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Indeterminacy in Two-Sector Models of Endogenous Growth with Leisure

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Abstract

This paper demonstrates that preference structure may play a pivotal role in generating indeterminacy in stylized models of endogenous growth. By examining two-sector models of endogenous growth with labor-leisure choice, we show that if the utility function of the representative family is not additively separable between consumption and pure leisure time, then indeterminacy may hold even if production technologies satisfy social constant returns. We first explore local indeterminacy in the context of a model with physical and human capital. We also examine global indeterminacy in a model without physical capital.

1 Introduction

The last decade has seen extensive investigations on indeterminacy of equilibrium in the representative agent models of economic growth. Most studies on this issue have examined models with external increasing returns. Early studies such as Benhabib and Farmer (1994) and Boldrin and Rustichini (1994) reveal that the degree of increasing returns should be sufficiently large to produce indeterminacy. The real business cycle theorists criticize this result and they claim that empirical validity of the business cycle theory based on indeterminacy and sunspots is dubious.[1] To cope with the criticism, the recent literature intends to find out the conditions under which indeterminacy emerges without assuming strong degree of increasing returns to scale: see, for example, Benhabib and Farmer (1996), Perli (1998) and Wen (1998).

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[1]Schmitt-Grohé (1997) presents a detailed examination of empirical plausibility of those studies.

The purpose of this paper is to make a contribution to such a research endeavour. In finding indeterminacy conditions, we put more emphasis on the role of preference structure rather than on that of production technologies. More specifically, we analyze two-sector endogenous growth models à la Lucas (1988 and 1990) that involve sector-specific externalities and labor-leisure choice. It is demonstrated that if the utility function of the representative family is not additively separable between consumption and pure leisure time, then indeterminacy may hold even if technologies of the final good and the new human capital production sectors satisfy social constant returns. We also explore models with quality leisure time in which effective leisure units are defined as the amount of time spent for leisure activities augmented by the level of human capital. In this formulation, we again verify that non-separability of the utility function may play a pivotal role in generating indeterminacy.

In the existing literature, Benhabib and Perli (1994) and Xie (1994) explore indeterminacy in the Lucas model. Xie (1994) presents a detailed analysis of transitional dynamics in the presence of indeterminacy by setting specific conditions on parameter values involved in the model. Since he treats a model without labor-leisure choice, indeterminacy needs strong increasing returns. Benhabib and Perli (1994) consider endogenous labor supply and show that indeterminacy may be observed with relatively small degree of increasing returns. They use an additively separable utility function, so that indeterminacy stems from specific production structure assumed in their model. In contrast to these contributions, the main discussion of this paper, without assuming social increasing returns, concentrates on the role of non-separable utility function.[2]

The central concern of this paper is closely related to two recent developments in the literature on indeterminacy in growth models. The first are the studies on the relation between non-separable utility and indeterminacy conducted by Bennett and Farmer (1998) and Pelloni and Waldmann (1998 and 1999). Bennett and Farmer (1998) introduce a non-separable utility function into the model of Benhabib and Farmer (1994) and find that a small degree of increasing returns would be enough for indeterminacy to hold. Pelloni and Waldmann (1998 and 1999), on the other hand, examine the role of non-separable utility in the one-sector endogenous growth model developed by Romer (1986). They show that indeterminacy can be observed in a simple Ak framework if there are sufficiently strong increasing returns. We push this line of research further to demonstrate that in two-sector endogenous growth models with non-separable utility indeterminacy would hold even in the absence of increasing returns.[3]

[2] See also Mitra (1998).

[3] In monetary dynamics literature, it has been well known that non-separable utility may yield complex dynamics. For example, as shown by Obstfeld (1984) and Matsuyama (1991), if the utility function is note separable between consumption and real money balances, there may exist multiple converging paths. In contrast, the representative agent models of growth without money have usually assumed additively separable utility functions when the models

The other development that is closely related to our analysis is made by Benhabib and Nishimura (1998 and 1999). These authors reveal that indeterminacy may hold in the neoclassical multi-sector growth models with social constant returns. The key condition for their finding is that relative factor intensities of the social technologies involving externalities may be opposite to that of the private technologies. Since the Lucas model we use assumes that the education sector employs human capital alone, there is no factor intensity reversal between the social and the private technologies. Therefore, the cause of indeterminacy with social constant returns in our discussion mainly comes from the preference side rather than from the production side emphasized by Benhabib and Nishimura (1998 and 1999).[4]

The paper is organized as follows. Section 2 sets up the base model with physical and human capital. Section 3 characterizes the dynamics of the model and presents local indeterminacy results. Section 4 explores models without physical capital and finds the global indeterminacy conditions. Concluding remarks are given in Section 5.

2 The Base Model

The analytical framework of this paper is essentially the same as that of Lucas (1988 and 1990). We introduce sector-specific externalities into the original model. Production side of the economy consists of two sectors. The first sector produces a final good that can be used either for consumption or for investment on physical capital. The production technology is given by

$$Y_1 = K^{\alpha} H_1^{\beta_1} K_E^{\varepsilon} H_{1E}^{\phi_1}, \quad \alpha,\ \beta_1 > 0,\ \alpha + \beta_1 + \varepsilon + \phi_1 = 1, \tag{1}$$

where Y_1 denotes the final good, K is stock of physical capital and H_1 is human capital devoted to the final good production. K_E^{α} and $H_{1E}^{\phi_1}$ represent sector-specific externalities associated with physical and human capital employed in this sector. The key assumption in (1) is that the production technology is socially constant returns to scale.

Following the Uzawa-Lucas setting, we assume that new human capital production needs human capital alone and its technology is specified as

$$Y_2 = \gamma H_2^{\beta_2} H_{2E}^{\phi_2}, \quad \gamma, \beta_2, \phi_2 > 0, \quad \beta_2 + \phi_2 = 1. \tag{2}$$

Here, H_2 is human capital used in the education sector, $H_{2E}^{\phi_2}$ stands for sector specific externalities. Again, the production technology of new human capital exhibits social constant returns.

consider endogenous labor supply..

[4]Mino (1999b) re-considers Benhabib-Nishimura proposition by using a two sector endogenous growth model in which both the final good and the new human capital producing sectors employ physical as well as human capital. It is shown that Benhabib and Nishimura result can be verified in the context of endogenous growth as well.

It is assumed that the total time available to the representative household is unity. Thus denoting the time length devoted to leisure by $l \in [0,1]$, the full employment condition for human capital is

$$H_1 + H_2 = (1-l)\,H,$$

where H is the total stock of human capital. As a result, if we define $v = H_1/H$, accumulation of physical and human capital respectively given by

$$\dot{K} = K^{\alpha}(vH)^{\beta_1} K_E^{\varepsilon} H_{1E}^{\phi_1}, -C - \delta K, \quad 0 < \delta < 1, \tag{3}$$

$$\dot{H} = \gamma\left[(1-v-l)\,H\right]^{\beta_2} H_{2E}^{\phi_2} - \eta H, \quad 0 < \eta < 1. \tag{4}$$

In the above, C denotes consumption, and δ and η are the depreciation rates of physical and human capital.

The objective function of the representative household is

$$U = \int_0^{\infty} u(C,l)\, e^{-\rho t} dt, \quad \rho > 0,$$

where the instantaneous utility function is given by the following:[5]

$$u(C,l) = \begin{cases} \dfrac{[C\Lambda(l)]^{1-\sigma} - 1}{1-\sigma}, & \sigma > 0, \ \sigma \neq 1, \\ \ln C + \ln \Lambda(l), & \text{for } \sigma = 1. \end{cases} \tag{5}$$

Function $\Lambda(l)$ is assumed to be monotonically increasing and strictly concave in l. We also assume that

$$\sigma\Lambda(l)\,\Lambda''(l) + (1-2\sigma)\,\Lambda'(l)^2 < 0. \tag{6}$$

This assumption, along with strictly concavity of $\Lambda(l)$, ensures that $u(C,l)$ is strictly concave in C and l.

The representative household maximizes U subject to (3), (4) and given initial levels of K and H by controlling C, v and l. In so doing, the household takes sequences of external effects, $\left\{K_E(t), H_{1E}^{\phi_1}(t), H_{2E}^{\phi_2}(t)\right\}_{t=o}^{\infty}$, as given. The current value Hamiltonian for the optimization problem can be set as

$$\begin{aligned} \mathcal{H} &= \frac{[C\Lambda(l)]^{1-\sigma} - 1}{1-\sigma} + p_1\left[K^{\alpha}(vH)^{\beta_1} K_E^{\varepsilon} H_{1E}^{\phi_1}, -C - \delta K\right] \\ &\quad + p_2\left[\gamma(1-v-l)^{\beta_2} H^{\beta_2} H_{2E}^{\phi_2} - \eta H\right], \end{aligned}$$

[5] As is well known, if the utility function involves pure leisure time as an argument, the functional form should be (5) in order to define feasible balanced-growth equilibrium.

where p_1 and p_2 are respectively denote the prices of consumption good and new human capital. Under given sequences of external effects, the necessary conditions for an optimum are the following:

$$C^{-\sigma}\Lambda(l)^{1-\sigma} = p_1, \tag{7}$$

$$C^{1-\sigma}\Lambda'(l)\Lambda(l)^{-\sigma} = \gamma p_2 \beta_2 (1-v-l)^{\beta_2 - 1} H^{\beta_2} H_{2E}^{\phi_2}, \tag{8}$$

$$p_1 \beta_1 K^{\alpha} v^{\beta_1 - 1} H^{\beta_1} K_E^{\varepsilon} H_{1E}^{\phi_1} = \gamma p_2 \beta_2 (1-v-l)^{\beta_2 - 1} H^{\beta_2} H_{2E}^{\phi_2}, \tag{9}$$

$$\dot{p}_1 = p_1 \left[\rho + \delta - \alpha K^{\alpha-1} (vH)^{\beta_1} K_E^{\varepsilon} H_{1E}^{\phi_1}\right], \tag{10}$$

$$\begin{aligned} \dot{p}_2 &= p_2 \left[\rho + \eta - \gamma\beta_2 (1-v-l)^{\beta_2} H^{\beta_2 - 1} H_{2E}^{\phi_2}\right] \\ &\quad - p_1 \left[\beta_1 K^{\alpha-1} v^{\beta_1} H^{\beta_1 - 1} K_E^{\varepsilon} H_{1E}^{\phi_1}\right], \end{aligned} \tag{11}$$

together with the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t} p_1 K = 0; \quad \lim_{t\to\infty} e^{-\rho t} p_2 H = 0. \tag{12}$$

3 Local Indeterminacy

3.1 Dynamic System

For analytical simplicity, the following discussion assumes that $\Lambda(l)$ is specified as

$$\Lambda(l) = \exp\left(\frac{l^{1-\theta} - 1}{1-\theta}\right), \quad \theta > 0, \ \theta \neq 1, \tag{13}$$

where $\Lambda(l) = l$ for $\theta = 1$. Given this specification, when $\sigma = 1$, the instantaneous utility function becomes

$$u(C,l) = \ln C + \frac{l^{1-\theta}}{1-\theta}$$

It is to be noted that, under this specification, condition (6) reduces to

$$(1-\sigma) l^{1-\theta} - \sigma\theta < 0. \tag{14}$$

If we assume that the number of firms is normalized to one, in equilibrium it holds that $K_E(t) = K(t)$ and $H_{iE}(t) = H_i(t)$ for all $t \geq 0$. Thus, keeping in mind that $\alpha + \beta_1 + \varepsilon + \phi_1 = 1$ and $\beta_2 + \phi_2 = 1$, from (7) and (8) we obtain

$$\frac{C\Lambda'(l)}{\Lambda(l)} = \frac{p_2 \gamma \beta_2 H}{p_1}.$$

Given (13), the above becomes

$$C = (p_2/p_1)\,\gamma\beta_2 l^{\theta} H. \tag{15}$$

Letting $x = K/vH$, (9) is written as

$$\frac{p_2}{p_1} = \frac{\beta_1}{\gamma\beta_2} x^{\alpha+\varepsilon}. \tag{16}$$

Equations (15) and (16) give $C = \beta_1 l^{\theta} x^{\alpha+\varepsilon} H$. Hence, using $x = K/vH$, the commodity market equilibrium conditions (3) and (4) yield the following growth equations of capital stocks:

$$\frac{\dot{K}}{K} = x^{\alpha+\varepsilon-1} - \frac{\beta_1 l^{\theta} x^{\alpha+\varepsilon}}{k} - \delta, \tag{3'}$$

$$\frac{\dot{H}}{H} = \gamma\left(1 - l - \frac{k}{x}\right) - \eta \tag{4'}$$

On the other hand, (10) gives the following:

$$\dot{p}_1/p_1 = \rho + \delta - \alpha x^{\alpha+\varepsilon-1}, \tag{9'}$$

Additionally, in view of (9), equation (11) becomes

$$\dot{p}_2/p_2 = \rho + \eta - \gamma\beta_2(1-l). \tag{10'}$$

As a result, by use of (9'),(10') and (16), x changes according to

$$\frac{\dot{x}}{x} = \frac{1}{\alpha+\varepsilon}\left[\eta - \delta + \alpha x^{\alpha+\varepsilon-1} - \beta_2\gamma(1-l)\right]. \tag{17}$$

Under (13), equation (6) is given by

$$C^{-\sigma} l^{-\theta} \exp\left((1-\sigma)\frac{l^{1-\theta}-1}{1-\theta}\right) = p_1.$$

Thus substituting (6) into (14) and taking time derivatives, we obtain

$$\left[(1-\sigma)\,l^{1-\theta} - \sigma\theta\right]\frac{\dot{l}}{i} = (1-\sigma)\frac{\dot{p}_1}{p_1} + \sigma\left(\frac{\dot{p}_2}{p_2} + \frac{\dot{H}}{H}\right). \tag{18}$$

Note that if the utility function is additively separable ($\sigma = 1$), the above becomes

$$\frac{\dot{l}}{l} = -\frac{1}{\theta}\left(\frac{\dot{p}_2}{p_2} + \frac{\dot{H}}{H}\right).$$

Namely, the optimal change in leisure time is negatively proportional to the change in aggregate value of human capital.

Using (4'), (9') and (10'), equation (18) yields the dynamic equation of leisure:

$$\frac{\dot{l}}{l} = \Delta(l)\left\{\alpha(1-\sigma)x^{\alpha+\varepsilon-1} + \sigma\gamma\frac{k}{x} - \sigma\gamma(1-\beta_2)(1-l) - \rho - (1-\sigma)\delta\right\}, \tag{19}$$

where $\Delta(l) = \left[\sigma\theta - (1-\sigma)l^{1-\theta}\right]^{-1}$, which has a positive value under the assumption of (14). Finally, (3') and (4') mean that the dynamic equations for the behavior of k $(= K/H)$ is given by

$$\frac{\dot{k}}{k} = x^{\alpha+\varepsilon-1} - \frac{\beta_1 l^{\theta} x^{\alpha+\varepsilon}}{k} - \delta + \eta - \gamma\left(1 - l - \frac{k}{x}\right). \tag{20}$$

Consequently, we find that (17), (19) and (20) constitute a complete dynamic system with respect to $k\,(= K/H)$, $x\,(= K/vH)$ and l.

3.2 Indeterminacy Conditions

Since the complete dynamic system derived above is highly nonlinear, the precise analytical conditions for generating indeterminacy are hard to obtain. The common strategy to deal with such a situation is to find numerical examples exhibiting indeterminacy by setting parameter values at empirically plausible magnitudes. In the following, rather than displaying the results of numerical experiments, we impose specific conditions on parameters in order to obtain analytical conditions for indeterminacy in a clearer manner. Following Xie's (1994) idea, we focus on the special case where $\sigma = \alpha$. As shown below, this condition enables us to reduce the three-dimensional dynamic system to a two-dimensional one. Additionally, we also assume that $\delta = \eta$, that is, physical and human capital depreciate at the identical rate. This assumption is made only for notational simplicity and the main results obtained below are not altered when $\delta \neq \eta$.

The assumption $\sigma = \alpha$ simplifies the argument as the following can be held:

Lemma 1 *If $\sigma = \alpha$ and $\theta = 1$, the consumption-physical capital ratio, C/K, stays constant over time.*

Proof. Let us define $z = \beta_1 x^{\alpha+\varepsilon} l / k \, (= C/K)$. If $\sigma = \alpha$ and $\theta = 1$, then (19) becomes

$$\frac{\dot{l}}{l} = x^{\alpha+\varepsilon-1} - z - \gamma(1-l) + \gamma \frac{k}{x}.$$

Therefore, by (19) and (20) we obtain:

$$\begin{aligned} \frac{\dot{z}}{z} &= (\alpha+\varepsilon)\frac{\dot{x}}{x} + \frac{\dot{l}}{l} - \frac{\dot{k}}{k} \\ &= z - \frac{\alpha + (1-\alpha)\delta}{\alpha}. \end{aligned}$$

Since this system is completely unstable, on the perfect-foresight competitive equilibrium path the following should hold for all $t \geq 0$:

$$z\left(= \frac{C}{K}\right) = \frac{\rho + (1-\alpha)\delta}{\alpha}.$$

Hence, consumption and physical capital change at the same rate even in the transition process. ■

The above result means that on the equilibrium path x is related to k and l in such a way that

$$x = \left(\left(\frac{\alpha + (1-\alpha)\delta}{\alpha}\right)\frac{k}{l}\right)^{\frac{1}{\alpha+\varepsilon}}.$$

Substituting this into (19) and (20), we obtain the following set of differential equations:

$$\frac{\dot{k}}{k} = \left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}} + \frac{\gamma}{\lambda}\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}} l - \gamma(1-l) - \lambda,$$

$$\frac{\dot{l}}{l} = (1-\alpha)\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}} + \frac{\gamma}{\lambda}\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}} l - \gamma(1-\beta_2)(1-l) - \lambda,$$

where $\lambda = [\rho + (1-\alpha)\delta]/\alpha$. To simplify further, denote $q = (\lambda k/l)^{1-\frac{1}{\alpha+\varepsilon}}$. Then the above system may be rewritten in the following manner:

$$\frac{\dot{q}}{q} = \left(\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}\right)[\gamma\beta_2(1-l) - \alpha q], \tag{21}$$

$$\frac{\dot{l}}{l} = \left(1 - \alpha + \frac{\gamma}{\lambda} l\right) q - \gamma(1-\beta_2)(1-l) - \lambda. \tag{22}$$

Under the conditions where $\sigma = \alpha$ and $\theta = 1$, this system is equivalent to the original dynamic equations given by (17), (19) and (20).

By inspection of (21) and (22), we find the following results:

Lemma 2 *If the dynamic system consisting of (21) and (22) has a stationary point with a saddle-point property, then the original dynamic system exhibits local determinacy. If a stationary point of (21) and (22) is a sink, then the original system involves local indeterminacy.*

Proof. If (21) and (22) exhibit a saddle point property, there (at least) locally exists a one-dimensional stable manifold around the steady state. Hence, the relation between q and l on the stable manifold can be expressed as $q = q(l)$. By displaying phase diagrams of (21) and (22), it is easy to confirm that if the stationary point is saddle, the stable arms has a negative slopes. By definition of q, it holds that

$$k = lq(l)^{\frac{\alpha+\epsilon}{\alpha+\epsilon-1}} \tag{23}$$

Since on the saddle path q is negatively related to l, the right hand side of the above monotonically increases with l. This implies that under a given initial level of k, the initial value of l is uniquely determined to satisfy (23). Thus converging path in the original system with respect to (k, x, l) is uniquely given as well. In contrast, if the stationary point of (21) and (22) is a sink, there are an infinite number of converging paths in (q, l) space Thus we cannot specify a unique initial values of l and x under a given initial level of k. ■

As for the uniqueness of balanced-growth equilibrium, we find the following conditions:

Lemma 3 *(i) There is a unique, feasible balanced growth equilibrium, if and only if*

$$\gamma(\beta_2 - \alpha) > \rho + (1-\alpha)\delta. \tag{a}$$

(ii) There may exist dual balanced-growth equilibria, if

$$\gamma(\beta_2 - \alpha) < \rho + (1-\alpha)\delta. \tag{b}$$

Proof. Condition $\dot{q} = 0$ in (21) yields $q = (\gamma\beta_2 l\alpha)(1-l)$. Thus conditions $\dot{l} = \dot{q} = 0$ are established if the following equation holds:

$$\xi(l) = \frac{\gamma\beta_2}{\alpha}\left(1 - \alpha + \frac{\gamma}{\lambda}l\right)(1-l) - \gamma(1-\beta_2)(1-l) - \lambda = 0.$$

Note that

$$\begin{aligned}
\xi(0) &= (\gamma\beta_2/\alpha)(1-\alpha) - \gamma(1-\beta_2) - \lambda \\
&= (1/\alpha)[\gamma(\beta_2-\alpha) - \rho - (1-\alpha)\delta] \\
\xi(1) &= -(1/\alpha)[\rho + (1-\alpha)\delta] < 0
\end{aligned}$$

If condition (a) is met, $\xi(0) > 0$ and $\xi(l)$ is monotonically decreasing with l for $l \in [0,1]$. Hence, $\xi(l) = 0$ has a unique solution in between 0 and 1. If (b) is satisfied, then $\xi(0) < 0$. Since $\xi(l) = 0$ is a quadratic equation, if $\xi(l) = 0$ has solutions for $l \in [0,1]$, there are two solutions. ■

Using the results shown above, we obtain the indeterminacy results for the special case of $\sigma = \alpha$:

Proposition 1 *Suppose that $\sigma = \alpha$ and $\theta = 1$. Then the balanced-growth equilibrium is locally indeterminate, if and only if the following conditions are satisfied:*

$$\left(1 - \beta_2 - \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon}\right)\bar{l} + \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon} + \frac{\rho + (1-\alpha)\delta}{\alpha} < 0, \tag{24}$$

$$\beta_2 - \alpha + \frac{\alpha\gamma\beta_2}{\rho + (1-\alpha)\delta}\left(2\bar{l} - 1\right) > 0, \tag{25}$$

where $\bar{l}$ denotes the steady-state value of leisure time.

Proof. Linearizing (21) and (22) at the stationary point and using the steady state conditions that satisfy $\dot{l} = \dot{q} = 0$, we find that signs of the trace and the determinant of the coefficient matrix of the linearized system fulfill:

$$\begin{aligned} & \text{sign (trace)} \\ = \; & \text{sign}\left\{\left(1 - \beta_2 - \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon}\right)\bar{l} + \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon} + \frac{\rho + (1-\alpha)\delta}{\alpha}\right\}, \end{aligned}$$

$$\text{sign (det)} = \text{sign}\left\{\beta_2 - \alpha + \frac{\alpha\gamma\beta_2}{\rho + (1-\alpha)\delta}\left(2\bar{l} - 1\right)\right\}.$$

Therefore, if (24) and (25) hold, then the trace and the determinant respectively have negative and positive values. This means that the linearized system has two stable eigenvalues, and thus in view of Lemma 2, the balanced growth equilibrium is locally indeterminate. ■

The above result implies the following fact:

Corollary 1 *If the system has dual steady states and if (24) is fulfilled, then one of the balanced-growth equilibria is locally determinate, while the other is locally indeterminate.*

Proof. Since there are two stationary points, the determinant of the coefficient matrix changes its sign depending on which steady state is chosen to evaluate each element of the matrix. Thus if (24) is held, one of the balanced-growth equilibrium satisfies (25) as well, so that it is locally indeterminate. This means that the other balanced growth equilibrium is a saddle point so that from Lemma 2 it is determinate. ■

Since the indeterminacy conditions displayed above contains an endogenous variable, $\bar{l}$, examination of numerical examples would be helpful. As an example, suppose that $\alpha = \sigma = 0.6$, $\varepsilon = 0.1$, $\beta_2 = 0.8$, $\rho = 0.05$, $\delta = \eta = 0.04$ and $\gamma = 0.18$. These examples satisfy condition (b) in Lemma 3. Actually, equation $\xi(l) = 0$ yields two feasible solutions: $\bar{l} = 0.125$ and 0.735. The corresponding growth rates in the steady state are 0.0034 and 0.082, respectively.[6] In this example, we can verify that (24) and (25) are met when $\bar{l} = 0.735$, while (25) does not hold when $\bar{l} = 0.125$. Consequently, the balanced growth equilibrium with a larger amount of leisure (so the low rate of economic growth) is locally indeterminate. In contrast, the high growth equilibrium is locally determinate.

We have assumed that $\sigma = \alpha < 1$, the utility function is not separable by the assumption. As demonstrated by Ladrón-de-Guevara et al. (1999), the pure leisure time model may contain multiple balanced growth equilibria even if we assume that there are no externalities and that utility function is separable.[7] This means that multiple steady states and indeterminacy may be established in our model even in the case that $\sigma = 1$. However, under plausible parameter values, we may confirm that indeterminacy is hard to obtain when we assume a separable utility.

4 Global Indeterminacy in a Model without Physical Capital

In this section we briefly examine a model without physical capital. Although the endogenous growth model that does not involve physical capital may lack reality, it is helpful for analyzing the global behavior of the economy. The production and preference structure are the same as before. Only difference is that there is no physical capital: both final good and new human capital producing sectors use human capital alone. Since the final good is used only for consumption, the market equilibrium condition for the first good is

$$C = (vH)^{\beta_1} H_{1E}^{\phi_1}, \quad \beta_1 \in (0,1), \quad \phi_1 > 0. \tag{26}$$

The production function of new human capital is (2) in the base model.

[6] Note that the transversality conditions (12) is expressed as $\bar{g}(1-\sigma) < \rho$ in the steady state. Thus our example does not violate the transversality condition.

[7] See also de Heck (1998) who explores multiplicity of the steady state in the neoclassical optimal growth model involving endogenous labor supply.

We first consider a model with pure leisure time where the utility function is given by (12). Again, we assume that the consumption good sector has a socially constant returns to scale technology so that $\beta_1+\phi_1=1$. The Hamiltonian function for the household's optimization problem is

$$\begin{aligned}\mathcal{H} &= \frac{[C\Lambda(l)]^{1-\sigma}-1}{1-\sigma}+p_1\left[(vH)^{\beta_1}H_{1E}^{\phi_1},-C\right]\\ &+p_2\left[\gamma(1-v-l)^{\beta_2}H^{\beta_2}H_{2E}^{\phi_2}-\eta H\right],\end{aligned}$$

where p_1 is the price of the consumption good. Noting that $\beta_1+\phi_1=\beta_2+\phi_2=1$ and that $H_{1E}=vH$ and $H_{2E}=(1-l-v)H$ for all $t\geq 0$, the necessary conditions for optimization are:

$$C^{-\sigma}\exp\left((1-\sigma)\frac{l^{1-\theta}-1}{1-\theta}\right)=p_1, \tag{27}$$

$$C^{1-\sigma}l^{-\theta}\exp\left((1-\sigma)\frac{l^{1-\theta}-1}{1-\theta}\right)=\gamma p_2\beta_2 H, \tag{28}$$

$$\beta_1 p_1=\gamma\beta_2 p_2, \tag{29}$$

$$\dot{p}_2=p_2\left[\rho+\eta-\gamma\beta_2(1-l)\right]. \tag{30}$$

Additionally, the transversality condition is given by $\lim_{t\to\infty}p_2e^{-\rho t}H=0$.

Using (27), (28) and (29), we obtain

$$C=\beta_1 l^{\theta}H. \tag{31}$$

On the other hand, in the presence of socially constant returns to scale, (26) becomes $C=vH$. Thus (31) gives the relation between $l$and v:

$$v=\beta_1 l^{\theta}. \tag{32}$$

Substituting (31) into (28) and taking logarithmic differentiation with respect to time, we obtain

$$-\sigma\theta\frac{\dot{l}}{l}-\sigma\frac{\dot{H}}{H}+(1-\sigma)l^{1-\theta}\frac{\dot{l}}{l}=\frac{\dot{p}_2}{p_2}.$$

Accordingly, from (4'), (30) and (32), the above yields a complete dynamic equation of leisure time l:

$$\dot{l}=l\Delta(l)\left[\gamma(\beta_2-\delta)(1-l)+\sigma\gamma\beta_1 l^{\theta}-\rho-(1-\sigma)\eta\right], \tag{33}$$

where $\Delta(l)=\sigma\theta-(1-\sigma)l^{1-\theta}>0$ by the concavity assumption. Equation (33) summarizes the entire model. Since the initial level of l is not specified, if (33) is stable around the stationary point, local indeterminacy emerges.

Inspection of (33) reveals the following results:

Lemma 4 *(i) There is a unique, balanced-growth equilibrium, if either (i-a) or (i-b) below is satisfied:*

$$\sigma > \max\left\{\beta_2, \frac{\rho+\eta}{\gamma\beta_1+\eta}\right\}, \quad \text{(i-a)}$$
$$\sigma < \min\left\{\beta_2, \frac{\rho+\eta}{\gamma\beta_1+\eta}\right\}. \quad \text{(i-b)}$$

(ii) There may exist dual balanced-growth equilibria, if either (ii-a) or (ii-b) is satisfied:

$$\frac{\rho+\eta}{\gamma\beta_2+\eta} < \sigma < \beta_2,\ \gamma(\beta_2-\sigma) > \rho+(1-\sigma)\eta \text{ and } \theta < 1, \quad \text{(ii-a)}$$
$$\frac{\rho+\eta}{\gamma\beta_2+\eta} < \sigma < \beta_2,\ \gamma(\beta_2-\sigma) < \rho+(1-\sigma)\eta \text{ and } \theta \geq 1. \quad \text{(ii-b)}$$

Proof. Define

$$\mu(l) = \gamma(\beta_2-\sigma)(1-l) + \sigma\beta_1 l^\theta - [\rho+(1-\sigma)\eta].$$

The balanced growth equilibrium level of l is a solution of $\mu(l) = 0$. Note that

$$\mu(0) = \gamma(\beta_2-\sigma) - [\rho+(1-\sigma)\eta],$$
$$\mu(1) = (\gamma\beta_1+\eta)\sigma - (\rho+\eta).$$

If condition (i-a) is held, it is easy to see that $\mu(l)$ is monotonically increasing and $\mu(1) > 0 > \mu(0)$. Thus $\mu(l) = 0$ has a unique solution $l \in (0,1)$. In the case of condition (i-b), we see that $\mu(0) > 0 > \mu(1)$ and $\mu(l)$ is monotonically decreasing. Hence, $\mu(l) = 0$ has only one solution in between 0 and 1. If $(\rho+\eta)/(\gamma\beta_2+\eta) < \sigma < \beta_2$, then $\mu(0)$ and $\mu(1)$ have the same sign. This means that if the balanced-growth path exists, there are at least two equilibria. Under conditions (ii-a), $\mu(0) < 0$, $\mu(1) < 0$ and $\mu(l)$ is strictly convex in l. Therefore, if $\mu(l) = 0$ has solutions, there are two solutions in between 0 and 1. Conversely, under conditions (ii-b), we find that $\mu(0) > 0$, $\mu(1) > 0$ and $\mu(l)$ is strictly concave, and hence $\mu(l) = 0$ also have dual solutions for $l \in (0,1)$. ■

Those results immediately yield the following proposition:

Proposition 2 *Given condition (i-a), the balanced-growth equilibrium is globally determinate, while it is globally indeterminate if condition (i-b) holds. If conditions (ii-a) are satisfied, the balanced-growth equilibrium with a lower level of l is locally indeterminate, while the other with a higher level of l is locally determinate. In case of (ii-b), the opposite results hold.*

Proof. Since condition (i-a) ensures that $\dot{l}/dl > 0$ for all $l \in [0,1]$, the balanced-growth equilibrium is globally determinate. Given condition (i-b), $\dot{l}/dl <$

0 for all $l \in [0, 1]$, so that global indeterminacy is established. In a similar manner, it is easy to see that results for the cases of (ii-a) and (ii-b) can be held. ■

Notice that If the utility function is additively separable between consumption and leisure ($\sigma = 1$), only condition (i-a) can be satisfied. Therefore, we never observe indeterminacy if we assume a separable utility function.

5 Concluding Remarks

This paper has demonstrated that preference structure may play a pivotal role in generating indeterminacy in endogenous growth models. Unlike the existing studies which explore the role of non-separable utility function in growth models, we have demonstrated that in the two-sector endogenous growth setting à la Lucas (1988), indeterminacy may emerge even in the absence of social increasing returns to scale. Since our model precludes the possibility of reversal of social and private factor intensity conditions emphasized by Benhabib and Nishimura (1998 and 1999), indeterminacy mainly stems from preference structure.

In this paper we have assumed that leisure activity of the representative household depends on pure time alone. An alternative formulation, which was suggested by Becker (1975), is that leisure activities need human capital as well[8]. In a longer version of the present paper (Mino 2000), it is demonstrated that if effective leisure depends on the level of human capital as well as on time, the economy has a unique balanced-growth equilibrium.[9] In this setting indeterminacy will not emerge under social constant returns. In the presence of social increasing returns, non-separability of the utility function, however, may be relevant for generating indeterminacy. These results suggest that not only form of the utility function but also specification of leisure activities would be relevant for the emergence of indeterminacy. Therefore, if we consider leisure as a home good produced by a more general technology than that we have assumed in the paper, we may have a larger possibility of indeterminacy under weaker restrictions on the production technology.

[8] A simple utility function that captures this idea is

$$u(c, lh) = \frac{\left(c^{\gamma}(lh)^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}.$$

[9] Ortigueira (1998) presents a detailed analysis of this class of model that does not involve externalities (so that indeterminacy will not emerge).

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