FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. We determine the radius of $p$-valent strongly starlike of order $\gamma$ for certain polynomials of the form $F(z) = f(z) \cdot [Q(z)]^\frac{g}{n}$.

1. Introduction

Let $A_p$ ($p$ is fixed integer $\geq 1$) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in the unit disk $D = \{z \in D : |z| < 1\}$. Let $\Omega$ denote the class of bounded function $w(z)$ analytic in $D$ and satisfying the conditions $w(0) = 0$ and $|w(z)| \leq |z|, z \in D$. We use $P$ to denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic in $D$ and a positive real part there.

For $0 \leq \alpha < p$ and $|\lambda| < \frac{\pi}{2}$, we denote by $S_p^\lambda(\alpha)$, the family of functions $g(z) \in A_p$ which satisfy

(1.1) \[
\frac{zg'(z)}{g(z)} < \frac{p + \{2(p-\alpha) \cos \lambda \cdot \exp(-i\lambda) - p\}z}{1-z}, \quad z \in D
\]

where $<$ means subordination. From the definition of subordination it follows that $g(z) \in A_p$ has a representation

\[
\frac{zg'(z)}{g(z)} = \frac{p + \{2(p-\alpha) \cos \lambda \cdot \exp(-i\lambda) - p\}w(z)}{1-w(z)}
\]

where $w(z) \in \Omega$. Clearly, $S_p^\lambda(\alpha)$ is subclass of $p$-valent $\lambda$-spiral functions of order $\alpha$. For $\lambda = 0$, we have the class $S_p^*(\alpha), 0 \leq \alpha < p$, of $p$-valent starlike functions of order $\alpha$, investigated by Goluzina [3].

1991 AMS Subject Classification : 30C45.

Key words and phrases. subordination, $p$-valent strongly starlike of order $\gamma$.

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As noted in a function is \( p \)-valent strongly starlike of order \( \gamma \), \( 0 < \gamma \leq 1 \) if

\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| \leq \frac{\pi}{2} \gamma.
\]

Basgöze(1969) has obtained sharp inequalities of univalence (starlikeness) for certain polynomials of the form \( F(z) = f(z) \cdot [Q(z)]^\beta \), where \( \beta \) is real and \( Q(z) \) is a polynomial of degree \( n > 0 \) all of whose zeros are outside or on the unit circle \( \{z \in D : |z| = 1\} \). Rajasekaran [5] extended Basgöze’s results for certain classes of analytic functions of the form. Recently, J. Patel [4] generalized some of the work of Rajasekaran and Basgöze for functions belonging to the class \( S_p^\lambda(\alpha) \). That is, determine the radius of starlikeness for some classes of \( p \)-valent analytic functions of the polynomial form \( F(z) \).

In the present paper, we will extend the results of J. Patel. Thus, we determine the radius of \( p \)-valent strongly starlike of order \( \gamma \) for the polynomials of the form \( F(z) \) in the such problems.

2. Some Lemmas

Before proving our next results, we need the following Lemmas.

**Lemma 1 (A. Gangadharan [2]).** For \( |z| \leq r < 1 \), \( |z_k| = R > r \), we have

\[
\left| \frac{z}{z-z_k} + \frac{r^2}{R^2-r^2} \right| \leq \frac{Rr}{R^2-r^2}.
\]

**Lemma 2 (Ratti [6]).** If \( \phi(z) \) is analytic in \( D \) and \( |\phi(z)| \leq 1 \) for \( z \in D \), then for \( |z| = r < 1 \),

\[
\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{1}{1-r}.
\]

**Lemma 3 (Causey and Merke’s [1]).** If \( p(z) = 1 + c_1 z + c_2 z + \cdots \in P \), then for \( |z| = r < 1 \),

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2}.
\]

This estimate is sharp.
Lemma 4 (J. Patel [4]). Suppose $g(z) \in S_{p}^{\lambda}(\alpha)$. Then for $|z| = r < 1$,

\[
\left| \frac{zg'(z)}{g(z)} - \left( p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}.
\]

The result is sharp.

Lemma 5 (A. Gangadharan [2]). If $R_{a} \leq \left( Re a \right) \sin \left( \frac{\pi}{2} \gamma \right) - \left( Im a \right) \cos \left( \frac{\pi}{2} \gamma \right)$, $Im a \geq 0$, the disk $|w - a| \leq Ra$ is contained in the sector $|\arg w| \leq \frac{\pi}{2} \gamma$, $0 < \gamma \leq 1$.

3. Main Theorem

**Theorem 1.** Suppose

(3.1) \quad F(z) = f(z)[Q(z)]^{\frac{\beta}{n}}

where $\beta$ is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$, $R \geq 1$. If $f(z) \in A_{p}$ satisfies

(3.2) \quad Re \left[ \left( \frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}} \right] > 0, \quad 0 < \delta \leq 1, \quad z \in D

and

(3.3) \quad Re \left[ \frac{g(z)}{h(z)} \right] > 0, \quad z \in D

for some $g(z) \in A_{p}$ and $h(z) \in S_{p}^{\lambda}(\alpha)$, then $F(z)$ is $p$-valent strongly starlike of order $\gamma$ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

(3.4) \quad \begin{align*}
& r^4 \left[ (p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) \right] \\
& + r^3 \left[ |\beta|R + 2(p - \alpha) \cos \lambda + 2(\delta + 1) \right] \\
& - r^2 \left[ (p(1 + R^2) + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha)R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) \right] \\
& - r |\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2(\delta + 1)R^2 \]
& + pR^2 \sin \frac{\pi}{2} \gamma.
\end{align*}
Proof. We choose a suitable branch of \((f(z)/g(z))^{\frac{1}{\delta}}\) so that \((f(z)/g(z))^{\frac{1}{\delta}}\) is analytic in \(D\) and takes the value 1 at \(z = 0\). Thus form (3.2) and (3.3), we have
\[
F(z) = p_1^\delta(z)p_2h(z)[Q(z)]_n^\beta
\]
where \(p_j(z) \in P \ (j = 1, 2)\).

Then we have
\[
\frac{zF'(z)}{F(z)} = \delta \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{zh'(z)}{h(z)} + \beta \sum_{k=1}^{n} \frac{z}{z-z_k}.
\]

Since \(h(z) \in S_\lambda^\alpha\), by Lemma 4, we have
\[
\left| \frac{zh'(z)}{h(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos \lambda}{1-r^2} \right\} \right| \leq \frac{2(p-\alpha)r \cos \lambda}{1-r^2}.
\]

Using (3.6) and (3.7) an Lemma 1, 3, we get
\[
\left| \frac{zF'(z)}{F(z)} - \left\{p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos \lambda}{1-r^2} \right\} \right| \leq \frac{2((p-\alpha)r \cos \lambda + r(\delta + 1))}{1-r^2} + \beta |Rr|.
\]

Using Lemma 5, we get that the about disk is contained in the sector \(|\arg w| < \frac{\pi}{2} \gamma\) provided the inequality
\[
\frac{2((p-\alpha)r \cos \lambda + r(\delta + 1))}{1-r^2} + \beta |Rr| \leq \left\{p + \frac{2(p-\alpha)r^2 \cos \lambda}{1-r^2} \right\} \sin \frac{\pi}{2} \gamma - \frac{2(p-\alpha)r^2 \sin \lambda \cos \lambda}{1-r^2} \cos \frac{\pi}{2} \gamma
\]
is satisfied. The above inequality simplifies to \(T(r) \geq 0\), where
\[
T(r) = r^4 \left[ (p-2(p-\alpha) \cos \lambda + \beta) \sin \frac{\pi}{2} \gamma + (p-\alpha) \sin 2\lambda \cos \frac{\pi}{2} \gamma \right]
+ r^3[[\beta|R + 2(p-\alpha) \cos \lambda + 2(\delta + 1)]
+ r^2 \left[ (-pR^2 - p + 2(p-\alpha)R^2 \cos \lambda - \beta) \sin \frac{\pi}{2} \gamma - (p-\alpha)R^2 \sin 2\lambda \cos \frac{\pi}{2} \gamma \right]
- r[[\beta|R + 2(p-\alpha)R^2 \cos \lambda + 2(\delta + 1)R^2] + pR^2 \sin \frac{\pi}{2} \gamma
\]
Since \(T(0) > 0\) and \(T(1) < 1\), there exists a real root of \(T(r) = 0\) in \((0, 1)\). Let \(R(\gamma)\) be the smallest positive root of \(T(r) = 0\) in \((0, 1)\). Then \(F\) is \(p\)-valent strongly starlike of order \(\gamma\) in \(|z| < R(\gamma)\).
**Remark.** For $R = 1$ and $\gamma = 1$, the result of theorem reduces to a result of J. Patel.

**Theorem 2.** Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satifies (3.2) for some $g(z) \in S^\lambda_p(\alpha)$, then $F(z)$ is $p$-valent strongly starlike of order $\gamma$ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$r^4 \left[ (p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) \right] + r^3 |\beta| R + 2(p - \alpha) \cos \lambda + 2\delta$$

$$- r^2 \left[ (p + R^2) + \beta \right] \sin \frac{\pi}{2} \gamma + 2(p - \alpha) R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right)$$

$$- r \left[ |\beta| R + 2(p - \alpha) R^2 \cos \lambda + 2\delta R^2 \right] + p R^2 \sin \frac{\pi}{2} \gamma.$$  

**Proof.** If $f(z) \in A_p$ satifies (3.2) for some $g(z) \in S^\lambda_p(\alpha)$, then

$$\frac{z F'(z)}{F(z)} = \delta \cdot \frac{z p'(z)}{p(z)} + \frac{z g'(z)}{g(z)} + \frac{\beta}{n} \sum_{k=1}^{n} \frac{z}{z - z_k}.$$  

Using Lemma 4, we get

$$\left| \frac{z g'(z)}{g(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda} r^2 \cos \lambda}{1 - r^2} \right\} \right| \leq \frac{2(p - \alpha) r \cos \lambda}{1 - r^2}.$$  

By (3.10) and (3.11) and Lemma 1, 3, we have

$$\left| \frac{z F'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda} r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right\} \right| \leq \frac{2\{(p - \alpha) r \cos \lambda + r \delta\}}{1 - r^2} + \frac{|\beta| R r}{R^2 - r^2}.$$  

The remaining parts of the proof can be proved by similar method given in the Theorem 1.

With $\lambda = 0, \beta = 0, \delta = 1, R = 1$ and $\gamma = 1$, Theorem 2 gives

**Corollary 1.** Suppose $f(z)$ is in $A_p$. If $Re \left( \frac{f(z)}{g(z)} \right) > 0$ for $z \in D$ and $g(z) \in S^*_p(\alpha)$, then $f(z)$ is $p$-valent starlike for

$$|z| < \frac{p}{(p + 1 - \alpha) + \sqrt{\alpha^2 - 2\alpha + 2p + 1}}.$$
Theorem 3. Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies

$$\left| \left( \frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}} - 1 \right| < 1, \quad 0 < \delta \leq 1, \quad p \sin \frac{\pi}{2} \gamma > \delta$$

and

$$\text{Re} \left( \frac{g(z)}{h(z)} \right) > 0, \quad z \in D$$

for some $g(z) \in A_p$ and $h(z) \in S_{\alpha}^p$, then $F(z)$ is $p$-valent strongly starlike of order $\gamma$ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$r^4 \left[ (p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) \right]$$

$$+ r^3 \left[ |\beta|R + 2(p - \alpha) \cos \lambda + 2 + \delta \right]$$

$$- r^2 \left[ (p(1 + R^2) + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) + \delta \right]$$

$$- r (|\beta|R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2) + p R^2 \sin \frac{\pi}{2} \gamma - \delta R^2.$$

Proof. We choose a suitable branch of $\left( \frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}}$ so that $\left( \frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}}$ is analytic in $D$ and takes the value 1 at $z = 0$. From (3.12), we deduce that

$$f(z) = g(z) \cdot (1 + w(z))^\delta,$$

where $w(z) \in \Omega$. So that

$$F(z) = p(z) \cdot h(z) \cdot (1 + z\phi(z))^\delta [Q(z)]^\frac{\beta}{n}$$

where $\phi(z)$ is analytic in $D$ and satisfies $|\phi(z)| \leq 1$ and $p \in P$ for $z \in D$.

We have

$$\frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)} + \delta \left( \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right) + \frac{\beta}{n} \sum_{k=1}^{n} \frac{z}{z - z_k}.$$  

Using Lemma 4 and (3.14), we have

$$\left| \frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right\} \right| \leq \frac{2(p - \alpha)r \cos \lambda + r}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}$$

So, using Lemma 5 and (3.15), the result can be proved by similar method given in the Theorem 1.
**Theorem 4.** Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies (3.12) for some $g(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is $p$-valent strongly starlike of order $\gamma$ in $|z| < R(\gamma)$, where $R(\gamma)$ is smallest positive root of the equation

$$
\begin{align*}
&\frac{r^4}{4} \left[(p+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)\cos\lambda\sin\left(\lambda - \frac{\pi}{2}\gamma\right)\right] \\
&+ r^3 \left[|\beta| R + 2(p-\alpha)\cos\lambda + \delta\right] \\
&- r^2 \left[(p(1+R^2)+\beta)\sin\frac{\pi}{2}\gamma + 2(p-\alpha)R^2\cos\lambda\sin\left(\lambda - \frac{\pi}{2}\gamma\right) + \delta\right] \\
&- r \left[(|\beta| R + 2(p-\alpha)R^2\cos\lambda + \delta R^2\right] \\
&+ pR^2 \sin\frac{\pi}{2}\gamma - \delta R^2.
\end{align*}
$$

(3.16)

**Proof.** We choose a suitable of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\tau}$ is analytic in $D$ and takes the value 1 at $z = 0$. Since $f(z) \in A_p$ (3.12) for some $g(z) \in S_p^\lambda(\alpha)$, we have

$$
F(z) = g(z)(1 + z\phi(z))[Q(z)]^\frac{\beta}{n}
$$

where $\phi(z)$ is analytic in $D$ and satisfies the condition $|\phi(z)| \leq 1$ for $z \in D$. Thus, we have

$$
(3.17) \quad \frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \delta \left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}\right) + \frac{\beta}{n} \sum_{k=1}^{n} \frac{z}{z-z_k}.
$$

Using Lemma 4 and (3.17), we get

$$
(3.18) \quad \left|\frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos\lambda}{1-r^2} \right\} \right| \leq \frac{2(p-\alpha)r \cos\lambda + \delta(1+r)}{1-r^2} + \frac{\beta |R|r}{R^2 - r^2}.
$$

Using Lemma 5 and (3.18) and similar method in the Theorem 1, we get the Theorem 4.

**Remark.** Some of the results of J. Patel can be obtained form the Theorem 4 by taking $R = 1, \gamma = 1$. 

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