A Variational Problem Governed by a Differential Inclusion in a Banach Space

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1 Introduction

Let $\mathcal{X}$ be a real separable reflexive Banach space. A correspondence (= multi-valued mapping) $\Gamma : [0, T] \times \mathcal{X} \rightarrow \mathcal{X}$ and a function $u : [0, T] \times \mathcal{X} \times \mathcal{X} \rightarrow \overline{\mathbb{R}}$ are assumed to be given. A double arrow $\rightarrow$ indicates the domain and the range of a correspondence. The compact interval $[0, T]$ is endowed with the Lebesgue measure $dt$. $\mathcal{L}$ denotes the $\sigma$-field of the Lebesgue-measurable sets of $[0, T]$.

Let $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ be the Sobolev space consisting of functions of $[0, T]$ into $\mathcal{X}$ (cf. Appendix) and let $\triangle(a)$ be the set of all the solutions in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ of a differential inclusion:

\[ (\ast) \quad \dot{x}(t) \in \Gamma(t, x(t)), \quad x(0) = a, \]

where $\dot{x}$ denotes the derivative of $x$ and $a$ is a fixed vector in $\mathcal{X}$. And consider a variational problem:

\[ (\#) \quad \text{Minimize}_{x \in \triangle(a)} \int_0^T u(t, x(t), \dot{x}(t)) dt. \]

The object of this paper is to discuss a couple of existence problems as follows:

(i) the existence of a solution for the differential inclusion $(\ast)$, and
(ii) the existence of an optimal solution for the variational problem $(\#)$.
In Maruyama [14] [15], I presented a solution of these problems in the special case $\mathcal{X} = \mathbb{R}^\ell$ by making use of the convenient properties of the weak convergence in the Sobolev space $W^{1,2}([0, T], \mathbb{R}^\ell)$; i.e. if a sequence $\{x_n\}$ in $W^{1,2}([0, T], \mathbb{R}^\ell)$, weakly converges to some $x^* \in W^{1,2}([0, T], \mathbb{R}^\ell)$, then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$
\begin{align*}
  z_n & \rightarrow x^* \quad \text{uniformly on } [0, T], \quad \text{(W)} \\
  \dot{z}_n & \rightarrow \dot{x}^* \quad \text{weakly in } L^2([0, T], \mathbb{R}^\ell).
\end{align*}
$$

However it deserves a special notice that this property does not hold in the space $W^{1,2}([0, T], \mathcal{X})$ if $\dim \mathcal{X} = \infty$. Taking account of this fact, I provided a new convergence result to overcome this difficulty in the case $\mathcal{X}$ is a real separable Hilbert space in Maruyama [17]. And I also gave a existence theory for the problems (i) and (ii) being based upon this new tool in the framework of a separable Hilbert space in Maruyama [17],[18].

The purpose of the present paper is to generalize my previous results to the case $\mathcal{X}$ is a real separable reflexive Banach space. Papageorgiou [19] also gave an elegant extension of my results in Maruyama [14],[15] to the infinite dimensional case. The present paper might be regarded as an alternative approach to Papageorgiou's theory.

Let me mention about another improvement added on this occasion. In Maruyama [17], I imposed a very restrictive requirement on the continuity of the correspondence $\Gamma$; i.e.

- the correspondence $z \mapsto \Gamma(t, z)$ is upper hemi-continuous for each fixed $t \in [0, T]$ with respect to the weak topology for the domain
- the strong topology for the range.

I have to admit frankly that this is a very unpleasant assumption. In the present paper, I propose the upper hemi-continuity of $z \mapsto \Gamma(t, x)$ with respect to the "weak-weak" combination of topologies instead of the "weak-strong" combination.

## 2 A Convergence Theorem in $W^{1,p}([0, T], \mathcal{X})$

As I have already said, any weakly convergent sequence $\{x_n\}$ in the Sobolev space $W^{1,2}([0, T], \mathbb{R}^\ell)$ has a subsequence which satisfies the property (W) in section 1.

On the other hand, let $\mathcal{X}$ be a real Banach space with the Radon-Nikodým property (RNP). Then any absolutely continuous function $f : [0, T] \rightarrow \mathcal{X}$ is Fréchet-differentiable a.e. (If the Banach space $\mathcal{X}$ does not have RNP, this property does not hold. For a counter-example, see Komura [13].) Let $\{x_n\}$ be a sequence in $W^{1,p}([0, T], \mathcal{X})$ which weakly converges to some $x^* \in W^{1,p}([0, T], \mathcal{X})$. 


We should keep in mind that it is not necessarily true that the sequence \( \{x_n\} \) has a subsequence \( \{z_n\} \) which satisfies the property (W) if \( \dim \mathcal{X} = \infty \) even in the case \( p = 2 \).

**Counter-Example** (Cecconi[9], pp.28-29) Let \( \mathcal{H} \) be a real separable Hilbert space and \( \{\varphi_n ; n=1,2, \cdots\} \) a complete orthonormal system of \( \mathcal{H} \). (cf. Yosida [28] P.89.) Define a sequence \( \{x_n : [0,T] \rightarrow \mathcal{H}\} \) by

\[
x_n(t) = t\varphi_n \quad (n=1,2, \cdots)
\]

We also define the function \( x^* : [0,1] \rightarrow \mathcal{H} \) by \( x^*(t) \equiv 0 \). Then \( x_n \)'s as well as \( x^* \) are elements of \( \mathcal{W}^{1,2}([0,T], \mathcal{H}) \). It follows from the Riemann-Lebesgue lemma that the sequence \( \{x_n\} \) weakly converges to \( x^* \) in \( \mathcal{W}^{1,2}([0,1], \mathcal{H}) \). However there is no subsequence of \( \{x_n\} \) which converges strongly (hence uniformly) to \( x^* \) in \( L^2([0,1], \mathcal{H}) \).

The following theorem cultivated to overcome this difficulty is a generalization of Theorem 1 of Maruyama [18]. Henceforth we denote by \( \mathcal{X}_s \) (resp.\( \mathcal{X}_w \)) a Banach space \( \mathcal{X} \) endowed with the strong (resp. weak) topology.

**THEOREM 1.** Let \( \mathcal{X} \) be a real separable reflexive Banach space. And consider a sequence \( \{x_n\} \) in the Sobolev space \( \mathcal{W}^{1,p}([0,T], \mathcal{X})(p \geq 1) \). Assume that

(i) the set \( \{x_n(t)\}_{n=1}^\infty \) is bounded (and hence relatively compact) in \( \mathcal{X}_w \) for each \( t \in [0,T] \), and

(ii) there exists some function \( \psi \in \mathcal{L}^p([0,T], (0, +\infty)) \) such that

\[
\|\dot{x}_n(t)\| \leq \psi(t) \quad \text{a.e.}
\]

Then there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and some \( x^* \in \mathcal{W}^{1,p}([0,T], \mathcal{X}) \) such that

(a) \( z_n \rightarrow x^* \) uniformly in \( \mathcal{X}_w \) on \( [0,T] \), and

(b) \( \dot{z}_n \rightarrow \dot{x}^* \) weakly in \( \mathcal{L}^p(0,T, \mathcal{X}) \).

Remark Since \( \mathcal{X} \) is separable and reflexive, the following results holds true. Assume that \( p \geq 1 \).

[1] \( \mathcal{L}^p([0,T], \mathcal{X}) \) is separable.
[II] $L^p([0, T], \mathcal{X})'$ is isomorphic to $L^q([0, T], \mathcal{X}')$, where $1/p + 1/q = 1$ and " , " denotes the dual space.

[III] Any absolutely continuous function $f : [0, T] \to \mathcal{X}$ is Fréchet-differentiable a.e. and the "fundamental theorem of calculus", i.e.

$$f(t) = f(0) + \int_0^t f(\tau) d\tau; t \in [0, T]$$

is valid.

Proof of Theorem 1. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since $\psi$ is integrable, there exists some $\delta > 0$ for each $\epsilon > 0$ such that

$$|| x_n(t) - x_n(s) || \leq \int_s^t || \dot{x}_n(\tau) || d\tau \leq \int_s^t \psi(\tau) d\tau \leq \epsilon$$

for all $n$ provided that $|t - s| \leq \delta$. This proves the equicontinuity of $\{x_n\}$ in the strong topology for $\mathcal{X}$. Hence $\{x_n\}$ is also equicontinuous in the weak topology for $\mathcal{X}$.

Taking account of this fact as well as the assumption (i), we can claim, thanks to the Ascoli-Arzelà theorem (cf. Schwartz[21] p.78), that $\{x_n\}$ is relatively compact in $C([0, T], \mathcal{X}_w)$ (the set of continuous functions of $[0, T]$ into $\mathcal{X}$) with respect to the topology of uniform convergence.

By the assumption (i), $\{x_n(0)\}$ is bounded in $\mathcal{X}$, say

$$\sup_n || x_n(0) || \leq C < +\infty.$$ 

And the assumption (ii) implies that

$$\left\| \int_0^t \dot{x}_n(\tau) d\tau \right\| \leq || \psi ||_1$$

for all $t \in [0, T]$.

Hence

$$\sup_n || x_n(t) || = \sup_n || x_n(0) + \int_0^t \dot{x}_n(\tau) d\tau || \leq C + || \psi ||_1$$

for all $t \in [0, T]$.

Thus each $x_n$ can be regarded as a mapping of $[0, T]$ into the set

$$M = \{ w \in \mathcal{X} | \| w \| \leq C + || \psi ||_1 \}.$$ 

The weak topology on $M$ is metrizable because $M$ is bounded and $\mathcal{X}$ is a
separable reflexive Banach space. Hence if we denote by $M_w$ the space $M$ endowed with the weak topology, then the uniform convergence topology on $C([0, T], M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $C([0, T], M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in C([0, T], X_w)$.

(b) Since

$$\| \dot{y}_n(t) \| \leqslant \psi(t) \quad \text{a.e.},$$

the sequence $\{w_n : [0, T] \to X\}$ defined by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)} ; \quad n = 1, 2, \ldots$$

is contained in the unit ball of $L^\infty([0, T], X)$ which is weak*-compact (as the dual space of $L^1([0, T], X')$) by Alaoglu's theorem. Note that the weak* topology on the unit ball of $L^\infty([0, T], X)$ is metrizable since $L^1([0, T], X')$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in L^\infty([0, T], X)$ in the weak* topology. We shall write $\dot{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : L^\infty([0, T], X) \to L^p([0, T], X)$ by

$$A : g \mapsto \psi \cdot g,$$

then $A$ is continuous in the weak* topology for $L^\infty$ and the weak topology for $L^p$. In order to see this, let $\{g_\lambda\}$ be a net in $L^\infty([0, T], X)$ such that $w^* - \lim_\lambda g_\lambda = g^* \in L^\infty([0, T], X)$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t) \rangle \, dt \to \int_0^T \langle \alpha(t), g^*(t) \rangle \, dt \quad \text{for all} \quad \alpha \in L^1([0, T], X').$$

Then it is quite easy to verify that

$$\int_0^T \langle \beta(t), \psi(t)g_\lambda(t) \rangle \, dt \to \int_0^T \langle \beta(t), \psi(t)g^*(t) \rangle \, dt \quad \text{for all} \quad \beta \in L^q([0, T], X'),$$

$$1/p + 1/q = 1$$

since $\psi \cdot \beta \in L^1([0, T], X')$. This proves the continuity of $A$.

Hence

$$\dot{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \quad \text{weakly in} \quad L^p([0, T], X'),$$

which implies
\[ \langle \theta, \int_{s}^{t} \dot{z}_{n}(\tau) d\tau \rangle = \int_{s}^{t} \langle \theta, \psi(\tau) \cdot w^{*}(\tau) \rangle d\tau \quad \text{for all } \theta \in \mathcal{X}'. \]  

(2)

On the other hand, since

\[ z_{n}(t) - z_{n}(s) = \int_{s}^{t} z_{n}(\tau) d\tau \quad \text{for all } n, \]

and

\[ z_{n}(t) - z_{n}(s) \to x^{*}(t) - x^{*}(s) \quad \text{in } \mathcal{X}_{w}, \]

we get

\[ \langle \theta, \int_{s}^{t} \dot{z}_{n}(\tau) d\tau \rangle = \langle \theta, z_{n}(t) - z_{n}(s) \rangle \to \langle \theta, x^{*}(t) - x^{*}(s) \rangle \quad \text{for all } \theta \in \mathcal{X}'. \]  

(3)

(2) and (3) imply that

\[ \langle \theta, x^{*}(t) - x^{*}(s) \rangle = \langle \theta, \int_{s}^{t} \psi(\tau) \cdot w^{*}(\tau) d\tau \rangle \quad \text{for all } \theta \in \mathcal{X}', \]

from which we can deduce the equality

\[ x^{*}(t) - x^{*}(s) = \int_{s}^{t} \psi(\tau) \cdot w^{*}(\tau) d\tau. \]  

(4)

By (1) and (4), we get the desired result:

\[ \dot{z}_{n} \to \dot{x}^{*} = \psi \cdot w^{*} \quad \text{weakly in } \mathcal{L}^{p}([0,T], \mathcal{X}). \]

\[ \square \]

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [1] (pp.13-14). However their reasoning does not seem to be perfectly sound.

3 Differential Inclusions

In this section, we prepare several lemmas which are to play crucial roles in the existence theory for differential inclusions.
Throughout this section, \( \mathcal{X} \) is assumed to be a real separable reflexive Banach space.

Let us begin by specifying some assumptions imposed on the correspondence \( \Gamma : [0, T] \times \mathcal{X}_w \rightarrow \mathcal{X}_w \). Special attentions should be paid to the fact that both of the domain and the range of \( \Gamma \) are endowed with the weak topologies.

**Assumption 1.** \( \Gamma \) is compact-convex-valued; i.e. \( \Gamma(t, x) \) is a non-empty, compact and convex subset of \( \mathcal{X}_w \) for all \( t \in [0, T] \) and all \( x \in \mathcal{X} \).

**Assumption 2.** The correspondence \( x \mapsto \Gamma(t, x) \) is upper hemi-continuous (abbreviated as \( \mathrm{u.h.c.} \)) for each fixed \( t \in [0, T] \); i.e. for any fixed \( (t, x) \in [0, T] \times \mathcal{X}_w \) and for any neighborhood \( V \) of \( \Gamma(t, x) \subset \mathcal{X}_w \), there exists some neighborhood \( U \) of \( x \) such that \( \Gamma(t, z) \subset V \) for all \( z \in U \).

**Assumption 3.** The graph of the correspondence \( t \mapsto \Gamma(t, x) \) is \( (\mathcal{L}, B(\mathcal{X}_w)) \)-measurable for each fixed \( x \in \mathcal{X} \) where \( B(\mathcal{X}_w) \) denotes the Borel \( \sigma \)-field on \( \mathcal{X}_w \). (For the concept of \"measurability\" of a correspondence, the best reference is Castaing-Valadier [8] Chap.III.)

**Assumption 4.** \( \Gamma \) is \( L^p \)-integrably bounded; i.e. there exists \( \psi \in L^p([0, T], (0, +\infty))(p > 1) \) such that \( \Gamma(t, x) \subset S_{\psi(t)} \) for every \( (t, x) \in [0, T] \times \mathcal{X} \), where \( S_{\psi(t)} \) is the closed ball in \( \mathcal{X} \) with the center \( 0 \) and the radius \( \psi(t) \).

The following lemma is essentially due to Castaing [5].

**LEMMA 1** (Castaing [5]) Suppose that a correspondence \( \Gamma : \mathcal{X} \rightarrow \mathcal{X} \) satisfies the Assumptions 1-3, and that a function \( x : [0, T] \rightarrow \mathcal{X} \) is Bochner-integrable. Then there exists a closed-valued correspondence \( \Sigma : [0, T] \rightarrow \mathcal{X}_w \) such that

\[
\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for all} \quad t \in [0, T],
\]

and the graph \( G(\Sigma) \) of \( \Sigma \) is \( (\mathcal{L}, B(\mathcal{X}_w)) \)-measurable.

**Proof.** Let \( \{x_n : [0, T] \rightarrow \mathcal{X}\} \) be a sequence of simple functions which satisfies that

\[
\| x_n(t) - x(t) \| \rightarrow 0 \quad \text{for each} \quad t \in [0, T] \quad \text{as} \quad n \rightarrow \infty.
\]

(For the existence of such a sequence, see Yosida [28] p.133.)

Define a correspondence \( \Gamma_n : [0, T] \rightarrow \mathcal{X}_w \) by

\[
\Gamma_n : t \mapsto \Gamma(t, x_n(t)); n = 1, 2, \cdots.
\]
Then it can be shown that the graph $G(\Gamma_n)$ of each $\Gamma_n$ is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. In order to confirm it, we denote by \( \{y_1, y_2, \cdots, y_k\} \) the image of \([0, T]\) by the simple function $x_n$; i.e.
\[
x_n([0, T]) = \{y_1, y_2, \cdots, y_k\}.
\]
Furthermore if we define a correspondence $\Phi_j : [0, T] \rightarrow \mathcal{X}_w$ ($j = 1, 2, \cdots, k$) by
\[
\Phi_j : t \mapsto \Gamma(t, y_j),
\]
then the graph $G(\Phi_j)$ of $\Phi_j$ is obviously $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. The graph $G(\Gamma_n)$ of $\Gamma_n$ can be expressed as
\[
G(\Gamma_n) = \bigcup_{j=1}^{k} G(\Phi_j|_{x_n^{-1}(\{y_j\})}),
\]
where $\Phi_j|_{x_n^{-1}(\{y_j\})}$ is the restriction of the correspondence $\Phi_j$ to the set $x_n^{-1}(\{y_j\}) = \{t \in [0, T] | x_n(t) = y_j\}$. Since $G(\Phi_j|_{x_n^{-1}(\{y_j\})})(j = 1, 2, \cdots, k)$ is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable, so is $G(\Gamma_n)$.

Since $\|x_n(t) - x(t)\| \rightarrow 0$ for each $t \in [0, T]$ as $n \rightarrow \infty$, the set \( \{x_1(t), x_2(t), \cdots; x(t)\} \) is weakly compact for each $t \in [0, T]$. Furthermore, by the Assumptions 1-2, the correspondence $\Gamma$ is compact-valued and u.h.c. in the second variable. Consequently the set
\[
\bigcup_{n=1}^{\infty} \Gamma(t, x_n(t))
\]
is relatively compact in $\mathcal{X}_w$ (for each $t \in [0, T]$). Taking account of the fact that the weak topology of a weakly compact subset of a separable Banach space is metrizable, we can conclude, by Baire's category theorem, that the set
\[
\Sigma(t) \equiv \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \overline{\Gamma(t, x_m(t))}
\]
is non-empty (for each $t \in [0, T]$), where $-w$ denotes the closure operation with respect to the weak topology.

The correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ is closed-valued and its graph is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. Finally the inclusion
\[
\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for each} \quad t \in [0, T]
\]
is clear because $\Gamma$ is compact-valued and u.h.c. $\square$

We can show the Next lemma in a similar way as in Maruyama[17], taking account of [III] of the Remark on page 4.

**Lemma 2** Let $A$ be a non-empty compact and convex set in $\mathcal{X}_w$, and $X$ a subset of $\mathcal{W}^{1,p}([0, T], \mathcal{X})(p > 1)$ defined by
$X = \{ x \in \mathcal{W}^{1,p} | \| \dot{x}(t) \| \leq \psi(t) \text{ a.e.}, \ x(0) \in A \}$, where $\psi \in L^p([0,T],(0, +\infty))$. Then $X$ is non-empty convex and compact in $\mathcal{X}_w$.

**Proof.** Since it is obvious that $X$ is non-empty and convex, we have only to show the weak compactness of $X$.

It is not hard to show the boundedness of $X$. Let $x$ be any element of $X$. Then $x$ can be represented in the form

$$x(t) = a + \int_0^t \dot{x}(\tau) d\tau; \ t \in [0, T]$$

($a$ is a point of $A$) by [III] of the Remark on page 3. It follows that

$$\| x(t) \| = \| a + \int_0^t \dot{x}(\tau) d\tau \| \leq \| a \| + \int_0^t \| \dot{x}(\tau) \| d\tau \leq \| a \| + \int_0^T \psi(\tau) d\tau \leq B + \int_0^T \psi(\tau) d\tau,$$

where $B = \sup_{a \in A} \| a \| < +\infty$. Consequently we have the evaluation:

$$\sup_{x \in X} \| x \|_p^p \leq [B + \int_0^T \psi(\tau)d\tau]^p \cdot T < +\infty,$$

where $\| \cdot \|_p$ denotes the $L^p$-norm. Since the right-hand side is independent of $x$, $X$ is bounded in $L^p$. On the other hand, the set $\{ \dot{x} | x \in X \}$ is also bounded by $\| \psi \|_p$. Therefore we can claim that $X$ is bounded in $\mathcal{W}^{1,p}$.

$\mathcal{W}^{1,p}$ is reflexive because $\mathcal{X}$ is reflexive and $p > 1$. Hence the bounded set $X$ is weakly relatively compact in $\mathcal{W}^{1,p}$.

To show the weak compactness of $X$, we need only to show the weak closedness of $X$. However $X$ is weakly closed if and only if $X$ is strongly closed since $X$ is convex. Let $\{x_n\}$ be a sequence in $X$ which strongly converges to $x^*$ in $\mathcal{W}^{1,p}$. Then $\{\dot{x}_n\}$ has a subsequence, say $\{\dot{x}_{n'}\}$, which converges to $\dot{x}^*$ a.e. Since $\| \dot{x}_{n'}(t) \| \leq \psi(t)$ a.e., it follows that

$$\| \dot{x}^*(t) \| \leq \psi(t) \text{ a.e.}$$

Finally it is clear that $x^*(0) \in A$. Then we obtain $x^* \in X$. This proves that $X$ is strongly closed in $\mathcal{W}^{1,p}$.

We denote by $\mathcal{B}(0; \mathcal{X}_w)$ a neighborhood base of the zero element of $\mathcal{X}_w$ which consists of convex sets. The following lemma plays a crucial role in the
subsequent arguments although its proof is easy.

**LEMMA 3** Suppose that the Assumptions 1-2 are satisfied. Let \((t^*, x^*)\) be any point of \([0, T] \times \mathcal{X}\). Define, for any \(V \in B(0; \mathcal{X}_w)\), a subset \(K(t^*; x^*, V)\), of \([0, T] \times \mathcal{X}\) by

\[
K(t^*; x^*, V) = \{(t, x) \in [0, T] \times \mathcal{X} | x \in x^* + V, t = t^*\}.
\]

Then we have

\[
\Gamma(t^*, x^*) = \bigcap_{V \in B(0; \mathcal{X}_w)} \overline{\mathrm{co}} \Gamma(K(t^*; x^*, V)).
\]

(Here we do not have to distinguish the convex closure with respect to the strong topology and that with respect to weak topology. So I simply denote it by \(\overline{\mathrm{co}}\).)

**LEMMA 4** Suppose that the Assumptions 1, 2 and 4 (with \(p > 1\)) are satisfied. Let \(A\) be a non-empty convex compact subset of \(\mathcal{X}_w\). Then the set

\[
H \equiv \{(a, x, y) \in A \times X \times X | \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e. and } x(0) = y(0) = a\}
\]

is weakly compact in \(A \times X \times X\). (The set \(X\) is defined in Lemma 2.)

**Proof.** Since we have already known that \(A \times X \times X\) is weakly compact in \(\mathcal{X} \times \mathcal{W}^{1,p} \times \mathcal{W}^{1,p}\), it is enough to show that \(H\) is a weakly closed subset of \(A \times X \times X\).

Since \(\mathcal{W}^{1,p}\) is a reflexive Banach space, the dual of which is separable, the weak topology on the bounded set \(X\) is metrizable. So we are permitted to use a sequence argument.

Let \(\{q_n \equiv (a_n, x_n, y_n)\}\) be a sequence in \(H\) which weakly converges to some \(q^* = (a^*, x^*, y^*)\) in \(A \times X \times X\). We have to show that \(q^* \in H\). And it is enough to check that

\[
\dot{y}^*(t) \in \Gamma(t, x^*(t)) \text{ a.e.}
\]

The set \(\{x_n(t)\}\) is relatively compact in \(\mathcal{X}_w\) (for each \(t \in [0, T]\)) since we have the evaluation:

\[
\|x_n(t)\| \leq \|a\| + \int_0^T \|\dot{x}_n(\tau)\| d\tau \leq \|a\| + \int_0^T \psi(\tau) d\tau
\]

by the Assumption 4. Hence, thanks to Theorem 1, \(\{q_n\}\) has a subsequence (no change in notation) such that

\[
x_n(t) \to x^*(t) \text{ uniformly in } \mathcal{X}_w, \text{ and} \quad (1)
\]

\[
y_n(t) \to \dot{y}^*(t) \text{ weakly in } \mathcal{L}^p. \quad (2)
\]
Then applying Mazur's theorem, we can choose, for each $j \in \mathbb{N}$, some finite elements
\[ \dot{y}_{n_{j}+1}, \dot{y}_{n_{j}+2}, \ldots, \dot{y}_{n_{j}+m(j)} \]
of $\{\dot{y}_{n}\}$ and numbers
\[ \alpha_{ij} \geq 0, 1 \leq i \leq m(j), \sum_{i=1}^{m(j)} \alpha_{ij} = 1 \]
such that
\[ \| \dot{y}^{*} - \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_{j}+i} \|_{\mathcal{P}} \leq \frac{1}{j}, n_{j+1} > n_{j} + m(j). \]
Denoting
\[ \eta_{j}(t) = \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_{j}+i}(t), \]
we obtain
\[ \eta_{j}(t) \in \co(\bigcup_{i=1}^{m(j)} \Gamma(t, x_{n_{j}+i}(t)). \]
Since $\{\eta_{j}\}$ has a subsequence which converges to $y^{*}$ a.e., we may assume, without loss of generality, that
\[ \| \eta_{j}(t) - y^{*}(t) \| \to 0 \text{ a.e.} \quad (3) \]
On the other hand, for each $V \in B(0; \mathcal{X}_{w})$, there exists some $n_{0}(V) \in \mathbb{N}$ such that
\[ x_{n}(t) \in x^{*}(t) + V \]
for all $n \geq n_{0}(V)$ and for all $t \in [0, T]$.
That is,
\[ (t, x_{n}(t)) \in K(t; x^{*}(t), V) \text{ for all } n \geq n_{0}(V) \text{ and for all } t \in [0, T]. \]
Hence we have
\[ \eta_{j}(t) \in \co(\Gamma(K(t; x^{*}(t), V)) \text{ a.e.} \]
for sufficiently large $j$. Passing to the limit, we obtain
\[ \dot{y}^{*}(t) \in \overline{\co(\Gamma(K(t; x^{*}(t), V)))} \text{ a.e.} \quad (4) \]
by (3). Since (4) holds true for all $V \in B(0; \mathcal{X}_w)$, it follows that

$$y^*(t) \in \cap_{V \in B(0, \mathcal{X}_w)} \text{co} \Gamma(K(t; x^*(t), V) = \Gamma(t, x^*) \quad \text{a.e.}$$

The last equality in (5) comes from Lemma 3. Thus we have proved that $(a^*, x^*, y^*) \in H$. □

4 Differential Inclusions (2)

$\mathcal{X}$ is still assumed to be a real separable reflexive Banach space in this section.

We are now going to find out a solution of $(\ast)$ in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X}), p > 1$. Define a set $\Delta(a)$ in $\mathcal{W}^{1,p}$ by

$$\Delta(a) = \{ x \in \mathcal{W}^{1,p} | x \text{ satisfies } (\ast) \quad \text{a.e.} \}$$

for a fixed $a \in \mathcal{X}$. The following theorem tells us that $\Delta(a) \neq \emptyset$ and that $\Delta$ depends continuously, in some sense, upon the initial value $a$.

**THEOREM 2.** Suppose that the correspondence $\Gamma$ satisfies the Assumptions 1-4. Let $A$ be a non-empty, convex and compact subset of $\mathcal{X}_w$. Then

(i) $\Delta(a^*) \neq \emptyset$ for any $a^* \in A$, and

(ii) the correspondence $\Delta : A \rightarrow \mathcal{W}^{1,p}$ is compact-valued and u.h.c. on $A_w$, in the weak topology for $\mathcal{W}^{1,p}$.

The proof is essentially the same as in Maruyama [17].

**Proof.** (i) Fix any $a^* \in A$. If we define a set $X(a^*) \subset X$ by $X(a^*) = \{ x \in X | x(0) = a^* \}$, then $X(a^*)$ is convex and weakly compact in $\mathcal{W}^{1,p}$. Furthermore we define a correspondence $\Phi : X(a^*)_w \rightarrow X(a^*)_w$ by

$$\Phi(x) = \{ y \in X(a^*) | \dot{y}(t) \in \Gamma(t, x(t)) \quad \text{a.e.} \}.$$ 

Then the problem is simply reduced to finding out a fixed point of $\Phi$.

1° $\Phi(x) \neq \emptyset$ for every $x \in X(a^*)$ — This fact can be proved through the Measurable Selection Theorem.

Let $x$ be any element of $X(a^*)$. Then by Lemma 1, there exists a closed-valued correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ such that $\Sigma(t) \subset \Gamma(t, x(t))$ for all $t \in [0, T]$, and its graph is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. We also note that $\mathcal{X}_w$ is a Souslin space. Thanks to Saint-Beuve's measurable selection theorem (Saint-Beuve [20]), $\Sigma$ admits a $(\mathcal{L}, B(\mathcal{X}_w))$-measurable selection $\sigma : [0, T] \rightarrow \mathcal{X}$. Since
$\mathcal{X}$ is separable, $\sigma$ is $(\mathcal{L}, B(\mathcal{X}_s))$-mesurable. (cf. Yosida [28] p.131.) By the Assumption 4, $\sigma$ is clearly integrable. If we define a function $y : [0, T] \rightarrow \mathcal{X}$ by

$$y(t) = a^* + \int_0^t \sigma(\tau)d\tau,$$

then $y \in \Phi(x)$.

2° $\Phi$ is convex-compact-valued. — This is not hard.

3° $\Phi$ is u.h.c. — If we define the $a^*$-selection $H_{a^*}$ of $H$ by $H_{a^*} = \{(a, x, y) \in H | a = a^*\}$, then $H_{a^*}$ is obviously weakly compact in $A \times X \times X$. And the graph $G(\Phi)$ of $\Phi$ is expressed as $G(\Phi) = \text{proj}_{X \times X}H_{a^*}$, the projection of $H_{a^*}$ into $X \times X$, which is also closed.

Summing up — $\Phi$ is convex-compact-valued and u.h.c. Applying now the Fan-Glicksberg Fixed-Point Theorem to the correspondence $\Phi$, we obtain an $x^* \in X(a^*)$ such that $x^* \in \Phi(x^*)$; i.e.

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \; \text{a.e. and} \; x^*(0) = a^*.$$

This proves (i).

(ii) Since the compactness of $\Delta(a)(a \in A)$ can be verified by applying Mazur’s theorem and making use of the Assumptions 1-2, we may omit the details. Hence we have only to show the u.h.c. of $\Delta$. However it is also obvious because the graph $G(\Delta)$ of $\Delta$ can be expressed as

$$G(\Delta) = \text{proj}_{A \times X} \{(a, x, y) \in H | x = y\},$$

which is closed in $A \times X$.

I am much indebted to Castaing-Valadier [7] for various important ideas embodied in the proof of Theorem 2.

**Remark.** Among other things, the assumption that the set $\Gamma(t, x)$ is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See De Blasi [10] and Tateishi [23].)

Here it may be suggestive for us to glimpse the special case in which $\Gamma$ is a (single-valued) mapping. A related result was obtained by Szep [23]. (I am indebted to Professor Tosio Kato for this reference.)

**COROLLARY 1.** Let $f : [0, T] \times X_w \rightarrow X_w$ be a (single-valued) mapping which satisfies the following three conditions.
(i) The function $x \mapsto f(t, x)$ is continuous for each fixed $t \in [0, T]$.

(ii) The function $t \mapsto f(t, x)$ is measurable for each fixed $x \in \mathcal{X}$.

(iii) There exists $\psi \in L^p([0, T], (0, +\infty)), p > 1$ such that $f(t, x) \in S_{\psi(t)}$ for every $(t, x) \in [0, T] \times \mathcal{X}$; i.e. $\sup_{x \in \mathcal{X}} \|f(t, x)\| \leq \psi(t)$ for all $t \in [0, T]$.

Then the differential equation

$$\dot{x} = f(t, x), \; x(0) = a \text{ (fixed vector in } \mathcal{X})$$

has at least a solution in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$. (A solution of $\dot{x} = f(t, x)$ is a function $x \in \mathcal{W}^{1,p}$ which satisfies $\dot{x} = f(t, x)$ a.e.)

5 Variational problem governed by an Differential Inclusion

Let $\mathcal{X}$ be a real separable reflexive Banach space throughout this section, too. Assume that $u : [0, T] \times \mathcal{X} \times \mathcal{X} \rightarrow (-\infty, +\infty]$ is a given proper function. Consider a variational problem:

$$\text{Minimize } x \in \Delta(a) \rightarrow J(x) = \int_{0}^{T} u(t, x(t), \dot{x}(t)) dt,$$

where $\Delta(a)$ is the set of all the solutions of the differential inclusion (1) discussed in the preceding sections.

In order to examine the existence of a solution of the problem (1), we have to check a couple of points as usual; i.e.

(I) the compactness of $\Delta(a)$ for some suitable topology, and

(II) the lower semi-continuity of the functional $J$ for the same topology.

Since we have already proved that $\Delta(a)$ is weakly compact in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ under certain conditions, we are concentrating on the second point (II) in this section. In this context, the theorem due to Castaing-Clauzure [6] provides the most crucial key. Related results are also obtained by Balder [2], Maruyama [16] and Valadier [25].

**DEFINITION** Let $(\Omega, \xi, \mu)$ be a measure space, $\mathcal{S}$ a topological space, and $V$ a real Banach space. A function $f : \Omega \times S \times V \rightarrow \mathbb{R}$ is assumed to be given. We denote by $\mathcal{M}(\Omega, S)$ the set of all the $(\xi \otimes B(S))$-measurable functions. ($B(S)$ denotes the Borel $\sigma$-field on $S$.) $f$ is said to have the lower compactness property if $\{f^{-}(\omega, \varphi_n(\omega), \theta_n(\omega))\}$ is weakly relatively compact in $L^{1}(\Omega, \mathbb{R})$ for any sequence $\{(\varphi_n, \theta_n)\}$ in $\mathcal{M}(\Omega, S) \times L^{p}(\Omega, V)(p \geq 1)$ which satisfies the following three conditions:
(a) \( \{ \varphi_n \} \) converges in measure to some \( \varphi^* \in \mathcal{M}(\Omega, S) \),
(b) \( \{ \theta_n \} \) converges weakly to some \( \theta^* \in \mathcal{L}^p(\Omega, \mathcal{V}) \), and
(c) there exists some \( C < +\infty \) such that
\[
\sup_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega))d\mu \leq C.
\]

The following theorem is a variation of a result due to Castaing-Clauzure \[6\] in the spirit of Ioffe \[12\]. See also Valadier \[27\].

**THEOREM 3** Let \( (\Omega, \xi, \mu) \) be a finite complete measure space, \( S \) a metrizable Souslin space, and \( \mathcal{V} \) a separable reflexive Banach space. Suppose that a proper function \( f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}} \) satisfies the following conditions:

(i) \( f \) is a normal integrand; i.e.
   (a) \( f \) is \( (\xi \otimes B(S) \otimes B(\mathcal{V}), B(\overline{\mathbb{R}})) \)-measurable, and
   (b) the function \( (\xi, v) \mapsto f(\omega, \xi, v) \) is lower semi-continuous for any fixed \( \omega \in \Omega \),

(ii) the function \( v \mapsto f(\omega, \xi, v) \) is convex for any fixed \( (\omega, \xi) \in \Omega \times S \), and

(iii) \( f \) has the lower compactness property.

Let \( \{ \varphi_n \} \) be a sequence in \( \mathcal{M}(\Omega, S) \) which converges in measure to some \( \varphi^* \in \mathcal{M}(\Omega, S) \). Let \( \{ \theta_n \} \) be a sequence in \( \mathcal{L}^p(\Omega, \mathcal{V})(1 \leq p < +\infty) \) which converges weakly to some \( \theta^* \in \mathcal{L}^p(\Omega, \mathcal{V}) \). Then we have
\[
\int_{\Omega} f(\omega, \varphi^*(\omega), \theta^*(\omega))d\mu \leq \liminf_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega))d\mu.
\]

**Remark 1°** A normal integrand \( f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}} \) which also satisfies the condition (ii) is called a convex normal integrand.

**2°** Ioffe \[8\] established a fundamental theorem on the lower semi-continuity of a nonlinear integral functional as above in the case both of \( S \) and \( \mathcal{V} \) are finite dimensional Euclidean spaces. Theorem 3 is an extension of Ioffe's result to the case of nonlinear integral functional defined on the space of Bochner integrable functions.

**LEMMA 5** Suppose that the Assumptions 1-4 are satisfied. Let \( \{ x_n \} \) be a sequence in \( \Delta(a) \subset \mathcal{W}^{1,p}([0, T], \mathcal{X}) (p > 1) \). Let \( u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \rightarrow \overline{\mathbb{R}} \) be a proper convex normal integrand with the lower compactness property. Then there exists a subsequence \( \{ x_n \} \) of \( \{ x_n \} \) and \( x^* \in \Delta(a) \) such that
\[ J(z^*) \leq \liminf_n J(z_n), \quad (1) \]

where

\[ J(x) = \int_0^T u(t, x(t), \dot{x}(t)) dt. \]

**Proof.** By the Assumption 4, all the images of \( x_n \)'s are contained in some closed ball \( \overline{B} \) with the center 0; i.e.

\[ x_n(t) \in \overline{B} \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad n. \]

Hence we may restrict the domain of \( u \) to \([0, T] \times \overline{B}_w \times \mathcal{X}\), provided that the sequence \( \{x_n\} \) is concerned. Denoting \( \bar{u} = u|_{[0,T] \times \overline{B} \times \mathcal{X}} \) we have to show that there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and some \( x^* \in \Delta(a) \) such that

\[ \int_0^T \bar{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf \int_0^T \bar{u}(t, z_n(t), \dot{z}_n(t)) dt, \]

which is equivalent to (1).

The set \( \overline{B} \) endowed with the weak topology is metrizable and compact. Hence it is a Polish space. According to Theorem 1, there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and \( x^* \in W^{1,p}(\Delta(a)) \) such that

(a) \( z_n \rightharpoonup x^* \) uniformly in \( \overline{B}_w \), and

(b) \( \dot{z}_n \rightharpoonup \dot{x}^* \) weakly in \( L^p([0,T], \mathcal{X}) \).

(a) implies, of course, that \( z_n \rightharpoonup x^* \) in measure. Thus applying Theorem 3, we obtain the relation

\[ \int_0^T \bar{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf \int_0^T \bar{u}(t, z_n(t), \dot{z}_n(t)) dt. \]

Finally we have to prove that \( x^* \in \Delta(a) \). By (a), it follows that

\[ \lim_{n \to \infty} \langle z_n(t), \eta(t) \rangle = \langle x^*(t), \eta(t) \rangle \]

for any \( t \in [0,T] \) and \( \eta \in L^q([0,T], \mathcal{X}') \), where \( 1/p + 1/q = 1 \). Since \( z_n(t) \in \overline{B} \), there exists some positive constant \( C < \infty \) such that

\[ | \langle z_n(t), \eta(t) \rangle | \leq C \| \eta(t) \|. \]
Hence we have, by the Bounded Convergence Theorem, that

\[
\lim_{n \to \infty} \int_0^T \langle z_n(t), \eta(t) \rangle \, dt = \int_0^T \langle x^*(t), \eta(t) \rangle \, dt
\]

for any \( \eta \in L^q([0, T], \mathcal{X}') \).

This proves that \( z_n \to z^* \) weakly in \( L^p \).

Combining this result with (b), we can conclude that \( \{z_n\} \) weakly converges to \( z^* \) in \( W^{1,p} \). Since \( \triangle(a) \) is weakly closed, \( z^* \in \triangle(a) \). \( \square \)

Let \( \{x_n\} \) be a minimizing sequence of the problem (\#). Then, by Lemma 5, \( \{x_n\} \) has a subsequence (without change of notation) such that

\[
J(x^*) \leq \liminf_{n} J(x_n)
\]

for some \( x^* \in \triangle(a) \). It is also obvious that

\[
\inf_{x \in \triangle(a)} J(x) = \liminf_{n} j(x_n) \leq J(x^*).
\]

Thus we have proved that \( x^* \) is a solution of the problem (\#). Summing up

**THEOREM 4** Suppose that Assumptions 1-4 with \( p > 1 \) are satisfied for a correspondence \( \Gamma : [0, T] \times \mathcal{X} \to \mathcal{X} \). Furthermore let \( u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \to \overline{\mathbb{R}} \) be a normal convex integrand with the lower compactness property. Then the problem (\#) has a solution.

**Appendix**

**Banach Space-valued Sobolev Spaces**

This appendix aims at a brief summary of the concepts and basic facts in the theory of Banach space-valued Sobolev spaces. (cf. Schwartz [22], Barbu [3].)

1. Let \( p = (p_1, p_2, \cdots, p_\ell) \) be an \( \ell \)-tuple of non-negative integers. The number \( |p| = p_1 + p_2 + \cdots + p_\ell \) is called the order of \( p \). We denote by \( D^p \) the differential operator

\[
D^p = \frac{\partial^{p_1 + p_2 + \cdots + p_\ell}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_\ell^{p_\ell}}
\]

Let \( \Omega \) be an open set of \( \mathbb{R}^\ell \) and \( K \) a compact subset of \( \Omega \). We denote by \( \mathcal{D}_K(\Omega) \) the set of all the infinitely differentiable real-valued functions \( \varphi : \Omega \to \mathbb{R} \) whose supports are contained in \( K \); i.e.
$D_K(\Omega) = \{ \varphi \in C^\infty(\Omega, \mathbb{R}) | \text{supp } \varphi \subset K \}.$

Under the topology generated by the family of seminorms:

$$p_{K,m}(\varphi) = \sup_{x \in K, |p| \leq m} |D^p \varphi(x)|,$$

$D_K(\Omega)$ becomes a locally convex Hausdorff topological vector space (LCHTVS).

The space $D(\Omega) = \bigcup \{ D_K(\Omega) | K \text{ is a compact subset of } \Omega \}$ is also a vector space. And the space $D(\Omega)$ endowed with the strict inductive limit topology defined by $\{ D_K(\Omega) | K \text{ is a compact subset of } \Omega \}$ is a LCHTVS, called the Schwartz space. It is well-known that a net $\{ \varphi_n \} \in D(\Omega)$ converges to some $\varphi^* \in D(\Omega)$ if and only if there exists some compact subset $K$ of $\Omega$ with

$$\text{supp } \varphi_n \subset K \text{ for all } n,$$

and

$$D^p \varphi_n \to D^p \varphi^* \text{ uniformly on } \Omega$$

for every index $p = (p_1, p_2, \ldots, p_\ell)$

2. Let $\mathcal{X}$ be a real Banach space. Any continuous linear operator $S : D(\Omega) \to \mathcal{X}$ is called a $\mathcal{X}$-valued distribution and the set of all the $\mathcal{X}$-valued distributions is denoted by $D'(\Omega | \mathcal{X})$.

If $f : \Omega \to \mathcal{X}$ is a locally Bochner-integrable function, the operator $S_f : D(\Omega) \to \mathcal{X}$ defined by

$$S_f : \varphi \mapsto \int_{\Omega} f(\omega)\varphi(\omega)d\omega, \ \varphi \in D(\Omega)$$

is an $\mathcal{X}$-valued distribution. ($d\omega$ is the Lebesgue measure on $\Omega$.) Identifying $f$ and $S_f$, we can safely say that any locally Bochner-integrable function is an $\mathcal{X}$-valued distribution.

The value of $S \in D'(\Omega | \mathcal{X})$ at $\varphi \in D(\Omega)$ is sometimes denoted by $\langle S, \varphi \rangle$ instead of $S(\varphi)$.

Let $S$ be an $\mathcal{X}$-valued distribution and $D^p$ an differential operator. Then the operator $D^p S : D(\Omega) \to \mathcal{X}$ defined by

$$\varphi \mapsto (-1)^{|p|} \langle S, D^p \varphi \rangle, \ \varphi \in D(\Omega)$$

is also an $\mathcal{X}$-valued distribution, called the distributional derivative (or the derivative in sense of distribution) of $S$; i.e.

$$\langle D^p S, \varphi \rangle = (-1)^{|p|} \langle S, D^p \varphi \rangle, \ \varphi \in D(\Omega).$$
An $\mathcal{X}$-valued distribution is infinitely differentiable in the sense of distribution.

3. The $\mathcal{X}$-valued Sobolev space $\mathcal{W}^{k,p}(\Omega, \mathcal{X})(p \geq 1)$ is the set of all the functions $f : \Omega \rightarrow \mathcal{X}$ such that its distributional derivative $D^s f$ exists and belongs to $L^p(\Omega, \mathcal{X})$ for all $s = (s_1, s_2, \ldots, s\ell)$ with $|s| \leq k$.

$\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is clearly a vector space. In fact, it becomes a Banach space under the norm:

$$
||f||_{k,p} = \left( \sum_{|s| \leq k} \int_{\Omega} ||D^s f(\omega)||^p d\omega \right)^{1/p}
$$

If $\mathcal{X}$ is a Hilbert space and $p = 2$, $\mathcal{W}^{k,2}(\Omega, \mathcal{X})$ is also a Hilbert space under the inner product:

$$
\langle f, g \rangle_{k,p} = \sum_{|s| \leq k} \int_{\Omega} \langle D^s f(\omega), D^s g(\omega) \rangle d\omega.
$$

Finally, we state three results which are to play some roles in this paper.

**FACT 1** If $\mathcal{X}$ is a separable Banach space, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})(p \geq 1)$ is also separable.

**FACT 2** If $\mathcal{X}$ is a separable reflexive Banach space and $p > 1$, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is reflexive.

Let $\Omega = (0, T)$. We denote by $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ the set of all the functions $f : [0 : T] \rightarrow \mathcal{X}$ such that

- The derivatives $D^j f$ (defined a.e.) are absolutely continuous for $j = 1, 2, \ldots, k - 1$, and
- $D^j f \in L^p([0, T], \mathcal{X})$ for $j = 0, 1, 2, \ldots, k$.

**FACT 3** Let $\mathcal{X}$ be a Banach space with the Radon-Nikodým property. Then the following two statements are equivalent for a function $f \in L^p([0, T], \mathcal{X})(p \geq 1)$.

(i) $f \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$.

(ii) There exists some $f_1 \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$ such that $f(t) = f_1(t)$ a.e. $\omega \in (0, T)$.

Thus we may assume, without loss of generality, that each element of $\mathcal{W}^{k,p}((0, T), \mathcal{X})$ is defined on the closed interval $[0, T]$ rather than $(0, T)$. When we wish to emphasize this aspect, we use the notation $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ rather than $\mathcal{W}^{k,p}((0, T), \mathcal{X})$. 
References


