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Cournot's Economics and the Tradition of Social Mathematics

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Abstract

Cournot's economics was deeply rooted in French mathematics in the nineteenth century. The main stream was Laplace-Lagrange's physico-mathematics. But in examining what drove Cournot to economics, we must not dismiss another stream of French mathematics in the nineteenth century, that is the tradition of social mathematics.

It was probability theory which supported social mathematics as a principal tool of analysis. However it is remarkable that the mathematical machinery which Cournot used in economics was *not* probability theory but, exclusively, mathematical analysis while he succeeded the tradition of social mathematics. The transformation of mathematical methodology of economics due to Cournot was made possible by his peculiar grasp of the law of large numbers.

Cournot understood the twofold meanings of it; one is the regularity as mass phenomena in the probabilistic sense, and the other is the regularity resulting from smoothing by aggregation. Thanks to the first law of large numbers, Cournot was able to get out of the world of probability. Moreover the second law of large numbers enabled him to find a way of applying mathematical analysis to economics. In this way, Cournot succeeded in paving the way to mathematical economics on the tradition of Laplace-Lagrange's analytic mechanics.

1 Introduction

In this paper, we would like to shed a new light on Cournot's economics from the viewpoint of mathematical methodology taking account of the historical background of French mathematics in the nineteenth century. Cournot published a pathbreaking work *Recherches sur les Principes Mathématiques de la Théorie des Richesses* in 1838, the content of which has exerted a profound influence on modern economists. In addition to his numerous contributions to economic theory, a special attention must be paid to his methodological novelties: (1) systematic applications of mathematical analysis to economics (2) philosophy which justified this applications.

Namely, in the history of economic theory, he was the first to apply mathematical analysis (classical analysis) to economics in a non-trivial way. Certainly, there were some authors prior to Cournot, who made use of arithmetical illustrations of economic problems. However we can find no such economist as Cournot, who established genuinely mathematical theorems in economic theory in the sense that they could not be obtained without mathematics[1].

[1] To be sure, the attempts to apply mathematics to economics can be traced back to the dates considerably

Cournot's economics reflected the tradition of French economics in the nineteenth century on the one hand [2], and was deeply rooted in French mathematics in this period on the other. His economics might be regarded as a landmark of the cross of these two intellectual backgrounds.

Of course, the main stream of French mathematics in the nineteenth century was Laplace-Lagrange's physico-mathematics. Cournot also started his career as a mathematician with an industrious study of mathematics in the spirit of Laplace-Lagrange. However Cournot did not restrict his research activity to this discipline, and stepped out into the field of economics. In examining what drove the young able mathematician to economics, we must not dismiss another stream of French mathematics in the nineteenth century, that is the tradition of social mathematics (*mathématique social*).

It was Condorcet who founded this discipline of mathematics in the Revolutionary period in France. He attempted to organize a systematic study of social phenomena by following the standards of natural sciences. Moreover, Condorcet made use of probability theory as the main tool in analyzing social phenomena.

We have to keep in mind that Cournot succeeded Condorcet's tradition of social mathematics as well as Laplace-Lagrange's tradition.

In France, economics was under the hegemony of J. B. Say's school[3], which distrusted and disliked the mathematization in economics. It is therefore hardly surprising that Cournot's work was usually neglected and sometimes even attacked. Moreover, most of French mathematicians did not pay any attention to it at best, and showed hostility to it at worst. These negative reactions from both economists and mathematicians seemed to add some tragic tint to the

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earlier than Cournot's *Recherches*. The authors during the period prior to 1838 applied mathematical analysis to empirical case-by-case problems and dealt with specific functional relations. Among others, we must mention such names as Daniel Bernoulli and Georg von Buquoy. For these early attempts, see Robertson (1949), Theocharis (1961, 1993) and Baumol & Goldfeld (1968).

[2] Cournot was a classmate of August Walras at École Normale. Although this episode indicates that Cournot seemed to be familiar with economic theories based on utility, or *rarété*, it did not serve as a departure point of Cournot's economics. Cournot exceptionally refereed to Canard's work. However his evaluation of Canard was rather negative as illustrated by the following quotation: "··· *Les Principes de l'Économie Politique*, by *Canard*, a small work published in the year X [of the French Republic, A.D. 1801], and crowned by the *Institut*. These pretended principles are so radically at fault, and the application of them is so erroneous, that the approval of a distinguished body of men was unable to preserve the work from oblivion. It is easy to see why essays of this nature should not incline such economists as Say and Ricardo to algebra." See Cournot (1838, p. 2)

[3] Jean Baptiste Say (1767-1832) published in 1803 his *Traité d'Économie Politique*, which made him the principal apostle of Adam Smith in Europe. He rejected all attempts to apply mathematics in economics for the reason that this discipline necessarily involved human free-will. His views weighed upon French economists as an impediment to any attempt to mathematize economics.

seemingly quiet life of Cournot.

We begin by summarizing the essence of criticisms against the mathematization of economics by French mathematicians in section 2. In section 3, we examine Cournot's position in the history of mathematics. We then explore the program of social mathematics in section 4. Some outstanding concrete results obtained in Condorcet's program are discussed in the Appendix.

We proceed, in section 5, to the central issue of Cournot's methodology. Here we give a positive credit to Cournot's methodology that justifies the application of mathematical analysis to economics. It is *the law of large numbers* which played a key role in this context. Being based upon his *peculiar twofold understanding* of this law, Cournot could get out of the probability world and find out the way to introduce mathematical analysis to economics. Regrettably we have to confess that there are very few materials which directly support our interpretation of Cournot's methodology. In this sense, our proposition should be regarded merely as a *hypothetical* one, by means of which we are trying to rationalize Cournot's approach. Even though our way of interpretation might be hypothetical, we expect it to shed a new light on the important unsettled questions in the history of economics: (1) how did Cournot get out of the world of probability, which had been the traditional and characteristic field of social mathematics? (2) how did Cournot succeed in introducing classical analysis to economics?

Walras inherited the mathematical method in economics from Cournot in his *Élément d'Économie Politique Pure* published in 1874 and 1877. However we can not overlook a gap between the methodological attitudes of Cournot and Walras. We try to give an interpretation of this gap in favor of Walras' side. In section 6, we comment on the preceding arguments in the recent literature by contrast ours. Finally, the summary of this paper is provided in the concluding remarks.

2 Criticisms against Mathematical Economics

Cournot-Walras' mathematical economics was suffered from heavy attacks from French mathematicians, among whom Bertrand [4] and Painlevé [5] seemed to be the representative figures in

[4]Joseph Louis François Bertrand (1822-1900), French mathematician, was a graduate and professor of mathematics at the École Polytechnique. He was remembered for his theories of probability and thermodynamics. And he was a member of the Collège de France from 1862 to 1900.

[5]Paul Painlevé (1863-1933), French mathematician and politician, was remembered for his work in tranformations and especially in differential equations and in theory of functions; one type of function became known as Painlevé's transcendants. He took special interest in the science of aviation. He was the first Frenchman to fly with Wilbur Wright in 1908. And Painlevé created the first course in aeronautical mechanics at the École

the anti-mathematical economics camp. Although Bertrand's criticisms were fragmentary (see de Bornier (1992, appendix)), Painlevé (1909) did not hesitate to exhibit a more exhaustive criticism in the introduction to the French edition of Jevons' *Theory of Political Economy*.

Anyway they shared the skeptical views as to the possibility of applying the Laplace-Lagrange mathematical program to economics. Their criticisms against mathematization of economics may be summarized as follows.

(1) Bertrand and Painlevé do not admit economics as one of the exact sciences like astronomy and physics because it necessarily involves human free-will. What economics can tackle with should be restricted only to the mass phenomena, which are endowed with some regularity through the cancellation of individual irregularities.

(2) Consequently, the only mathematical tool for economists is probability theory.

(3) Even in this restricted area of economics, the only well-defined measurable magnitudes should be admitted as scientific concepts. However, very regrettably, some concepts like utility devoid of this qualification are used in economics.

Bertrand, in his review article on Cournot's *Recherches* and Walras' *Théorie Mathématique de la Richesse Social* (1883), expressed all his hostility towards mathematical economics. He neither admitted economics as an exact science like physics nor took trouble to understand the content of economics. His accusation was being based upon his conviction that mathematical method was not applicable to any inexact science like economics.

Bertrand also rejected Walras' paper in 1871, as a referee of *Revue des deux Mondes* on charges of a non-scientific character of it (see Letter from Bertrand to Walras dated 20 January 1876 in Jaffé (1965, Letter No. 345)). Furthermore, Bertrand, in his review article on Fechner's psychology (1899), criticized Fechner's program as a whole on the same grounds.

Bertrand's *Calcul des Probabilités* should be regarded as an extended version of his lectures given at the Collège de France. In the preface of this book, he denounced J. S. Mill's view that the application of probability theory to judicial decisions was the scandal of mathematics[6]

Supérieure d'Aéronautique. Painlevé was a member of the French chamber of Deputies, becoming Minister of Public Instituion in 1915, Minister of War in 1917, Prime Minister in 1917 and 1925. He taught first at the Faculty of Science at Lille in 1887-92, and then at the École Normale Supérieure of Paris, where he became professor in 1903.

[6]It means the misapplication of the calculus of probabilities into the problems without taking into fuller consideration the special circumstances of the case. Mill says that "some mathematicians have set out from the proposition that the judgement of any one judge or jurymen is, at least in some small degree, more likely to be right than wrong, and have concluded that the chance of a number of persons concurring in a wrong verdict is diminished the more number is increased; so that if the judges are only made sufficiently numerous, the correctness

(see Bertrand (1899 p. XLIII)). Here it deserves a special notice that Bertrand admitted the possibility of applying probability theory to social phenomena, although he did never admit the application of mathematical analysis to them.

In Painlevé's view, astronomy was the only candidate for the exact science that could predict the phenomena in the future precisely, for instance the positions of celestial bodies and its movements. Laplace's *Traité de Mécanique Céleste* (1799) and Lagrange's *Mécanique Analytique* (1788) were considered to be the exemplary works in this really exact science. Physics and chemistry did not reach the same standard as that of astronomy. Painlevé categorized them as typical samples of quantitative-statistical sciences, the research objects of which were solely mass phenomena which showed certain regularity resulting from the cancellation of individual irregularity. How about economics? He claimed that if we were able to comprehend the psychological situations of all economic agents, economics could compare with astronomy. Actually, however, this supposition was never realized. Painlevé, therefore, brutally rejected the attempt of mathematical economics. Since the research field of economics as a quantitative-statistical science was restricted to the regularities in mass phenomena, the only mathematical tool for economics was probability theory. This criticism was shared by almost all the French mathematicians against mathematical economics[7] [8].

But, as we shall see below, Cournot had a methodology which could be beyond these kinds of criticisms.

of the judgement may be reduced almost to certainly." However, he insists that "the cause of error, whether arising from the intricacy of the case or from some common prejudice or mental infirmity, if it acted upon one judge, would be extremely likely to affect all the others in the same manner, or at least a majority, and thus render a wrong instead of a right decision more probable, the more the number was increased." John Steuart Mill, *System of Logic*. p. 354

[7] Furthermore even in this restricted realm of economics, the only well-defined measurable magnitudes were admitted as scientific concepts. Painlevé also criticized economic theory from this viewpoint. He could not admit a non-measurable magnitude like utility. It is well-known that Walras gave a perfect answer to this kind of criticism in his correspondence with Poincaré (see Jaffé (1977) and Walras (1909)).

[8] Jules Henri Poincaré (1854-1912). French mathematician and theoretical astronomer, made substantial contributions to several branches of mathematics. In his writing on probability, he anticipated the concept of ergodicity that is basic to statistical mechanics. In celestial mechanics, Poincaré made important contributions to the theory of orbits, particulary the classical three-body problem. In his solution, he developed powerful new mathematical techniques, including the theories of asymptotic expansions and integral invariants. Always deeply interested in the philosophy of science, he wrote *La Science et l'Hypothèse* (1903), *La Valeur de la Science* (1905) and *Science et Méthode* (1908), all of which reached a wide public.

3 Cournot's Position in the History of Mathematics

In this section, we give a brief sketch of Cournot's life, focusing on his academic activity. His scientific works can be classified into three categories; mathematics, economics and philosophy. Although they are closely interdependent, philosophical works will be laid aside here and we will be exhaustively concerned with the relationship between mathematics and economics in Cournot's research activity. For a more extensive account of Cournot's life, the reader is refereed to the excellent papers by Fisher (1898, 1938), Moore (1905) and Nichol (1938).

Cournot was born at Gray (Haute-Saône) in 1801. His early studies took place at Besançon. In 1821, he was admitted to the École Normale Supérieure in Paris. However it was soon abolished for political reasons. So he had to transfer to Sorbonne in 1822, from which he was graduated in 1823. He spent the happiest life as a student of mathematics during this period at Sorbonne. Among his teachers were some of the most outstanding figures of his time, including Lagrange [9], Laplace [10] and Fourier.[11]. Cournot had read Laplace's *Exposition du Systéme du Monde* (1796). And Cournot kept a close friendship with Dirichlet [12], who was to become the successor to Gauss [13] at Göttingen. In 1829, Cournot received a doctorate in science. His scientific works involving his thesis on mechanics and astronomy attracted much attention of Poisson [14]. He was

[9] **Joseph Louis Lagrange** (1736-1813), Italian-French mathematician, made great contributions to number theory and analysis. He also developed mechanics, using the calculus of 4-dimensional space. He published papers on the three-body problem, which concerns the evolution of three particles mutually attracted according to Newton's law of gravity, differential equations, prime number theory, probability and mechanics.

[10] **Pierre Simon Laplace** (1749-1827), French mathematician, astronomer and physicist, was best known for his investigations into the stability of the solar system. His publication of the *Mécanique Céleste* was regarded as marking the cumulation of the Newtonian view of gravitation. Laplace was Minister of the Interior under Napoleon, a great admirer of men of science. He was professor at the École Normale and École Polytechnique.

[11] **Jean-Baptiste Joseph Fourier** (1768-1830) exerted strong influence on mathematical physics through his *Théorie Analytique de la Chaleur* (1822). He showed how the conduction of heat in solid bodies may be analyzed in terms of infinite mathematical series, which is today called Fourier series. Fourier extended this concept into the so-called Fourier integral. In 1798, Fourier accompanied Napoleon on his expedition to Egypt. He was engaged in extensive research on Egyptian antiquities until 1801.

[12] **Gustav Peter Lejeune Dirichlet** (1805-1859), French mathematician, made valuable contributions to number theory, analysis and mechanics. In number theory he proved the existence of an infinite number of primes in any arithmetic series $a+b, 2a+b, 3a+b, \ldots, na+b$, in which a and b are not divisible by one another. Dirichlet developed the general theory of units in algebraic number theory. In mechanics he investigated the equilibrium of systems and potential theory, which led him to the Dirichlet problem.

[13] **Carl Friedrich Gauss** (1777-1855), German mathematician, made substantial contributions to several branches of mathematics and revolutionized the mathematical techniques of his time. Among his many achievements were the discovery of the method of least squares, the discovery of non-Euclidean geometry, the first proof of the fundamental theorem of algebra, and the development of the basic theorems of number theory.

[14] **Siméon-Denis Poisson** (1781-1840), French mathematician, was known for his work on definite integrals,

professor at the École Polytechnique and later at the University of Paris. And Poisson arranged Cournot's appointment to the chair of mathematical analysis at Lyons in 1834, but he taught there for only one year. After that, he became involved more and more in the works of university administration.

Cournot's mathematical works appeared between 1840 and 1850. The most prominent works among them are *Traité Élémentaire de la Théorie des Fonctions et du Calcul Infinitesimal* (1841) and *Exposition de la Théorie des Chances et des Probabilités* (1843).

The letter from Poisson to Cournot indicated that the content of *Exposition* had already been completed in outline as ealry as January 1836 (see *Exposition,* pp. vi-vii). Moreover he translated Sir J. F. W. Herschel's *Astronomy* and H. Kater and D. Lardner's *Éléments de Méchanique* in 1834.

The story of Cournot's academic activity thus outlined seemed to suggest the peculiar position of Cournot in the history of mathematics; in fact Cournot stood at the very crossroad of two branches of mathematics, probability theory and mathematical analysis.

It was probability theory which supported Condorcet's social mathematics as a principal tool of analysis, and even Bertrand and Painlevé might approve the probabilistic theorizations of social phenomena. However it is remarkable that the mathematical machinery which Cournot used in economics was *not* probability theory but, exclusively, mathematical analysis while he succeeded the tradition of social mathematics. What was the grounds of his decision to adopt analysis and leave off probability?

In this regard, Ménard (1987, p. 530) insists that Cournot was inspired by nineteenth-century physics, which he considered to be the exemplary science. Even if we can admit his argument, we must not dismiss that some probabilistic elements still remain in Cournot's *Recherches*; Cournot clearly stated that an individual demand function depended upon a variety of needs, fortunes and caprices. So if he had attempted to deal with it directly, Cournot would have depended upon probabilistic theory. Thus it still remains to inquire the questions how Cournot could get out of the probability world and find a way of introducing mathematical analysis to economics. This is an important unsettled questions in the history of economics. We examine this point in detail in the subsequent sections.

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electromagnetic theory and probability. Poisson's *Traité de Mécanique* (1811-3), which was concerned with the application of mathematics to electricity, magnetism and mechanics, was the standard work in mechanics for many years. Poisson also contributed to celestial mechanics by extending the work of Laplace and Lagrange on the stability of planetary orbits. Poisson's other works include *Théorie Nouvelle de l'Action Capillaire* (1831) and *Théorie Mathématique de la Chaleur* (1835).

4 The Tradition of Condorcet's Social Mathematics

The starting point of social mathematics goes back to Marie Jean Antoine Nicholas Caritat Marquis de Condorcet (1743-1794). Condorcet was born at Riebemont-sur-Aisine, in Picardy. By 1763, Condorcet's mathematical genius had come into blossom and developed a new field in integral calculus. Condorcet published *Calcul de Intégral* in 1765. In 1769, Condorcet was admitted to the *Academie des Sciences* under the powerful patronage of d'Alembert [15]. Later Condorcet became acquainted with Turgot, who was also to influence Condorcet's career since then. Being the trusted aide of Turgot, Condorcet worked as Inspecteur de Monnaies until 1790 and as commissioner to the Treasuary from 1791 to 1792. At first glance, these busy works seemed to disturb Condorcet's mathematical activity, but he planned to establish new genre in mathematics, that is social mathematics [16]. It consisted of two projects, which were closely interconnected.

(1) It aimed at the creation of social sciences, the method of which was to be comparable with that of the natural sciences.

(2) It made use of probability theory as a principal tool of analysis.

Social mathematics fascinated some of the mathematicians in the Revolution era, such as Lagrange, Laplace and Vandermonde, who expected the emergence of a new branch of mathematics different from Laplace-Lagrange's physico-mathematics. In particular, Vandermonde [17] felt a deep sympathy with it and gave a lecture on mathematical economics at École Normale. Social mathematics encompassed epistemology, psychology of brain and behavior, economics, actuarial science and the theory of voting.

[15] **Jean le Rond d'Alembert** (1717-1783), French mathematician, was one of the most leading figures in France during the mid-eighteenth century. He worked on partial differential equations, solving the vibrating string problem and the general wave equation. At his age of twenty-four, d'Alembert had been elected to the Academie des Sciences, and he became its *secrétaire perpetuel* in 1754. From 1751 to 1772, he collabolated in the twenty-eight volumes of the celebrated *Encyclopédie*, for which he wrote the much-admired "Discours préliminaire," as well as most of the mathematical and scientific articles. As a result of his activities, and of his friendship with Voltaire and others among the "philosophes," he was one of those who paved the way for the French Revolution.

[16] Baker analyzed historically the particular course of Condorcet's intellectual development in the general context of Enlightment to which he contributed. See Baker (1975).

[17] **Alexandre Théophile Vandermond** (1735-1796), French mathematician, made an important contributions to algebra. In his time, algebra was known as the science of solution of equations. The problem was that of finding methods for expressing the solutions of algebraic equations in terms of the coefficients using arithmetic operations and extraction of roots of arbitrary degree. No one knew how to solve by radicals the equation $x^n - 1 = 0$ for $n > 10$ up until 1770. Vandermonde analyzed the case $n = 11$ in his paper of 1770. The work of Vandermond devoted to algebraic equations introduced the first group-theoretic theorem, that is substituions.

Condorcet provided his program with a concrete shape in his *Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix* (henceforth *Essai*).

Laplace and Poisson as well as Condorcet regarded jurisprudence as a natural field of application of probability.

However, social mathematics was declining in prosperity with the change of the times.

Under the restriction of the academic freedom in the years of Napoleon's despotism, Laplace changed his attitude. He seemed to have no solid political conviction. As soon as Napoleon fell from power, Laplace collected the remaining copies of his treatise from bookstores and hastened to replace the dedication to Napoleon with the one to Louis XVIII. And Condorcet's life came to a tragic end in the Terror. Nontheless, Condorcet's influence barely persisted in French mathematicians beyond his time.

In his remarkable work *Essai*, Condorcet constructed a theory of voting, which have been still attracting considerable attentions (see McLean & Hewitt (1994), Young (1988)). He illustrated a paradoxical property of voting, *cyclic* majority, under majority rule. He proposed a solution for ranking any number of alternatives even when cyclic majority was involved. In *Essai*, Condorcet began by introducing the notion *homo suffragans*, as the impartial and intelligent voter, which was to play a central part in his scheme.

Laplace's *Théorie Analytique des Probabilités* is a landmark in the history of probability theory, because it was not only a compilation of preceding probability theory but also it paved a new development. An important contribution by himself is the rigorous deduction of the DeMoivre-Laplace central limit theorem on the convergence of the binominal distribution to the normal distribution. Laplace applied it to the solution of a number of urn problems [18].

In his *Essai Philosophique sur les Probabilités,* he admitted the possibility of applications of probability theory to both of natural and social sciences. In chapter 11 of the *Théorie Analytique des Probabilités*, Laplace studied the probabilistic estimation of testimonies of witness and verdicts of law courts.

It must be emphasized here that these problems due to Condorcet and Laplace have essentially the probabilistic nature in the sense that the stochastic element does not vanish even if a considerable number of observations of the same kind would be made.

[18] We mention here one such problem. Consider that there are two urns A and B, each containing n white balls and n black balls. These balls are moved cyclically, one by one, from one urn to another. Then what is the probability $z_{x,r}$ that urn A will contain x white balls after r moves? (chapter 3.) Laplace solved this problem by using generating functions. Laplace also applied the central limit theorem when he calculated the value of life annuities. (chapter 9.)

Poisson published *Recherches sur la Probabilité des Judgments en Matière Criminelle et en Matière Civile* in 1837, the analytical method of which was due to Condorcet and Laplace. In this book, Poisson regarded the law of large numbers as the key in every application of the probability theory. Then how did he understand the law of large numbers?

As well known, the classical result of it, due to Bernoulli, can be stated as follows.

The Bernoulli law of large numbers (1713): Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of mutually independent random variable with the same distribution. Denote the sum by $S_n = \sum_{k=1}^{n} X_k$.

If the expectation $m = E(X_n)$ exists, then for any $\epsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - m\right| \leqq \epsilon\right) \longrightarrow 1 \quad as \quad n \longrightarrow \infty.$$

In other words, the probability P that the frequency S_n/n in n repeated trials differs only within certain limits from the expectation $m = E(X_n)$ converges to unity as the number of trials increases indefinitely. Historically, this law passed to the Borel strong law of large numbers. It can be stated as follows.

The Borel strong law of large numbers (1909): Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of mutually independent random variables and that obey an identical distribution. If $E(|X_n|)$ exists, then S_n/n *almost surely* converges to the expectation $m = E(|X_n|)$.

Poisson (1837, pp. 137-143) described his law of large numbers as follows.

The Poisson law of large numbers (1837):

[1] Suppose that in n trials an event E occurs with probabilities $p_1, p_2, \cdots, p_n$ and the opposite event F occurs with probabilities $q_1, q_2, \cdots, q_n$.

Let S_n be the number that an event E occurs in n trials, then the difference between

$$m_n = \frac{1}{n}(p_1 + p_2 + \cdots + p_n)$$

and the relative frequency S_n/n converges to zero as the number n of trials increases indefinitely.

[2] Suppose that mutually exclusive *causes* $C_1, C_2, \cdots, C_n$ bring about an event E and that the probabilities of the occurrence are $\gamma_1, \gamma_2, \cdots, \gamma_n$. Now also suppose that the probability that an event E occurs is c_i in the presence of cause C_i. Let

$$\gamma = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_n c_n.$$

Then S_n/n converges to γ as $n \longrightarrow \infty$.

[3] Suppose that in each trial a certain quantity A correspondes one or another of its values $a_1, a_2, \cdots, a_\lambda$. Let c_{ij} be the probability of A assuming value a_j given the cause C_i and let γ_i be the probability that C_i is at work. Then the mean of a_j over n trials would be

$$\alpha_j = \gamma_1 c_{1j} + \gamma_2 c_{2j} + \cdots + \gamma_\lambda c_{\gamma j}.$$

Then S_n/n converges to the expectation $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_\lambda\alpha_\lambda$ as $n \longrightarrow \infty$.

In essence, the Poisson law of large numbers was the generalization of the Bernoulli law of large numbers. It can be also verified that the Borel strong law of large number implies the Poisson law of large numbers.

Poisson tried to keep even the most capricious phenomena in the reach of rigorous analysis, having recourse to this law.

It is certain that Cournot, as a Poisson's fervent disciple, was acquainted with the law of large numbers. Sheynin (1978) and also Martin (1996) insist that Cournot did not mention the law of large numbers in his *Exposition.* They argue that Cournot underestimated the law of large numbers. Sheynin says; (1) Probably because of Bienaymé's (1875) criticism Cournot just did not mention this law at all (Sheynin, (1978) p. 274) and (2) Bienaymé's criticism was being based on his conviction that this law just did not exist (Sheynin, (1978) p. 273).

However, Sheynin completely misunderstands the content of Bienaymé's criticism. What Bienaymé actually says is simply that we should not regard the law of large numbers as a new mathematical discovery due to Poisson and hence we should refer to this law as the principle or the theorem of Bernoulli.

A careful examination of Cournot's *Exposition* reveals the contradiction of Cournot's text and Sheynin-Martin's claim. In fact, Cournot exclusively referred to this law as "the principle of Bernoulli" rather than as "the law of large numbers". This change of vocabulary seems to be a stumbling block which impeded Sheynin and Martin to grasp Cournot's attitude to the law correctly.

Cournot gave a full explanation of the principle of Bernoulli, in such as 86 and 87 of his *Exposition.* It follows from the law of large numbers that the relative frequencies of occurrence of events in consecutive series of trials are approximately equal to one another. Cournot applied this law of large numbers to the ratio of birth of boys and girls observed in various countries. In a large number of cases, Cournot considers that from past events we can try to obtain clues that may guide us in our conjecture about the causes on which these events depend. For example, bearing in mind that the ratio of birth of boys and girls is the almost same in Egypt and in the

Cape of Good Hope, we see that climate has a negligible effect in this respect (chapter 13).

Consequently, it is certain that Cournot was in favor of the law of large numbers. *Nevertheless Cournot's method of economics was solely mathematical analysis and not probability theory.* How shall we understand this?

As we pointed out at the end of section 1, it seems very hard to find out any materials in Cournot's works which directly answer this question. What we are trying to present here is a *hypothetical* interpretation of Cournot's mathematical methodology, by means of which we may rationalize his attitude.

5 Cournot's Methodology for Mathematical Economics

In our views, the law of large numbers seemed to play an important role behind Cournot's economics. Cournot understood the law in two meanings, which should be distinguished by nature. However it is this twofold understanding of the law of large numbers which enabled Cournot to find out the way to introduce mathematical analysis to economics. The first meaning of this law is the regularity as mass phenomena in the probabilistic sense, and the second one is the regularizing effect resulting from *smoothing by aggregation.*

(1) regularity as mass phenomena in the probabilistic sense

Cournot started his economic analysis with an aggregated average demand function but not with individual demand functions. An individual demand function may be influenced by some random elements. We now consider an economy with ℓ goods and denote the price vector by $p \in \mathbf{R}^{\ell}_{+}$. Therefore the individual i's demand function ξ_i must be expressed as $\xi_i : \mathbf{R}^{\ell}_{+} \times \Omega \longrightarrow \mathbf{R}^{\ell}$, which depends upon two variables p and ω. Here Ω denotes a probability space, an element ω of which stands for a variety of needs, fortunes and caprices according to Cournot. (See the quotation below.) He rejected to deal with an individual demand function directly. If Cournot had attempted to start with an individual demand function, he would have depended upon probability theory. By virtue of the law of large numbers, if ξ_i's are independent and subject to the same distribution, then the average of individual demand functions converges *almost surely* to the expected value of ξ_i. Strictly speaking, the limiting relation

$$\frac{1}{n}\sum_{i=1}^{n} \xi_i(p,\omega) \longrightarrow E[\xi_i(p,\cdot)] \qquad as \quad n \longrightarrow \infty$$

holds true for almost all $\omega \in \Omega$. Roughly speaking, under these conditions the average of ξ_i's converges to its expected value with probability 1. In this sense, the limit of the average of individual demand functions no longer depends on $\omega \in \Omega$. In other words, aggregated average demand function would be markedly independent of the anomalies of chance if n is very large.

In order to justify our formulation investigated so far, it suffices to remember that the Borel strong law of large numbers[19] essentially generalizes what Poisson and Cournot understood as the law of large numbers. Thus, we can understand the way of Cournot's thinking without changing the essential feature.

If we rewrite this limit $F(p)$, this function $p \longmapsto F(p)$ is the starting point for Cournot.

Thus a door was opened for Cournot to get out of the probability world provided that he agreed to start with an aggregated average demand function. Cournot thus distinguished the field to which probability theory should be applied from the field in which probability theory can be dispensed with.

(2) smoothing by aggregation

Moreover, Cournot recognized the concept of so-called smoothing by aggregation. Even if the behavior of an individual is irregular, discontinuous or nonsmooth, as can be seen in casual observations, the aggregation of sufficiently many functions which differ slightly to each other gives rise to a rather regular, continuous and smooth function through the cancellation of the irregularity. Thus we can assume that an aggregated average demand function is smooth and continuous [20].

This principle of *smoothing by aggregation* enabled Cournot to apply classical mathematical analysis to an aggregated average demand function.

Cournot (1838, pp. 49-50) notes;

> We assume that the function $F(p)$, which expresses the law of demand or of the market, is a continuous function, *i.e.* a function which does not pass suddenly from one value to another, but which takes in passing all intermediate values. It might be otherwise if the number of consumers were very limited: thus in a certain household the same quantity of firewood will possibly be used whether wood costs 10 franc or 15 francs the stere, and the consumption may suddenly be diminished if the price

[19] This type of the law of large numbers explicitly uses the concept of *measure*, which was unknown in Poisson-Cournot's time.

[20] A modern treatment of this problem was promoted by Debreu's paper. See Debreu (1972).

> of the stere rises above the latter figure. But the wider the market extends, and the more combinations of needs, of fortunes or even of caprices, are varied among consumers, the closer the function $F(p)$ will come to varying with p in a continuos manner.

Incidentally, the quotation shows that Cournot could not distinguish the concept of *continuity* from that of *differentiability* of a function. The distinction of these two concepts was made clear by Cauchy [21] in his rigorous treatment of analysis.

Thanks to the first law of large numbers, Cournot was able to find out an approach to economics which was traceable without bothering about probabilistic randomness. Moreover thanks to the second one, Cournot succeeded in finding a way of applying mathematical analysis to economics.

(3) Cournot versus Walras: a modern evaluation

Walras started his theoretical research with the critical examination of Cournot's economic theory. He often mentioned August Walras together with Cournot, as the authors to whom he had owed most in establishing his economics. And Walras applied mathematical analysis directly to an individual utility function. Consequently, Walras' attitude on the methodology of economics seems to contradict to Cournot's in that Walras applied mathematical analysis directly to an individual utility function. How can we evaluate this gap between the two? Although he also referred to the law of large numbers to justify the mathematization of economics [22], we can not expect that Walras had a methodological consideration in mind as deep as

[21] **Augustin-Louis Caucy** (1789-1857), French mathematician, was a pioneer in analysis and the theory of substituion groups. Cauchy's greatest contributions to mathematics was characterized by the clear and rigorous methods he introduced. Cauchy clarified the principles of calculus and put them on a satisfactory basis by developing the theory of functions of a complex variable. Cauchy's contributions on mathematics was embodied in his three treatises: *Cours d'Analyse de l'École Royale Polytechnique* (1821), *Résumé des Leçon sur le Calcul Infinitésimal* (1823) and *Leçon sur les Applications du Calcul Infinitésimal à la Géometrié* (1826-28). Cauchy was a prolific worker, providing a total of 789 papers on mathematics and sciences.

[22] Walras put it;" There is nothing to indicate that the individual demand curves $\cdots$, or the individual demand equations $\cdots$, are *continuous*, in other words, that an infinitesimally small increase in p_α produces an infinitesimally small decrease in d_α. On the contrary, these functions are often discontinuous. In the case of oats, for example, surely our first holder of wheat will not reduce his demand gradually as the price rises, but he will do it in some intermittent way every time he decides to keep one horse less in his stable. His individual demand curve will, in reality, take the form of a step curve $\cdots$. All the other individual demand curve will take the same general form. And yet the aggregate demand curves $\cdots$ can, for all practical purposes, be considered as continuous by virtue of the so-called *law of large numbers*."

Cournot. It seems almost impossible for us to reconstruct Walras' attitude toward this problem and vindicate it being based upon his own words. So we had better evaluate Walras' attitude from the viewpoint of modern economic theory.

There are several justifications for Walras' attitude to start from utility theory rather than a demand function.

(1) In the analysis starting from an observable demand function, it is a given datum and not an object of explanation. The properties mentioned about it are the facts to be verified statistically or inductively. However if we seek to *explain* these properties of a demand function instead of just describing or assuming them away, we have to make clear how it is generated by more elementary factors like a preference relation or a utility function.

Moreover when we are required to examine whether the properties of a demand function are *general* ones or *exceptional* ones, we have again to start from a utility as a generating factor.

(2) According to Sonnenschein-Debreu's theorem[23], any function which satisfies the continuity, homogeneity and Walras' law can be an excess demand function deduced from some utility functions through consumers' maximization behaviors. On the grounds of this result, it seems to be a reasonable way for us to analyze an excess demand function by supposing as if it were generated by the utility maximization behaviors of some *ideal* consumers. Such a reasoning may be regarded as an experiment in thought, which is even more important since it is difficult to carry out an ordinary experiment in economics.

(3) If we assume the strong axiom of revealed preference and a demand function which satisfies the Lipschitz condition with respect to income, we can derive a continuous utility function which generates this demand function[24]. If we can admit those assumptions, we have a good reason to start from a utility theory.

Even if we can justify the use of a utility function, how can we justify the supposition of its differentiability?

(α) At first, a continuous utility function can be approximated by a smooth function (Stone-Weierstrass' theorem). It may be sometimes justified to suppose the differentiability of a utility function as a way of approximation.

(β) It is one of the important task of demand theory to identify the direction of changes of demands corresponding to changes of prices. In the real world, such a change is observed in a

See Walras' *Élément*. translated by Jaffé (1977), section 52. p. 95.

[23]See Sonnenschein (1972, 1973) and Debreu (1972)

[24]See Uzawa (1960)

discrete or finite quantity. However we can safely make use of a smooth utility function as a theoretical devise which explains how the demands depend upon prices, provided that it predicts the same direction of changes in the real world [25].

Thus we have found several justifications for the application of mathematical analysis to a utility function. They also reinforce the validity of Walras' methodological view on mathematical economics.

6 Comments on the Recent Literature

We are now in a position to evaluate the preceding investigations on this topic in the recent literature in comparison with the one developed in this paper.

The mathematical contributions by those who succeeded the tradition of social mathematics have been studied in several literature. Among others, Sheynin (1976/77, 1978) reviews Laplace's and Poisson's mathematical works on probability respectively. But we can not agree with his assertion that Cournot did not mention the law of large numbers in his *Exposition*. On the contrary, we insist that he referred to the same law by another terminology, the principle of Bernoulli and he was in favor of this law.

In addition, detailed explanation of Laplace's theory of probability is given in Kolmogorov and Yushkevich's (1992) survey of the history of probability theory in the nineteenth century, which is a reliable basic reference from mathematical point of view.

Stigler (1986) provides a history of statistics, with special attentions to the application of probability to social sciences and discusses Cournot's works on probability in detail. Martin (1996) provides a comprehensive study on Cournot's work in the field of probability, focusing on its philosophical aspects, which we omit in this paper.

Ingrao & Israel (1990. chapter 2) precisely evaluate the role of social mathematics in the history of economics. They also correctly point out that the relationship between Cournot and the tradition of social mathematics; "social mathematics appeared in his writings as veiled allusions, with few direct references. Reading between the lines, one could infer that he was not unaware of the texts, or at least the projects, of his predecessors (p. 79)." But they seem to regard that the influence of social mathematics was reflected only on his attitude toward mathematizing economics. They did not try to elucidate that Cournot marked a significant turning-point in introducing the application of mathematical analysis into the program of social

[25] This argument seemes to be in accordance with Samuelson's spirit. See Samuelson (1947).

mathematics. Thus, we are naturally led to ask the question: (1) how did Cournot get out of the world of probability, which had been the traditional field of social mathematics? (2) how did Cournot succeed in introducing classical analysis to economics?

Ménard (1987) also tried to gave an answer to the question; why is mathematical economics possible? He stressed Cournot's cautious attitude toward the use of statistics in economics. "In adopting this attitude, Cournot was inspired by nineteenth-century physics, which he considered to be the exemplary science (p. 530)." Ménard affirms that Cournot erected the first economic model accoding to the image of classical physics (p. 141)." Even if we can admit his argument, Cournot clearly stated that "so many moral causes capable of neither enumeration nor measurement affect the law of demand" (*Recherches,* p. 47). Thus, some probabilistic elements still remain in his *Recherdhes.* Therefore, it still remains to inquire the question how Cournot could get out of the probability world, but Ménard failed to gave a decisive answer to this question. In our views, Ménard misses the first meaning of the law in our sense, that is to say the regularity in the probabilistic sense, which enabled Cournot to get out of the probability world. By this very failure to recognize the law of large numbers in this sense, the explanation of Cournot's methodology justifying the application of mathematical analysis to economics remains more or less unsatisfactory.

Ménard recognized the second meaning of the law of large numbers in our sense played a role in Cournot's economics; "The law of large numbers, the principle of compensation, and the calculation of averages had to be able to explain the shape and characteristics of the demand curve (p. 533)." On this point, De Villé and Ménard (1989) said that Cournot thought that the extension of market activities within several major countries as well as the development of international trade would expand the role of regularities and reduce the weight of irrational behaviors (p. 498). And Ménard considered that these reguralities in the economic world enabled Cournot to introduce classical analysis into economics. We completely agreed with this conclusion, but it must be emphasized that smoothing by aggregation has nothing to do with the probabilistic law of large numbers. Therefore, although Ménard succeeded to explain the reason why Cournot found a way of introducing mathematical analysis to economics, it still remains to inquire that how Cournot could get out of the probability world.

7 Concluding Remarks

In this paper, we have investigated Cournot's methodological foundations for applying mathematical analysis to economics. We explored the reason of French mathematicians' critical view against the mathematization of economics. The only mathematical tool they admitted as applicable to economics was probability theory. The origin of this thought can be traced back to Condorcet's social mathematics. Although Cournot also succeeded the tradition of social mathematics, the mathematical tool chosen in Cournot's economics was not probability theory but mathematical analysis. The transformation of mathematical methodology of economics due to Cournot was made possible by his peculiar grasp of the law of large numbers.

Cournot understood the twofold meanings of it; one is the regularity as mass pheneomena in the probabilistic sense, and the other is the regularity resulting from smoothing by aggregation. Thanks to the first law of large numbers, Cournot was able to get out of the world of probability. Moreover the second law of large numbers enabled him to find a way of applying mathematical analysis to economics. In this way, Cournot succeeded in paving the way to mathematical economics on the tradition of Laplace-Lagrange's analytic mechanics.

Walras, on the other hand, applied mathematical analysis even to a utility function. We evaluated the methodological difference between Cournot and Walras. And we presented a way of justification for Walras' approach from a viewpoint of modern economic theory.

Appendix

In this Appendix, we now exemplify some results obtained in the discipline of social mathematics.

(a) Condorcet

Let us assume $2q+1$ voters, who vote independently each other. Let v stand for the probability that his vote is correct in view of the truth. On the contrary, w represents the probability that his vote is an error. Let V^q and W^q stand for the probabilities that the social decision is correct and erroneous respectively. Then we must have

$$V^q = v^{2q+1} +_{2q+1} C_1 v^{2q} w +_{2q+1} C_2 v^{2q-1} w^2 + \cdots +_{2q+1} C_q v^{q+1} w^q,$$

and

$$W^q = w^{2q+1} +_{2q+1} C_1 w^{2q} v +_{2q+1} C_2 w^{2q-1} v^2 + \cdots +_{2q+1} C_q w^{q+1} v^q.$$

Since we obtain

$$V^{q+1} = v^{2q+3} +_{2q+3} C_1 v^{2q+2} w +_{2q+3} C_2 v^{2q+1} w^2 + \cdots +_{2q+3} C_{q+1} v^{q+2} w^{q+1},$$

by a similar way, it follows that

$$V^{q+1} - V^q =_{2q+1} C_{q+1} -_{2q+1} C_q v^{q+1} w^{q+2}.$$

Taking account of the relations $_{2q+1}C_{q+1} =_{2q+1} C_q$ and

$$V^{q+1} - V^q =_{2q+1} C_q v^{q+1} w^{q+1} \times (v - w),$$

we have

$$V^q = v + (v - w) \times \{vw +_3 C_1 v^2 w^2 + \cdots +_{2q-1} C_{q-1} v^q w^q\}.$$

Similarly, we have

$$W^q = w + (w - v) \times \{wv +_3 C_1 w^2 v^2 + \cdots +_{2q-1} C_{q-1} w^q v^q\}.$$

According to Condorcet, this result implies that the probability that social decision by majority is correct converges to 1 as the number of voters increases if $v > w$. However since this assumption ($v > w$) is not necessarily realistic, he advocated the necessity of a good education in order to overcome this difficulties.

(b) Laplace

For example, consider the situation that a ticket is drawn from an urn containing n tickets numbered from 1 to n. There are two witnesses, whose testimonies coincide with each other. The degree of confidence of both testimonies are p and p' respectively. A witness testimonies that the ticket drawn is the one numbered, say, i and both make no mistake for the sake of simplicity: (a) both do not deceive and (b) both deceive. Let us consider the case (a). In this case, the ticket numbered i is actually drawn, the probability of which is $1/n$. Thus the probability of the case (a) is pp'/n. On the contrary, in the case of (b), the probability that the ticket numbered i is drawn and both deceive is

$$\frac{(1-p)(1-p')}{n(n-1)}.$$

Consequently, the probability that the ticket numbered i is drawn turns out to be

$$\frac{pp'}{pp' + \frac{(1-p)(1-p')}{(n-1)}}.$$

In the case of $n = 2$ and $p = p'$, this formula will be

$$\frac{p^2}{p^2 + (1-p)^2}.$$

Proceeding from this simple consideration, Laplace applied this result to the discussion of the work of tribunals.

Let us assume that the probability p of a just verdict is the same for each judge and that exceeds one-half. Under these assumptions, the probability that r judges unanimously bring in a correct verdict turns out to be

$$\frac{p^r}{p^r + (1-p)^r}.$$

This is one of the main results obtained in Laplace's work. The concrete value of p must be estimated from statistical data. Once this estimation is done, the optimal number of judges, say r, is to be solved.

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