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SURVEY ON INTEGRAL TRANSFORMS IN THE UNIVALENT FUNCTION THEORY

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ABSTRACT. The main object of this article is a survey covering both recent and older results on the topic. A number of further generalizations, relevant to the conjectures and open problems, are also considered.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form :

$$f(z)=\sum_{n=0}^{\infty}a_{n+1}z^{n+1} \qquad (a_1:=1), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}.$$

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are *univalent* in the unit disk $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *starlike of order* α $(0\leq\alpha<1)$ in $\mathcal{U}$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathcal{U}\ ;\ 0\leq\alpha<1). \tag{1.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α in $\mathcal{U}$.

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A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *convex of order* α in $\mathcal{U}$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathcal{U}\ ;\ 0\leq\alpha<1). \tag{1.3}$$

We denote by $\mathcal{K}(\alpha)$ the class of all functions in $\mathcal{S}$ which are convex of order α in $\mathcal{U}$.

It follows readily from (1.2) and (1.3) that

$$f(z)\in\mathcal{K}(\alpha)\iff zf'(z)\in\mathcal{S}^*(\alpha) \qquad (0\leq\alpha<1). \tag{1.4}$$

If we let $D=z\frac{d}{dz}$, the equation (1.4) means that

$$f\in\mathcal{K}(\alpha)\iff Df\in\mathcal{S}^*(\alpha). \tag{1.5}$$

We note also that

$$\mathcal{S}^*(\alpha)\subseteq\mathcal{S}^*(0)\equiv\mathcal{S}^* \text{ and } \mathcal{K}(\alpha)\subseteq\mathcal{K}(0)\equiv\mathcal{K} \quad (0\leq\alpha<1), \tag{1.6}$$

where $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex in $\mathcal{U}$, respectively.

The first integral transform defined a subclass of $\mathcal{S}$ was introduced by J.W. Alexander in 1915. In [1], Alexander showed that the operator

$$F_0(f)(z)\equiv F_0(z)=\int_0^z\frac{f(t)}{t}dt \tag{1.7}$$

maps $\mathcal{S}^*$ onto $\mathcal{K}$. From (1.4), it is clear that

$$f\in\mathcal{S}^*(\alpha)\iff F_0(f)\in\mathcal{K}(\alpha).$$

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *close-to-convex* in $\mathcal{U}$ if there exists a convex function $g(z)$ such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right)>0 \quad (z\in\mathcal{U}). \tag{1.8}$$

We shall denote by $\mathcal{C}$ the class of close-to-convex functions in $\mathcal{U}$.

Let $a,b,$ and c be complex numbers with $c\neq 0,-1,-2,\cdots$. Then the *Gaussian hypergeometric function* ${}_2F_1(z)$ is defined by

$$\begin{aligned}{}_2F_1(z) &\equiv {}_2F_1(a,b;c;z)\\ &:=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\end{aligned} \tag{1.9}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(1.10)\qquad (\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\cdots\}). \end{cases}$$

If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then there is a probability measure $\mu(t)$ on $[0,1]$ (*cf., e.g.,* Whittaker and Watson [37,p.293]) such that

$$(1.11)\qquad {}_2F_1(a,b;c;z) = \int_0^1 (1-zt)^{-a}\, d\mu(t)$$

In [22], Miller and Mocanu determined conditions for the Gaussian hypergeometric function to be starlike in $\mathcal{U}$ and later by [7,Choi et al.].

For the functions $f_j(z)(j=1,2)$ defined by

$$(1.12)\qquad f_j(z) := \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \qquad (a_{j,1} := 1;\ j=1,2),$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(1.13)\qquad (f_1 * f_2)(z) := \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1} \qquad (a_{j,1} := 1;\ j=1,2).$$

From the definition of Hadamard product, it is easy to see that

$$(1.14)\qquad F_0(f)(z) = -\log(1-z) * f(z)$$

and

$$(1.15)\qquad Df(z) = \frac{z}{(1-z)^2} * f(z).$$

Now define the function $\phi(a,c,z)$ by

$$(1.16)\qquad \phi(a,c,z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0,-1,-2,\cdots;\ z \in \mathcal{U}),$$

so that $\phi(a, c, z)$ is an *incomplete* Beta function with

$$\phi(a, c, z) = z\ {}_2F_1(1, a; c; z). \tag{1.17}$$

Note that

$$-\log(1 - z) = \phi(1, 2, z) \tag{1.18}$$

and

$$\frac{z}{(1-z)^2} = \phi(2, 1, z). \tag{1.19}$$

Corresponding to the function $\phi(a, c, z)$, Carlson and Shaffer [6] defined a linear operator $\mathcal{L}(a, c)$ on $\mathcal{A}$ by the convolution [6, p. 738, Eqution (2.2)]:

$$\mathcal{L}(a, c)f(z) = \phi(a, c, z) * f(z) \qquad (f \in \mathcal{A}). \tag{1.20}$$

Clearly, $\mathcal{L}(a, c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c, a)$ is the inverse of $\mathcal{L}(a, c)$, provided that

$$a \neq 0, -1, -2, \cdots .$$

In [11], Kim and Srivastava investigated several interesting properties of Carlson-Shaffer linear operator associated with various subclasses of univalent functions.

A function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $V(a, c; \alpha)$ if $\mathcal{L}(a, c)f$ is an element of $\mathcal{S}^*(\alpha)$. Further, a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $W(a, c; \alpha)$ if $zf'(z)$ is an element of $V(a, c; \alpha)$. Then it is easily verified that

$$W(a, c; \alpha) = \mathcal{L}(1, 2)V(a, c; \alpha) = \mathcal{L}(c, a)\mathcal{K}(\alpha),$$

$$V(a, c; \alpha) = \mathcal{L}(2, 1)W(a, c; \alpha) = \mathcal{L}(c, a)\mathcal{S}^*(\alpha),$$

$$\mathcal{K}(\alpha) = W(a, a; \alpha) = \mathcal{L}(1, 2)V(a, a; \alpha),$$

and

$$\mathcal{S}^*(\alpha) = V(a, a; \alpha) = \mathcal{L}(2, 1)W(a, a; \alpha),$$

See [36, Srivastava and Owa] for the further information of these classes.

Ruscheweyh [32] introduced an operator $\mathcal{D}^\lambda : \mathcal{A} \to \mathcal{A}$ defined by the convolution:

$$\mathcal{D}^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda \geq -1;\ z \in \mathcal{U}), \tag{1.21}$$

which implies that

$$\mathcal{D}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.22}$$

Making use of (1.17) and (1.21), we also have

$$\mathcal{D}^\lambda f(z) = \mathcal{L}(\lambda+1, 1)f(z).$$

Since $\mathcal{D}^0 f = f$ and $\mathcal{D}^1 f = Df$, from (1.22) we have

$$f \in \mathcal{S}^*(\alpha) \iff \mathrm{Re}\{\frac{\mathcal{D}^1 f(z)}{\mathcal{D}^0 f(z)}\} > \alpha$$

and

$$f \in \mathcal{K}(\alpha) \iff \mathrm{Re}\{\frac{\mathcal{D}^2 f(z)}{\mathcal{D}^1 f(z)}\} > \frac{\alpha+1}{2}.$$

Hence Ruscheweyh gave the following problem in his paper [32] :

Problem. Determine the smallest values δ_n, such that the condition $\mathrm{Re}(\frac{\mathcal{D}^{n+1}f}{\mathcal{D}^n f}) > \delta_n$, $z \in \mathcal{U}$, guarantees the univalence of $f \in \mathcal{A}$. It is known that $\delta_0 = 0$, $\delta_1 = 1/4$.

2. History and Problems of Linear Integral Transforms

This section is based on the survey article of Rønning [30]. The main object of this section is to study integral transforms of the type

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \tag{2.1}$$

where $\lambda : [0,1] \to \mathbb{R}$, $\lambda(t) \geq 0$ and $\int_0^1 \lambda(t)dt = 1$.

For examples, the following authors defined linear integral transforms with special types of $\lambda(t)$.

(1) Bernardi [3] :

$$\lambda(t) = (c+1)t^c, \quad c > -1. \tag{2.2}$$

$c = 0$: The Alexander (Biernacki) transform (see(1.7)).
$c = 1$: The Libera transform [19].

(2) Komatu [16] :

$$\lambda(t) = \frac{(c+1)^\delta}{\Gamma(\delta)}t^c(\log(1/t))^{\delta-1}, \quad c > -1, \ \delta \geq 0. \tag{2.3}$$

(3) Carlson and Shaffer [6] :

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1}(1-t)^{c-b-1}, \quad c > b > 0. \tag{2.4}$$

(4) Hohlov [10] (or Kim and Rønning [13]) :

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right) \tag{2.5}$$
$$(a > 0,\ b > 0,\ c > a+b-1).$$

Remark 1. We see that

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{z}{1-tz} dt * f(z).$$

Then, with λ as in (2.5) we can write

$$V_\lambda(f)(z) = z\,{}_2F_1(a, b; c; z) * f(z). \tag{2.6}$$

Remark 2. In (2.1), if we take λ as in (2.4), from (2.6) it is easy to see that

$$V_\lambda(f)(z) = \mathcal{L}(b, c) f(z),$$

where $\mathcal{L}(b, c)$ is defined by (1.20).

Let F_0 be the Alexander operator defined by (1.7). In 1960 Biernacki [4] claimed that $f \in \mathcal{S}$ implies $F_0 \in \mathcal{S}$, but this turned out to be wrong. A counterexample was given by Krzyż and Lewandowski [17] who proved that

$$f(z) = \frac{z}{(1-iz)^{1-i}}$$

is spiral-like in $\mathcal{U}$, and hence in $\mathcal{S}$, but that the corresponding F_0 is in fact infinite-valent in $\mathcal{U}$ (*cf.* [Duren, 8]). From this fact, we have the following open problem :

Problem 2.1. Find the radius of univalence in the set $\{F_0(f) : f \in \mathcal{S}\}$.

Merkes and Wright [20] proved that $F_0(\mathcal{C}) \subset \mathcal{C}$ and also Libera [19] proved that if f is a member of $\mathcal{K}$, $\mathcal{S}^*$, or $\mathcal{C}$ then Libera transform F_1 belongs to the same class, where

$$F_1(z) = 2 \int_0^1 f(tz) dt. \tag{2.7}$$

This result was extended by Bernardi [3] and he defined the more general transform F_c by (2.2). In fact,

$$F_c(z) = (c+1)\int_0^1 t^{c-1} f(tz)dt, \quad c > -1. \tag{2.8}$$

Also for the Libera transform it is so that there is an $f \in \mathcal{S}$ such that F_1 is infinite-valent in $\mathcal{U}$.

In 1979, Komatu [15] presented two conjectures and considered the linear integral transform

$$\begin{aligned} F_0^\delta(z) &= \frac{2^\delta}{\Gamma(\delta)}\int_0^1 (\log(1/t))^{\delta-1} f(tz)dt \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{n^\delta} z^n, \end{aligned} \tag{2.9}$$

where $f(z)$ is given by (1.1). Note that this transform is a special case of the transform defined by (2.3).

Conjecture. If f is a member of $\mathcal{S}^*$ or $\mathcal{K}$, then Komatu transform F_0^δ belongs to same class at least for $\delta \geq 1$.

In 1983, Lewis [18] proved that

$$f_\delta(z) = z + \sum_{n=2}^{\infty} \frac{1}{n^\delta} z^n$$

is convex for all $\delta \geq 0$. Since $F_0^\delta(z) = f_\delta(z) * f(z)$, by the convolution properties for the Polya-Schoenberg conjecture [31] (or [8, p.248]), the Komatu conjecture is true for all $\delta \geq 0$.

If we let

$$H_{a,b,c}(f)(z) = z\,{}_2F_1(a,b;c;z) * f(z) \tag{2.10}$$

for $f \in \mathcal{A}$, by using the Gauss summation theorem, Hohlov [10] determined the conditions to guarantee that $H_{a,b,c}(f)$ will be univalent in $\mathcal{U}$ for a function f in $\mathcal{S}$. We note that the Hohlov operator is a natural choice for studying the geometric properties of it because of its interaction with geometric function theory for the special operator popularly known as Bernardi operator. In fact the Bernardi operator F_η in (2.8) is a special case of the Hohlov operator $H_{a,b,c}$ when $a = 1$, $b = 1+\eta$, $c = 2+\eta$ with $\operatorname{Re}\eta > -1$.

For $0 \leq \gamma \leq 1$ we define the class

$$\mathcal{P}_\gamma(\beta) = \{f \in \mathcal{A} | \exists \varphi \in \mathbb{R} | \mathrm{Re}\{e^{i\varphi}((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta)\} > 0,\ z \in \mathcal{U}\}.$$

Noshiro-Warschawski Theorem gives that

$$\mathrm{Re} f'(z) > 0 \Rightarrow f(z) \in \mathcal{S} \qquad (z \in \mathcal{U}).$$

This means that

$$\mathrm{Re}\frac{f(z)}{z} > 0 \Rightarrow F_0(f) \in \mathcal{S} \qquad (z \in \mathcal{U}).$$

The behaviour of (2.8) was investigated by Singh and Singh [34] who proved that $-1 < c \leq 0$, $F_c \in \mathcal{S}^*$ if $\mathrm{Re} f'(z) > 0$ in $\mathcal{U}$. Note that this result gives no information about the case $c > 0$, so the Libera transform acting on $\mathcal{P}_1(0)$ is not covered by this result. In 1986 Mocanu proved that

$$\mathrm{Re} f'(z) > 0 \Rightarrow F_1 \in \mathcal{S}^*,$$

and Nunokawa [26] improved this result.

In[35], Singh and Singh also proved that

$$\mathrm{Re}\{f'(z) + zf''(z)\} > -\frac{1}{4} \Rightarrow f(z) \in \mathcal{S}^* \qquad (z \in \mathcal{U}).$$

This result implies that

$$\mathrm{Re}\{f'(z)\} > -\frac{1}{4} \Rightarrow F_0(f) \in \mathcal{S}^* \qquad (z \in \mathcal{U}).$$

After Miller and Mocanu published their papers (*cf.* [23], [24]), many authors have used differential subordination techniques, and these have not given sharp results. A new approach was taken by Fournier and Ruscheweyh in their paper [9], using the duality theory for convolutions. They found the sharp bound $\beta = \beta_c$ such that $F_c(\mathcal{P}_1(\beta)) \subset \mathcal{S}^*$. For examples, they gave that

$$\beta_0 = \frac{1 - 2\log(2)}{2 - 2\log(2)} = -0.629\ldots, \qquad \beta_1 = \frac{3 - 4\log(2)}{2 - 4\log(2)} = -0.294\ldots,$$

$$\beta_2 = \frac{4 - 6\log(2)}{5 - 6\log(2)} = -0.188\ldots.$$

This appears to be an adequate tool when dealing with these types of integral transforms, and tends to give sharp bounds (see also [29]).

Recently, by using the similar technique, Kim and Rønning [13] improved

Theorem 1. *Let*

$$L_{\Lambda_\gamma}(h) = \inf_{z\in\mathcal{U}} \int_0^1 t^{1/\gamma-1}\Lambda_\gamma(t)\left(\mathrm{Re}\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt,$$

where

$$\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}}ds, \quad \gamma > 0.$$

Let β be given by

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g_\gamma(t)dt,$$

where g is the solution to

$$\frac{d}{dt}(t^{1/\gamma}(1+g(t))) = \frac{2}{\gamma}\frac{t^{1/\gamma-1}}{(1+t)^2}, \quad g(0) = 1.$$

Then

$$V_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}^* \iff L_{\Lambda_\gamma}(h) \geq 0,$$

where

$$h(z) = \frac{z(1+\frac{x-1}{2}z)}{(1-z)^2}, \qquad |x| = 1.$$

Using Theorem 1 and (2.5) we obtain

Corollary 1. *Let $1/2 \leq \gamma \leq 1$ and $g_\gamma(t)$ be defined as above. Define $\beta = \beta(a,b,c,\gamma)$ by*

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)\Gamma(c-a-b+1)}\times$$
$$\int_0^1 t^{b-1}(1-t)^{c-a-b}{}_2F_1(c-a,1-a;c-a-b+1;1-t)g_\gamma(t)dt.$$

Then for $f \in \mathcal{P}_\gamma(\beta)$, $0 < a \leq 1$, $0 < b \leq 2$ and $c \geq a+b$ we have $H_{a,b,c}(f) \in \mathcal{S}^*$. The value of β is sharp.

But Corollary 1 does not give the answer of all the cases in (2.5). Hence we suggest the open problems associated with our paper [13] :

Problem 2.2 In Corollary 1, determine the value β if $a > 0$, $b > 0$ and $c > a+b-1$.

Problem 2.3 Find conditions on β and $\lambda(t)$ such that

$$V_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}.$$

3. Problems of Non-Linear Integral Transforms

During the last several years, many authors have defined and developed several types of non-linear integral transforms which maps subsets of $\mathcal{S}$ into $\mathcal{S}$. In 1978, Miller, Mocanu and Reade [25] defined a univalent integral operator of the form

$$I(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}dt\right]^{1/\beta}$$

and provides extensions and sharpening of all previous results.

Also Miller and Mocanu [21] wrote short history of univalent integral operatos in the introduction of their paper. From the paper we can see the history of non-linear integral transforms roughly. Hence, in this section, we shall restrict to give problems of non-linear integral transforms.

From 1963 many papers have appeared concerning the non-linear integral transform

$$J_{\alpha}(f)(z) \equiv J_{\alpha}(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt, \tag{3.1}$$

where α is complex anf f is in a subclass of $\mathcal{S}$. In particular, if $0 \le \alpha \le 1$ then $J_{\alpha}(\mathcal{S}^*(\beta)) \subset \mathcal{K}(\alpha\beta + (1-\alpha))$.

Also Merkes and Wright [20] proved that

$$\text{(i)}\ -1 \le \alpha \le 3 \Rightarrow J_{\alpha}(\mathcal{K}) \subset \mathcal{C}.$$
$$\text{(ii)}\ -\frac{1}{2} \le \alpha \le \frac{3}{2} \Rightarrow J_{\alpha}(\mathcal{S}^*) \subset \mathcal{C}.$$
$$\text{(iii)}\ -\frac{1}{2} \le \alpha \le 1 \Rightarrow J_{\alpha}(\mathcal{C}) \subset \mathcal{C}.$$

In general, it is well-known ([14]) that if $|\alpha| \le \frac{1}{4}$, then $J_{\alpha}(\mathcal{S}) \subset \mathcal{S}$. But it remains many open problems associated with the inclusion theorems of the operator J_{α}.

Problem 3.1. Find the exact region of the exponents α which lead to the univalence of the operator J_{α}.

Problem 3.2. If $0 \le \alpha \le 1$, determine $\beta = \beta(\alpha)$such that $J_{\alpha}(\mathcal{P}_1(\beta)) \subset \mathcal{S}^*$. For example, if $\alpha = 1$, then $\beta = \frac{1-2\log 2}{2-2\log 2} = -0.624...$ (see [9]).

It is well known that if f is univalent in $|z| < 1$, then

$$I_{\alpha}(f)(z) \equiv I_{\alpha}(z) = \int_0^z [f'(t)]^{\alpha} dt \tag{3.2}$$

is univalent for each complex α of sufficient small modulus. Pfaltzgraff [28] derived this result with $|\alpha| \le \frac{1}{4}$. On the other hand, Royster [31] showed that I_α need not be univalent for any α with $|\alpha| > \frac{1}{3}$.

From the definitions (3.1) and (3.2) we see that

$$I_\alpha \circ J_1 = J_\alpha \quad \text{and} \quad I_\alpha = J_\alpha \circ D,$$

where $D = z\frac{d}{dz}$. For the non-linear integral operators I_α and J_α, Nunokawa also investigated some inclusion theorems associated with several subclasses of univalent functions (*cf.* [27]).

Let $f(z)$ be a locally univalent function on $\mathcal{U}$. we define the order of f by

$$\text{ord}(f) = \sup_{\zeta \in \mathcal{U}} \left| -\overline{\zeta} + \frac{1 - |\zeta|^2}{2} \frac{f''(\zeta)}{f'(\zeta)} \right|. \tag{3.3}$$

Then

$$\text{ord}(f) = \sup_{\zeta \in \mathcal{U}} |a_2(\zeta)|, \tag{3.4}$$

where $a_2(\zeta)$ is a second coefficient of disk automorphism

$$F(z,\zeta) = \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + a_2(\zeta)z^2 + \ldots.$$

Since

$$\frac{I_\alpha''(z)}{I_\alpha'(z)} = \alpha \frac{f''(z)}{f'(z)}, \tag{3.5}$$

In [28], Pfaltzgraff used usefully the equation (3.4) in his proof. Also from (3.1) we have

$$\frac{J_\alpha''(z)}{J_\alpha'(z)} = \alpha \frac{J_1''(z)}{J_1'(z)},$$

but J_1 does not preserve the univalence of f. Note that $J_1(z) = F_0(z)$, where $F_0(z)$ is defined by (1.7). In Section 2, we already mentioned that the radius of univalence of the operator F_0 is still unknown. Hence we have the following question :

Problem 3.3. If $f \in \mathcal{S}$, find $\text{ord}(J_1(f))$.

Remark 3. If $f \in \mathcal{S}^*$, then $F_0(f) \in \mathcal{K}$, so that $\mathrm{ord}(J_1(f)) = 1$.

For $f \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, we define

$$G_\alpha(f)(z) \equiv G_\alpha(z) = \int_0^z (f'(t))^\alpha \varphi(t)dt, \quad z \in \mathcal{U},$$

where $\varphi(z) = \frac{1+z}{1-z}$. Then

$$\frac{G'_\alpha(z)}{I'_\alpha(z)} = \varphi(z) = \frac{1+z}{1-z}, \tag{3.6}$$

where I_α is defined by (3.2). From (3.5) and (1.6), we are easy to see that if $0 \le \alpha \le 1$, then

$$I_\alpha(\mathcal{K}) \subset \mathcal{K}(1-\alpha) \subset \mathcal{K}. \tag{3.7}$$

Hence (3.6) and (3.7) imply that if $0 \le \alpha \le 1$ and $f \in \mathcal{K}$, then we see that $I_\alpha(f) \in \mathcal{K}$ and

$$\mathrm{Re}\left(\frac{G'_\alpha(z)}{I'_\alpha(z)}\right) > 0, \qquad (z \in \mathcal{U}).$$

This means that if $0 \le \alpha \le 1$, then from (1.8) we have

$$G_\alpha(\mathcal{K}) \subset \mathcal{C}.$$

In general,

$$I_\alpha(\mathcal{K}) \subset \mathcal{K} \Rightarrow G_\alpha(\mathcal{K}) \subset \mathcal{C}. \tag{3.8}$$

Problem 3.4. Find the largest value of $|\alpha|$ such that $I_\alpha(\mathcal{K}) \subset \mathcal{K}$.

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