A Variational Problem Governed by a Differential Inclusion in a Banach Space

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1 Introduction

Let $\mathcal{X}$ be a real separable reflexive Banach space. A correspondence (multi-valued mapping) $\Gamma : [0, T] \times \mathcal{X} \rightharpoonup \mathcal{X}$ and a function $u : [0, T] \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ are assumed to be given. A double arrow $\rightharpoonup$ indicates the domain and the range of a correspondence. The compact interval $[0, T]$ is endowed with the Lebesgue measure $dt$. $\mathcal{L}$ denotes the $\sigma$-field of the Lebesgue-measurable sets of $[0, T]$.

Let $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ be the Sobolev space consisting of functions of $[0, T]$ into $\mathcal{X}$ (cf. Appendix) and let $\Delta(a)$ be the set of all the solutions in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ of a differential inclusion:

\[(*) \quad \dot{x}(t) \in \Gamma(t, x(t)), \quad x(0) = a,\]

where $\dot{x}$ denotes the derivative of $x$ and $a$ is a fixed vector in $\mathcal{X}$. And consider a variational problem:

\[(#) \quad \text{Minimize}_{x \in \Delta(a)} \int_{0}^{T} u(t, x(t), \dot{x}(t)) \, dt.\]

The object of this paper is to discuss a couple of existence problems as follows:

(i) the existence of a solution for the differential inclusion $(*)$, and
(ii) the existence of an optimal solution for the variational problem $(#)$. 
In Maruyama [14] [15], I presented a solution of these problems in the special case $\mathcal{X} = \mathbb{R}^\ell$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$; i.e. if a sequence $\{x_n\}$ in $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$, weakly converges to some $x^* \in \mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \to x^* \quad \text{uniformly on } [0, T],$$

$$\dot{x}_{n_k} \to \dot{x}^* \quad \text{weakly in } L^2([0, T], \mathbb{R}^\ell).$$

However it deserves a special notice that this property does not hold in the space $\mathcal{W}^{1,2}([0, T], \mathcal{X})$ if $\dim \mathcal{X} = \infty$. Taking account of this fact, I provided a new convergence result to overcome this difficulty in the case $\mathcal{X}$ is a real separable Hilbert space in Maruyama [17]. And I also gave a existence theory for the problems (i) and (ii) being based upon this new tool in the framework of a separable Hilbert space in Maruyama [17],[18].

The purpose of the present paper is to generalize my previous results to the case $\mathcal{X}$ is a real separable reflexive Banach space. Papageorgiou [19] also gave an elegant extension of my results in Maruyama [14],[15] to the infinite dimensional case. The present paper might be regarded as an alternative approach to Papageorgiou's theory.

Let me mention about another improvement added on this occasion. In Maruyama [17], I imposed a very restrictive requirement on the continuity of the correspondence $\Gamma$; i.e.

- the correspondence $z \mapsto \Gamma(t, z)$ is upper hemi-continuous for each fixed $t \in [0, T]$ with respect to the weak topology for the domain and the strong topology for the range.

I have to admit frankly that this is a very unpleasant assumption. In the present paper, I propose the upper hemi-continuity of $z \mapsto \Gamma(t, z)$ with respect to the "weak-weak" combination of topologies instead of the "weak-strong" combination.

2 A Convergence Theorem in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$

As I have already said, any weakly convergent sequence $\{x_n\}$ in the Sobolev space $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$ has a subsequence which satisfies the property (W) in section 1.

On the other hand, let $\mathcal{X}$ be a real Banach space with the Radon-Nikodým property (RNP). Then any absolutely continuous function $f : [0, T] \to \mathcal{X}$ is Fréchet-differentiable a.e. (If the Banach space $\mathcal{X}$ does not have RNP, this property does not hold. For a counter-example, see Komura [13].) Let $\{x_n\}$ be a sequence in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ which weakly converges to some $x^* \in \mathcal{W}^{1,p}([0, T], \mathcal{X})$.
We should keep in mind that it is not necessarily true that the sequence \( \{x_n\} \) has a subsequence \( \{z_n\} \) which satisfies the property (W) if \( \dim \mathcal{X} = \infty \) even in the case \( p = 2 \).

**Counter-Example** (Cecconi[9], pp.28-29) Let \( \mathcal{H} \) be a real separable Hilbert space and \( \{\varphi_n; n = 1, 2, \cdots\} \) a complete orthonormal system of \( \mathcal{H} \). (cf. Yosida [28] P.89.) Define a sequence \( \{x_n : [0,T] \rightarrow \mathcal{H}\} \) by
\[
x_n(t) = t\varphi_n \quad (n = 1,2, \cdots).
\]
We also define the function \( x^*: [0,1] \rightarrow \mathcal{H} \) by \( x^*(t) \equiv 0 \). Then \( x_n \)'s as well as \( x^* \) are elements of \( \mathcal{W}^{1,2}([0,T], \mathcal{H}) \). It follows from the Riemann-Lebesgue lemma that the sequence \( \{x_n\} \) weakly converges to \( x^* \) in \( \mathcal{W}^{1,2}([0,1], \mathcal{H}) \). However there is no subsequence of \( \{x_n\} \) which converges strongly (hence uniformly) to \( x^* \) in \( \mathcal{L}^2([0,1], \mathcal{H}) \).

The following theorem cultivated to overcome this difficulty is a generalization of Theorem 1 of Maruyama [18].

Henceforth we denote by \( \mathcal{X}_s \) (resp.\( \mathcal{X}_w \)) a Banach space \( \mathcal{X} \) endowed with the strong (resp. weak) topology.

**THEOREM 1.** Let \( \mathcal{X} \) be a real separable reflexive Banach space. And consider a sequence \( \{x_n\} \) in the Sobolev space \( \mathcal{W}^{1,p}([0,T], \mathcal{X})(p \geqq 1) \). Assume that

(i) the set \( \{x_n(t)\}_{n=1}^{\infty} \) is bounded (and hence relatively compact) in \( \mathcal{X}_w \) for each \( t \in [0,T] \), and

(ii) there exists some function \( \psi \in \mathcal{L}^p([0,T], (0, +\infty)) \) such that
\[
\|\dot{x}_n(t)\| \leqq \psi(t) \quad \text{a.e.}
\]

Then there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and some \( x^* \in \mathcal{W}^{1,p}([0,T], \mathcal{X}) \) such that

(a) \( z_n \rightarrow x^* \) uniformly in \( \mathcal{X}_w \) on \( [0,T] \), and

(b) \( \dot{z}_n \rightarrow \dot{x}^* \) weakly in \( \mathcal{L}^p(0,T], \mathcal{X}) \).

**Remark** Since \( \mathcal{X} \) is separable and reflexive, the following results holds true. Assume that \( p \geqq 1 \).

[I] \( \mathcal{L}^p([0,T], \mathcal{X}) \) is separable.
[II] \( \mathcal{L}^p([0, T], \mathcal{X})' \) is isomorphic to \( \mathcal{L}^q([0, T], \mathcal{X}') \), where \( 1/p + 1/q = 1 \) and "'" denotes the dual space.

[III] Any absolutely continuous function \( f : [0, T] \to \mathcal{X} \) is Fréchet-differentiable a.e. and the "fundamental theorem of calculus", i.e.

\[
f(t) = f(0) + \int_{0}^{t} \dot{f}(\tau) d\tau ; \ t \in [0, T]
\]
is valid.

**Proof of Theorem 1.** (a) To start with, we shall show the equicontinuity of \( \{x_n\} \). Since \( \psi \) is integrable, there exists some \( \delta > 0 \) for each \( \varepsilon > 0 \) such that

\[
\| x_n(t) - x_n(s) \| \leq \int_{s}^{t} \| \dot{x}_n(\tau) \| d\tau \leq \int_{s}^{t} \psi(\tau) d\tau \leq \varepsilon \quad \text{for all } n
\]

provided that \( |t - s| \leq \delta \). This proves the equicontinuity of \( \{x_n\} \) in the strong topology for \( \mathcal{X} \). Hence \( \{x_n\} \) is also equicontinuous in the weak topology for \( \mathcal{X} \).

Taking account of this fact as well as the assumption (i), we can claim, thanks to the Ascoli-Arzelà theorem (cf. Schwartz[21] p.78), that \( \{x_n\} \) is relatively compact in \( C([0, T], \mathcal{X}_w) \) (the set of continuous functions of \( [0, T] \) into \( \mathcal{X}_w \)) with respect to the topology of uniform convergence.

By the assumption (i), \( \{x_n(0)\} \) is bounded in \( \mathcal{X} \), say

\[
\sup_n \| x_n(0) \| \leq C < +\infty.
\]

And the assumption (ii) implies that

\[
\| \int_{0}^{t} \dot{x}_n(\tau) d\tau \| \leq \| \psi \|_1 \quad \text{for all } t \in [0, T].
\]

Hence

\[
\sup_n \| x_n(t) \| = \sup_n \| x_n(0) + \int_{0}^{t} \dot{x}_n(\tau) d\tau \| \leq C + \| \psi \|_1
\]

for all \( t \in [0, T] \).

Thus each \( x_n \) can be regarded as a mapping of \( [0, T] \) into the set

\[
M = \{ w \in \mathcal{X} \mid \| w \| \leq C + \| \psi \|_1 \}.
\]

The weak topology on \( M \) is metrizable because \( M \) is bounded and \( \mathcal{X} \) is a
separable reflexive Banach space. Hence if we denote by $M_w$ the space $M$ endowed with the weak topology, then the uniform convergence topology on $\mathcal{C}([0,T], M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $\mathcal{C}([0,T], M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in \mathcal{C}([0,T], \mathcal{X}_w)$.

(b) Since

$$\|\dot{y}_n(t)\| \leq \psi(t) \quad \text{a.e.},$$

the sequence $\{w_n : [0,T] \to \mathcal{X}\}$ defined by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)}; \quad n = 1, 2, \cdots$$

is contained in the unit ball of $\mathcal{L}^\infty([0,T], \mathcal{X})$ which is weak*-compact (as the dual space of $\mathcal{L}^1([0,T], \mathcal{X}')$) by Alaoglu's theorem. Note that the weak* topology on the unit ball of $\mathcal{L}^\infty([0,T], \mathcal{X})$ is metrizable since $\mathcal{L}^1([0,T], \mathcal{X}')$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in \mathcal{L}^\infty([0,T], \mathcal{X})$ in the weak* topology. We shall write $\hat{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : \mathcal{L}^\infty([0,T], \mathcal{X}) \to \mathcal{L}^p[0,T], \mathcal{X})$ by

$$A : g \mapsto \psi \cdot g,$$

then $A$ is continuous in the weak* topology for $\mathcal{L}^\infty$ and the weak topology for $\mathcal{L}^p$.

In order to see this, let $\{g_\lambda\}$ be a net in $\mathcal{L}^\infty([0,T], \mathcal{X})$ such that $w^* - \lim_{\lambda} g_\lambda = g^* \in \mathcal{L}^\infty([0,T], \mathcal{X})$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t) \rangle dt \to \int_0^T \langle \alpha(t), g^*(t) \rangle dt \quad \text{for all } \alpha \in \mathcal{L}^1([0,T], \mathcal{X}')$$

Then it is quite easy to verify that

$$\int_0^T \langle \beta(t), \psi(t)g_\lambda(t) \rangle dt \to \int_0^T \langle \psi(t)\beta(t), g_\lambda(t) \rangle dt$$

$$\to \int_0^T \langle \psi(t)\beta(t), g^*(t) \rangle dt$$

for all $\beta \in \mathcal{L}^q([0,T], \mathcal{X}')$, $1/p + 1/q = 1$.

since $\psi \cdot \beta \in \mathcal{L}^1([0,T], \mathcal{X}')$. This proves the continuity of $A$.

Hence

$$\hat{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \quad \text{weakly in } \mathcal{L}^p([0,T], \mathcal{X})$$

which implies

$$\hat{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \quad \text{weakly in } \mathcal{L}^p([0,T], \mathcal{X})$$
\( \langle \theta, \int_{s}^{t} \dot{z}_{n}(\tau) d\tau \rangle = \int_{s}^{t} \langle \theta, \psi(\tau) \cdot w^{*}(\tau) \rangle d\tau \) for all \( \theta \in \mathcal{X}' \). 

(2)

On the other hand, since

\[ z_{n}(t) - z_{n}(s) = \int_{s}^{t} z_{n}(\tau) d\tau \text{ for all } n, \]

and \( z_{n}(t) - z_{n}(s) \to x^{*}(t) - x^{*}(s) \text{ in } \mathcal{X}_{w}, \) we get

\[ \langle \theta, \int_{s}^{t} \dot{z}_{n}(\tau) d\tau \rangle = \langle \theta, z_{n}(t) - z_{n}(s) \rangle \to \langle \theta, x^{*}(t) - x^{*}(s) \rangle \text{ for all } \theta \in \mathcal{X}'. \]

(3)

(2) and (3) imply that

\[ \langle \theta, x^{*}(t) - x^{*}(s) \rangle = \langle \theta, \int_{s}^{t} \psi(\tau) \cdot w^{*}(\tau) d\tau \rangle \text{ for all } \theta \in \mathcal{X}', \]

from which we can deduce the equality

\[ x^{*}(t) - x^{*}(s) = \int_{s}^{t} \psi(\tau) \cdot w^{*}(\tau) d\tau. \]

(4)

By (1) and (4), we get the desired result:

\[ \dot{z}_{n} \to \dot{x}^{*} = \psi \cdot w^{*} \text{ weakly in } \mathcal{L}^{p}([0,T], \mathcal{X}). \]

\[ \square \]

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [1] (pp.13-14). However their reasoning does not seem to be perfectly sound.

3 Differential Inclusions (1)

In this section, we prepare several lemmas which are to play crucial roles in the existence theory for differential inclusions.
Throughout this section, $\mathcal{X}$ is assumed to be a real separable reflexive Banach space.

Let us begin by specifying some assumptions imposed on the correspondence $\Gamma : [0, T] \times \mathcal{X} \rightarrow \mathcal{X}$. Special attentions should be paid to the fact that both of the domain and the range of $\Gamma$ are endowed with the weak topologies.

**Assumption 1.** $\Gamma$ is compact-convex-valued; i.e. $\Gamma(t, x)$ is a non-empty, compact and convex subset of $\mathcal{X}$ for all $t \in [0, T]$ and all $x \in \mathcal{X}$.

**Assumption 2.** The correspondence $x \mapsto \Gamma(t, x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t \in [0, T]$; i.e. for any fixed $(t, x) \in [0, T] \times \mathcal{X}$ and for any neighborhood $V$ of $\Gamma(t, x) \subset \mathcal{X}$, there exists some neighborhood $U$ of $x$ such that $\Gamma(t, z) \subset V$ for all $z \in U$.

**Assumption 3.** The graph of the correspondence $t \mapsto \Gamma(t, x)$ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}))$-measurable for each fixed $x \in \mathcal{X}$ where $\mathcal{B}(\mathcal{X})$ denotes the Borel $\sigma$-field on $\mathcal{X}$.

(For the concept of "measurability" of a correspondence, the best reference is Castaing-Valadier [8] Chap.III.)

**Assumption 4.** $\Gamma$ is $\mathcal{L}^{p}$-integrably bounded; i.e. there exists $\psi \in \mathcal{L}^{p}([0, T], (0, +\infty))(p > 1)$ such that $\Gamma(t, x) \subset S_{\psi(t)}$ for every $(t, x) \in [0, T] \times \mathcal{X}$, where $S_{\psi(t)}$ is the closed ball in $\mathcal{X}$ with the center $0$ and the radius $\psi(t)$.

The following lemma is essentially due to Castaing [5].

**LEMMA 1** (Castaing [5]) Suppose that a correspondence $\Gamma : \mathcal{X} \rightarrow \mathcal{X}$ satisfies the Assumptions 1-3, and that a function $x : [0, T] \rightarrow \mathcal{X}$ is Bochner-integrable. Then there exists a closed-valued correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}$ such that

$$\Sigma(t) \subset \Gamma(t, x(t)) \text{ for all } t \in [0, T],$$

and the graph $G(\Sigma)$ of $\Sigma$ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}))$-measurable.

**Proof.** Let $\{x_n : [0, T] \rightarrow \mathcal{X}\}$ be a sequence of simple functions which satisfies that

$$\|x_n(t) - x(t)\| \rightarrow 0 \text{ for each } t \in [0, T] \text{ as } n \rightarrow \infty.$$

(For the existence of such a sequence, see Yosida [28] p.133.) Define a correspondence $\Gamma_n : [0, T] \rightarrow \mathcal{X}$ by

$$\Gamma_n : t \mapsto \Gamma(t, x_n(t)); n = 1, 2, \cdots.$$
Then it can be shown that the graph $G(\Gamma_n)$ of each $\Gamma_n$ is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. In order to confirm it, we denote by $\{y_1, y_2, \ldots, y_k\}$ the image of $[0, T]$ by the simple function $x_n$; i.e.

$$x_n([0, T]) = \{y_1, y_2, \ldots, y_k\}.$$  

Furthermore if we define a correspondence $\Phi_j : [0, T] \mapsto \mathcal{X}_w$ ($j = 1, 2, \ldots, k$) by

$$\Phi_j : t \mapsto \Gamma(t, y_j),$$

then the graph $G(\Phi_j)$ of $\Phi_j$ is obviously $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. The graph $G(\Gamma_n)$ of $\Gamma_n$ can be expressed as

$$G(\Gamma_n) = \bigcup_{j=1}^{k} G[\Phi_j|_{x_n^{-1}(\{y_j\})}],$$

where $\Phi_j|_{x_n^{-1}(\{y_j\})}$ is the restriction of the correspondence $\Phi_j$ to the set $x_n^{-1}(\{y_j\}) = \{t \in [0, T] | x_n(t) = y_j\}$. Since $G[\Phi_j|_{x_n^{-1}(\{y_j\})}] (j = 1, 2, \ldots, k)$ is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable, so is $G(\Gamma_n)$.

Since $\|x_n(t) - x(t)\| \to 0$ for each $t \in [0, T]$ as $n \to \infty$, the set $\{x_1(t), x_2(t), \ldots; x(t)\}$ is weakly compact for each $t \in [0, T]$. Furthermore, by the Assumptions 1-2, the correspondence $\Gamma$ is compact-valued and u.h.c. in the second variable. Consequently the set

$$\bigcup_{n=1}^{\infty} \Gamma(t, x_n(t))$$

is relatively compact in $\mathcal{X}_w$ (for each $t \in [0, T]$). Taking account of the fact that the weak topology of a weakly compact subset of a separable Banach space is metrizable, we can conclude, by Baire's category theorem, that the set

$$\Sigma(t) \equiv \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \overline{\Gamma(t, x_m(t))}^w$$

is non-empty (for each $t \in [0, T]$), where $-w$ denotes the closure operation with respect to the weak topology.

The correspondence $\Sigma : [0, T] \mapsto \mathcal{X}_w$ is closed-valued and its graph is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. Finally the inclusion

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for each} \quad t \in [0, T]$$

is clear because $\Gamma$ is compact-valued and u.h.c. \hfill $\Box$

We can show the Next lemma in a similar way as in Maruyama[17], taking account of [III] of the Remark on page 4.

**Lemma 2** Let $A$ be a non-empty compact and convex set in $\mathcal{X}_w$, and $X$ a subset of $\mathcal{W}^{1,p}([0, T], \mathcal{X})(p > 1)$ defined by
$X = \{x \in \mathcal{W}^{1,p} | \| \dot{x}(t) \| \leq \psi(t) \text{ a.e., } x(0) \in A\},$

where $\psi \in \mathcal{L}^p([0,T],(0, +\infty))$. Then $X$ is non-empty convex and compact in $\mathcal{X}_w$.

**Proof.** Since it is obvious that $X$ is non-empty and convex, we have only to show the weak compactness of $X$.

It is not hard to show the boundedness of $X$. Let $x$ be any element of $X$. Then $x$ can be represented in the form

$$x(t) = a + \int_0^t \dot{x}(\tau) d\tau; \quad t \in [0, T]$$

($a$ is a point of $A$) by [III] of the Remark on page 3. It follows that

$$\| x(t) \| = \| a + \int_0^t \dot{x}(\tau) d\tau \| \leq \| a \| + \int_0^t \| \dot{x}(\tau) \| d\tau$$

$$\leq \| a \| + \int_0^T \psi(\tau) d\tau \leq B + \int_0^T \psi(\tau) d\tau,$$

where $B = \sup_{a \in A} \| a \| < +\infty$. Consequently we have the evaluation:

$$\sup_{x \in X} \| x \|_p \leq [B + \int_0^T \psi(\tau) d\tau]^p \cdot T < +\infty,$$

where $\| \cdot \|_p$ denotes the $\mathcal{L}^p$-norm. Since the right-hand side is independent of $x$, $X$ is bounded in $\mathcal{L}^p$. On the other hand, the set $\{\dot{x} | x \in X\}$ is also bounded by $\| \psi \|_p$. Therefore we can claim that $X$ is bounded in $\mathcal{W}^{1,p}$.

$\mathcal{W}^{1,p}$ is reflexive because $\mathcal{X}$ is reflexive and $p > 1$. Hence the bounded set $X$ is weakly relatively compact in $\mathcal{W}^{1,p}$.

To show the weak compactness of $X$, we need only to show the weak closedness of $X$. However $X$ is weakly closed if and only if $X$ is strongly closed since $X$ is convex. Let $\{x_n\}$ be a sequence in $X$ which strongly converges to $x^*$ in $\mathcal{W}^{1,p}$. Then $\{\dot{x}_n\}$ has a subsequence, say $\{\dot{x}_{n'}\}$, which converges to $\dot{x}^*$ a.e. Since $\| \dot{x}_{n'}(t) \| \leq \psi(t)$ a.e., it follows that

$$\| \dot{x}^*(t) \| \leq \psi(t) \quad \text{a.e.}$$

Finally it is clear that $x^*(0) \in A$. Then we obtain $x^* \in X$. This proves that $X$ is strongly closed in $\mathcal{W}^{1,p}$. 

We denote by $\mathcal{B}(0; \mathcal{X}_w)$ a neighborhood base of the zero element of $\mathcal{X}_w$ which consists of convex sets. The following lemma plays a crucial role in the
subsequent arguments although its proof is easy.

**LEMMA 3** Suppose that the Assumptions 1-2 are satisfied. Let \((t^*, x^*)\) be any point of \([0, T] \times \mathcal{X}\). Define, for any \(V \in B(0; \mathcal{X}_w)\), a subset \(K(t^*; x^*, V)\), of \([0, T] \times \mathcal{X}\) by

\[
K(t^*; x^*, V) = \{(t, x) \in [0, T] \times \mathcal{X} \mid x \in x^* + V, \ t = t^*\}.
\]

Then we have

\[
\Gamma(t^*, x^*) = \bigcap_{V \in B(0; \mathcal{X}_w)} \overline{\text{co}}\Gamma(K(t^*; x^*, V)).
\]

(Here we do not have to distinguish the convex closure with respect to the strong topology and that with respect to weak topology. So I simply denote it by \(\overline{\text{co}}\).)

**LEMMA 4** Suppose that the Assumptions 1, 2 and 4 (with \(p > 1\)) are satisfied. Let \(A\) be a non-empty convex compact subset of \(\mathcal{X}_w\). Then the set

\[
H \equiv \{(a, x, y) \in A \times \mathcal{X} \times \mathcal{X} \mid \dot{y}(t) \in \Gamma(t, x(t)) \ \text{a.e. and} \ \ x(0) = y(0) = a\}
\]

is weakly compact in \(A \times \mathcal{X} \times \mathcal{X}\). (The set \(X\) is defined in Lemma 2.)

**Proof.** Since we have already known that \(A \times \mathcal{X} \times \mathcal{X}\) is weakly compact in \(\mathcal{X} \times \mathcal{W}^{1,p} \times \mathcal{W}^{1,p}\), it is enough to show that \(H\) is a weakly closed subset of \(A \times \mathcal{X} \times \mathcal{X}\).

Since \(\mathcal{W}^{1,p}\) is a reflexive Banach space, the dual of which is separable, the weak topology on the bounded set \(X\) is metrizable. So we are permitted to use a sequence argument.

Let \(\{q_n \equiv (a_n, x_n, y_n)\}\) be a sequence in \(H\) which weakly converges to some \(q^* = (a^*, x^*, y^*)\) in \(A \times \mathcal{X} \times \mathcal{X}\). We have to show that \(q^* \in H\). And it is enough to check that

\[
\dot{y}^*(t) \in \Gamma(t, x^*(t)) \ \text{a.e.}
\]

The set \(\{x_n(t)\}\) is relatively compact in \(\mathcal{X}_w\) (for each \(t \in [0, T]\)) since we have the evaluation:

\[
\|x_n(t)\| \leq ||a|| + \int_0^t \|\dot{x}_n(\tau)\| \ d\tau \leq ||a|| + \int_0^T \psi(\tau) d\tau
\]

by the Assumption 4. Hence, thanks to Theorem 1, \(\{q_n\}\) has a subsequence (no change in notation) such that

\[
x_n(t) \rightarrow x^*(t) \ \text{uniformly in} \ \mathcal{X}_w, \ \text{and} \quad (1)
\]

\[
y_n(t) \rightarrow \dot{y}^*(t) \ \text{weakly in} \ \mathcal{L}^p. \quad (2)
\]
Then applying Mazur's theorem, we can choose, for each $j \in \mathbb{N}$, some finite elements

$$\dot{y}_{n_j+1}, \dot{y}_{n_j+2}, \ldots, \dot{y}_{n_j+m(j)}$$

of $\{\dot{y}_n\}$ and numbers

$$\alpha_{ij} \geq 0, 1 \leq i \leq m(j), \sum_{i=1}^{m(j)} \alpha_{ij} = 1$$

such that

$$\| \dot{y}^* - \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_j+i} \|_p \leq \frac{1}{j}, n_{j+1} > n_j + m(j).$$

Denoting

$$\eta_j(t) = \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_j+i}(t),$$

we obtain

$$\eta_j(t) \in \text{co}(\cup_{i=1}^{m(j)} \Gamma(t, x_{n_j+i}(t)).$$

Since $\{\eta_j\}$ has a subsequence which converges to $y^*$ a.e., we may assume, without loss of generality, that

$$\| \eta_j(t) - y^*(t) \| \rightarrow 0 \quad \text{a.e.}$$

(3)

On the other hand, for each $V \in B(0; \mathcal{X}_w)$, there exists some $n_0(V) \in \mathbb{N}$ such that

$$x_n(t) \in x^*(t) + V$$

for all $n \geq n_0(V)$ and for all $t \in [0, T]$.

That is,

$$(t, x_n(t)) \in K(t; x^*(t), V) \quad \text{for all } n \geq n_0(V) \text{ and for all } t \in [0, T].$$

Hence we have

$$\eta_j(t) \in \text{co}\Gamma(K(t; x^*(t), V) \quad \text{a.e.}$$

for sufficiently large $j$. Passing to the limit, we obtain

$$\dot{y}^*(t) \in \overline{\text{co}}\Gamma(K(t; x^*(t), V)) \quad \text{a.e.}$$

(4)
by (3). Since (4) holds true for all $V \in B(0; X_w)$, it follows that
\[ y^*(t) \in \bigcap_{V \in B(0, X_w)} \overline{\text{co}} \Gamma(K(t; x^*(t), V) = \Gamma(t, x^*(t)) \quad \text{a.e.} \]

The last equality in (5) comes from Lemma 3. Thus we have proved that 
\[ (a^*, x^*, y^*) \in H. \]

\[ \square \]

4 Differential Inclusions \hspace{1cm} (2)

$\mathcal{X}$ is still assumed to be a real separable reflexive Banach space in this section.

We are now going to find out a solution of $(\ast)$ in the Sobolev space $W^{1,p}([0, T], \mathcal{X}), p > 1$. Define a set $\triangle(a)$ in $W^{1,p}$ by
\[ \triangle(a) = \{ x \in W^{1,p} \mid x \text{ satisfies } (\ast) \quad \text{a.e.} \} \]
for a fixed $a \in \mathcal{X}$. The following theorem tells us that $\triangle(a) \neq \emptyset$ and that $\triangle$ depends continuously, in some sense, upon the initial value $a$.

THEOREM 2. Suppose that the correspondence $\Gamma$ satisfies the Assumptions 1-4. Let $A$ be a non-empty, convex and compact subset of $\mathcal{X}_w$. Then

(i) $\triangle(a^*) \neq \emptyset$ for any $a^* \in A$, and

(ii) the correspondence $\triangle : A \rightarrow W^{1,p}$ is compact-valued and u.h.c. on $A_w$, in the weak topology for $W^{1,p}$.

The proof is essentially the same as in Maruyama [17].

Proof. (i) Fix any $a^* \in A$. If we define a set $X(a^*) \subset X$ by $X(a^*) = \{ x \in X \mid x(0) = a^* \}$, then $X(a^*)$ is convex and weakly compact in $W^{1,p}$. Furthermore we define a correspondence $\Phi : X(a^* )_{w} \rightarrow X(a^*)_{w}$ by
\[ \Phi(x) = \{ y \in X(a^*) \mid \dot{y}(t) \in \Gamma(t, x(t)) \quad \text{a.e.} \}. \]

Then the problem is simply reduced to finding out a fixed point of $\Phi$.

\[ 1^\circ \quad \Phi(x) \neq \emptyset \text{ for every } x \in X(a^*) \quad \text{— This fact can be proved through the} \]

Measurable Selection Theorem.

Let $x$ be any element of $X(a^*)$. Then by Lemma 1, there exists a closed-valued correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ such that $\Sigma(t) \subset \Gamma(t, x(t))$ for all $t \in [0, T]$, and its graph is $(\mathcal{L}, B(\mathcal{X}_w))$-measurable. We also note that $\mathcal{X}_w$ is a Souslin space. Thanks to Saint-Beuve's measurable selection theorem (Saint-Beuve [20]), $\Sigma$ admits a $(\mathcal{L}, B(\mathcal{X}_w))$-measurable selection $\sigma : [0, T] \rightarrow \mathcal{X}$. Since
\( \mathcal{X} \) is separable, \( \sigma \) is \((\mathcal{L}, \mathcal{B}(\mathcal{X}))\)-mesurable. (cf. Yosida [28] p.131.) By the Assumption 4, \( \sigma \) is clearly integrable. If we define a function \( y : [0, T] \to \mathcal{X} \) by

\[
y(t) = a^* + \int_0^t \sigma(\tau) d\tau,
\]

then \( y \in \Phi(x) \).

2° \( \Phi \) is convex-compact-valued. — This is not hard.

3° \( \Phi \) is u.h.c. — If we define the \( a^* \)-selection \( H_{a^*} \) of \( H \) by

\[
H_{a^*} = \{(a, x, y) \in H \mid a = a^* \},
\]

then \( H_{a^*} \) is obviously weakly compact in \( A \times X \times X \). And the graph \( G(\Phi) \) of \( \Phi \) is expressed as \( G(\Phi) = \text{proj}_{X \times X} H_{a^*} \), the projection of \( H_{a^*} \) into \( X \times X \), which is also closed.

Summing up — \( \Phi \) is convex-compact-valued and u.h.c. Applying now the Fan-Glicksberg Fixed-Point Theorem to the correspondence \( \Phi \), we obtain an \( x^* \in X(a^*) \) such that \( x^* \in \Phi(x^*) \); i.e.

\[
\dot{x}^*(t) \in \Gamma(t, x^*(t)) \text{ a.e. and } x^*(0) = a^*.
\]

This proves (i).

(ii) Since the compactness of \( \Delta(a) (a \in A) \) can be verified by applying Mazur’s theorem and making use of the Assumptions 1-2, we may omit the details. Hence we have only to show the u.h.c. of \( \Delta \). However it is also obvious because the graph \( G(\Delta) \) of \( \Delta \) can be expressed as

\[
G(\Delta) = \text{proj}_{A \times X} \{(a, x, y) \in H \mid x = y \},
\]

which is closed in \( A \times X \).

\( \square \)

I am much indebted to Castaing-Valadier [7] for various important ideas embodied in the proof of Theorem 2.

Remark. Among other things, the assumption that the set \( \Gamma(t, x) \) is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See De Blasi [10] and Tateishi [23].)

Here it may be suggestive for us to glimpse the special case in which \( \Gamma \) is a (single-valued) mapping. A related result was obtained by Szep [23]. (I am indebted to Professor Tosio Kato for this reference.)

**Corollary 1.** Let \( f : [0, T] \times \mathcal{X}_w \to \mathcal{X}_w \) be a (single-valued) mapping which satisfies the following three conditions.

-\[ \text{END} \]
(i) The function $x \mapsto f(t, x)$ is continuous for each fixed $t \in [0, T]$.

(ii) The function $t \mapsto f(t, x)$ is measurable for each fixed $x \in X$.

(iii) There exists $\psi \in L^p([0, T], (0, +\infty)), p > 1$ such that $f(t, x) \in S_\psi(t)$ for every $(t, x) \in [0, T] \times X$; i.e. $\sup_{x \in X} ||f(t, x)|| \leq \psi(t)$ for all $t \in [0, T]$.

Then the differential equation

$$(**)
\dot{x} = f(t, x), x(0) = a \text{ (fixed vector in X)}$$

has at least a solution in $W^{1,p}([0, T], X)$. (A solution of $(**)$ is a function $x \in W^{1,p}$ which satisfies $(**)$ a.e.)

5 Variational problem governed by an Differential Inclusion

Let $X$ be a real separable reflexive Banach space throughout this section, too. Assume that $u : [0, T] \times X \times X \to (-\infty, +\infty]$ is a given proper function. Consider a variational problem:

(†) Minimize$_{x \in \triangle(a)} J(x) = \int_{0}^{T} u(t, x(t), \dot{x}(t)) dt$

where $\triangle(a)$ is the set of all the solutions of the differential inclusion $(*)$ discussed in the preceding sections.

In order to examine the existence of a solution of the problem (†), we have to check a couple of points as usual; i.e.

(I) the compactness of $\triangle(a)$ for some suitable topology, and

(II) the lower semi-continuity of the functional $J$ for the same topology.

Since we have already proved that $\triangle(a)$ is weakly compact in $W^{1,p}([0, T], X)$ under certain conditions, we are concentrating on the second point (II) in this section. In this context, the theorem due to Castaing-Clauzure [6] provides the most crucial key. Related results are also obtained by Balder [2], Maruyama [16] and Valadier [25].

DEFINITION Let $(\Omega, \xi, \mu)$ be a measure space, $S$ a topological space, and $\nu$ a real Banach space. A function $f : \Omega \times S \times \nu \to \overline{\mathbb{R}}$ is assumed to be given. We denote by $\mathcal{M}(\Omega, S)$ the set of all the $(\xi \otimes B(S))$-measurable functions. ($B(S)$ denotes the Borel $\sigma$-field on $S$.) $f$ is said to have the lower compactness property if $\{f^-(\omega, \varphi_n(\omega), \theta_n(\omega))\}$ is weakly relatively compact in $L^p(\Omega, \overline{\mathbb{R}})$ for any sequence $\{\varphi_n, \theta_n\}$ in $\mathcal{M}(\Omega, S) \times L^p(\Omega, \nu)(p \geq 1)$ which satisfies the following three conditions:
(a) \( \{ \varphi_n \} \) converges in measure to some \( \varphi^* \in \mathcal{M}(\Omega, S) \),

(b) \( \{ \theta_n \} \) converges weakly to some \( \theta^* \in \mathcal{L}^p(\Omega, \mathcal{V}) \), and

(c) there exists some \( C < +\infty \) such that

\[
\sup_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega))d\mu \leq C.
\]

The following theorem is a variation of a result due to Castaing-Clauzure [6] in the spirit of Ioffe [12]. See also Valadier [27].

**Theorem 3** Let \((\Omega, \xi, \mu)\) be a finite complete measure space, \( S \) a metrizable Souslin space, and \( \mathcal{V} \) a separable reflexive Banach space. Suppose that a proper function \( f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}} \) satisfies the following conditions:

(i) \( f \) is a normal integrand; i.e.

(a) \( f \) is \( (\xi \otimes B(S) \otimes B(\mathcal{V}), B(\overline{\mathbb{R}})) \)-measurable, and

(b) the function \( (\xi, v) \mapsto f(\omega, \xi, v) \) is lower semi-continuous for any fixed \( \omega \in \Omega \),

(ii) the function \( v \mapsto f(\omega, \xi, v) \) is convex for any fixed \( (\omega, \xi) \in \Omega \times S \), and

(iii) \( f \) has the lower compactness property.

Let \( \{ \varphi_n \} \) be a sequence in \( \mathcal{M}(\Omega, S) \) which converges in measure to some \( \varphi^* \in \mathcal{M}(\Omega, S) \). Let \( \{ \theta_n \} \) be a sequence in \( \mathcal{L}^p(\Omega, \mathcal{V})(1 \leq p < +\infty) \) which converges weakly to some \( \theta^* \in \mathcal{L}^p(\Omega, \mathcal{V}) \). Then we have

\[
\int_{\Omega} f(\omega, \varphi^*(\omega), \theta^*(\omega))d\mu \leq \liminf_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega))d\mu.
\]

**Remark 1** A normal integrand \( f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}} \) which also satisfies the condition (ii) is called a convex normal integrand.

2° Ioffe [8] established a fundamental theorem on the lower semi-continuity of a nonlinear integral functional as above in the case both of \( S \) and \( \mathcal{V} \) are finite dimensional Euclidean spaces. Theorem 3 is an extension of Ioffe's result to the case of nonlinear integral functional defined on the space of Bochner integrable functions.

**Lemma 5** Suppose that the Assumptions 1-4 are satisfied. Let \( \{ x_n \} \) be a sequence in \( \Delta(a) \subset \mathcal{W}^{1,p}([0, T], \mathcal{X})(p > 1) \). Let \( u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \rightarrow \overline{\mathbb{R}} \) be a proper convex normal integrand with the lower compactness property. Then there exists a subsequence \( \{ z_n \} \) of \( \{ x_n \} \) and \( z^* \in \Delta(a) \) such that
\[ J(x^*) \leq \liminf_n J(z_n), \quad (1) \]

where

\[ J(x) = \int_0^T u(t, x(t), \dot{x}(t)) dt. \]

**Proof.** By the Assumption 4, all the images of \( x_n \)'s are contained in some closed ball \( \overline{B} \) with the center 0; i.e.

\[ x_n(t) \in \overline{B} \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad n. \]

Hence we may restrict the domain of \( u \) to \([0, T] \times \overline{B} \times \mathcal{X}\), provided that the sequence \( \{x_n\} \) is concerned. Denoting \( \overline{u} = u|_{[0,T] \times \overline{B} \times \mathcal{X}} \) we have to show that there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and some \( x^* \in \Delta(a) \) such that

\[ \int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t)) dt, \]

which is equivalent to (1).

The set \( \overline{B} \) endowed with the weak topology is metrizable and compact. Hence it is a Polish space. According to Theorem 1, there exists a subsequence \( \{z_n\} \) of \( \{x_n\} \) and \( x^* \in W^{1,p}([0,T], \mathcal{X}) \) such that

(a) \( z_n \rightharpoonup x^* \) uniformly in \( \overline{B}_w \), and

(b) \( \dot{z}_n \rightharpoonup \dot{x}^* \) weakly in \( L^p([0, T], \mathcal{X}) \).

(a) implies, of course, that \( z_n \rightarrow x^* \) in measure. Thus applying Theorem 3, we obtain the relation

\[ \int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t)) dt. \]

Finally we have to prove that \( x^* \in \Delta(a) \). By (a), it follows that

\[ \lim_{n \rightarrow \infty} \langle z_n(t), \eta(t) \rangle = \langle x^*(t), \eta(t) \rangle \]

for any \( t \in [0, T] \) and \( \eta \in L^q([0, T], \mathcal{X}') \), where \( 1/p + 1/q = 1 \). Since \( z_n(t) \in \overline{B} \), there exists some positive constant \( C < \infty \) such that

\[ |\langle z_n(t), \eta(t) \rangle| \leq C \|\eta(t)\|. \]
Hence we have, by the Bounded Convergence Theorem, that

\[
\lim_{n \to \infty} \int_0^T \langle z_n(t), \eta(t) \rangle \, dt = \int_0^T \langle x^*(t), \eta(t) \rangle \, dt
\]

for any \( \eta \in L^q([0, T], \mathcal{X}') \).

This proves that \( z_n \to x^* \) weakly in \( L^p \).

Combining this result with (b), we can conclude that \( \{z_n\} \) weakly converges to \( x^* \) in \( W^{1,p} \). Since \( \Delta(a) \) is weakly closed, \( x^* \in \Delta(a) \).

Let \( \{x_n\} \) be a minimizing sequence of the problem (>). Then, by Lemma 5, \( \{x_n\} \) has a subsequence (without change of notation) such that

\[
J(x^*) \leq \liminf J(x_n)
\]

for some \( x^* \in \Delta(a) \). It is also obvious that

\[
\inf_{x \in \Delta(a)} J(x) = \liminf_{n} j(x_n) \leq J(x^*).
\]

Thus we have proved that \( x^* \) is a solution of the problem (>). Summing up

THEOREM 4  Suppose that Assumptions 1-4 with \( p > 1 \) are satisfied for a correspondence \( \Gamma : [0, T] \times \mathcal{X} \to \mathcal{X} \). Furthermore let \( u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \to \overline{R} \) be a normal convex integrand with the lower compactness property. Then the problem (>\>) has a solution.

Appendix
Banach Space-valued Sobolev Spaces

This appendix aims at a brief summary of the concepts and basic facts in the theory of Banach space-valued Sobolev spaces. (cf. Schwartz [22], Barbu [3].)

1. Let \( p = (p_1, p_2, \ldots, p_\ell) \) be an \( \ell \)-tuple of non-negative integers. The number \( |p| = p_1 + p_2 + \cdots + p_\ell \) is called the order of \( p \). We denote by \( D^p \) the differential operator

\[
D^p = \frac{\partial^{p_1+p_2+\cdots+p_\ell}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_\ell^{p_\ell}}
\]

Let \( \Omega \) be an open set of \( \mathbb{R}^\ell \) and \( K \) a compact subset of \( \Omega \). We denote by \( \mathcal{D}_K(\Omega) \) the set of all the infinitely differentiable real-valued functions \( \phi : \Omega \to \mathbb{R} \) whose supports are contained in \( K \); i.e.
\[ D_K(\Omega) = \{ \varphi \in C^\infty(\Omega, \mathbb{R}) | \text{supp} \varphi \subset K \}. \]

Under the topology generated by the family of seminorms:

\[ p_{K,m}(\varphi) = \sup_{x \in K, |\xi| \leq m} |D^\xi \varphi(x)|, \quad m = 1, 2, \ldots, \]

\[ D_K(\Omega) \]

becomes a locally convex Hausdorff topological vector space (LCHTVS).

The space \( D(\Omega) = \cup \{ D_K(\Omega) | K \text{ is a compact subset of } \Omega \} \) is also a vector space. And the space \( D(\Omega) \) endowed with the strict inductive limit topology defined by \( \{ D_K(\Omega) | K \text{ is a compact subset of } \Omega \} \) is a LCHTVS, called the Schwartz space. It is well-known that a net \( \{ \varphi_\alpha \} \) in \( D(\Omega) \) converges to some \( \varphi^* \in D(\Omega) \) if and only if there exists some compact subset \( K \) of \( \Omega \) with

\[ \text{supp} \varphi_\alpha \subset K \quad \text{for all} \quad \alpha, \]

and

\[ D^p \varphi_\alpha \rightarrow D^p \varphi^* \quad \text{uniformly on} \quad \Omega \]

for every index \( p = (p_1, p_2, \ldots, p_\ell) \).

2. Let \( \mathcal{X} \) be a real Banach space. Any continuous linear operator \( S : D(\Omega) \rightarrow \mathcal{X} \) is called a \( \mathcal{X} \)-valued distribution and the set of all the \( \mathcal{X} \)-valued distributions is denoted by \( D'(\Omega | \mathcal{X}) \).

If \( f : \Omega \rightarrow \mathcal{X} \) is a locally Bochner-integrable function, the operator \( S_f : D(\Omega) \rightarrow \mathcal{X} \) defined by

\[ S_f : \varphi \mapsto \int_\Omega f(\omega) \varphi(\omega) d\omega, \quad \varphi \in D(\Omega) \]

is an \( \mathcal{X} \)-valued distribution. (\( d\omega \) is the Lebesgue measure on \( \Omega \).) Identifying \( f \) and \( S_f \), we can safely say that any locally Bochner-integrable function is an \( \mathcal{X} \)-valued distribution.

The value of \( S \in D'(\Omega | \mathcal{X}) \) at \( \varphi \in D(\Omega) \) is sometimes denoted by \( \langle S, \varphi \rangle \) instead of \( S(\varphi) \).

Let \( S \) be an \( \mathcal{X} \)-valued distribution and \( D^p \) an differential operator. Then the operator \( D^p S : D(\Omega) \rightarrow \mathcal{X} \) defined by

\[ \varphi \mapsto (-1)^{|\rho|} \langle S, D^\rho \varphi \rangle, \quad \varphi \in D(\Omega) \]

is also an \( \mathcal{X} \)-valued distribution, called the distributional derivative (or the derivative in sense of distribution) of \( S \); i.e.

\[ \langle D^p S, \varphi \rangle = (-1)^{|\rho|} \langle S, D^\rho \varphi \rangle, \quad \varphi \in D(\Omega). \]
An $\mathcal{X}$-valued distribution is infinitely differentiable in the sense of distribution.

3. The $\mathcal{X}$-valued Sobolev space $\mathcal{W}^{k,p}(\Omega, \mathcal{X})(p \geq 1)$ is the set of all the functions $f : \Omega \to \mathcal{X}$ such that its distributional derivative $D^s f$ exists and belongs to $L^p(\Omega, \mathcal{X})$ for all $s = (s_1, s_2, \cdots, s_\ell)$ with $|s| \leq k$.

$\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is clearly a vector space. In fact, it becomes a Banach space under the norm:

$$||f||_{k,p} = (\sum_{|s| \leq k} \int_\Omega ||D^s f(\omega)||^p d\omega)^{1/p}$$

If $\mathcal{X}$ is a Hilbert space and $p = 2$, $\mathcal{W}^{k,2}(\Omega, \mathcal{X})$ is also a Hilbert space under the inner product:

$$\langle f, g \rangle_{k,p} = \sum_{|s| \leq k} \int_\Omega \langle D^s f(\omega), D^s g(\omega) \rangle d\omega.$$

Finally, we state three results which are to play some roles in this paper.

**FACT 1** If $\mathcal{X}$ is a separable Banach space, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})(p \geq 1)$ is also separable.

**FACT 2** If $\mathcal{X}$ is a separable reflexive Banach space and $p > 1$, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is reflexive.

Let $\Omega = (0, T)$. We denote by $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ the set of all the functions $f : [0 : T] \to \mathcal{X}$ such that

a. The derivatives $D^j f$ (defined a.e.) are absolutely continuous for $j = 1, 2, \cdots, k - 1$, and

b. $D^j f \in L^p([0, T], \mathcal{X})$ for $j = 0, 1, 2, \cdots, k$.

**FACT 3** Let $\mathcal{X}$ be a Banach space with the Radon-Nikodým property. Then the following two statements are equivalent for a function $f \in L^p([0, T], \mathcal{X})(p \geq 1)$.

(i) $f \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$.

(ii) There exists some $f_1 \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$ such that $f(t) = f_1(t)$ a.e. $\omega \in (0, T)$.

Thus we may assume, without loss of generality, that each element of $\mathcal{W}^{k,p}((0, T), \mathcal{X})$ is defined on the closed interval $[0, T]$ rather than $(0, T)$. When we wish to emphasize this aspect, we use the notation $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ rather than $\mathcal{W}^{k,p}((0, T), \mathcal{X})$. 


References


