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A note on the languages recognized by commutative asynchronous automata*

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Abstract

The languages recognized by commutative asynchronous automata are studied and described here. It turns out that over a finite nonvoid alphabet $X$ with $|X| = k$, the languages recognized by commutative asynchronous automata constitute such a Boolean algebra which is isomorphic to the Boolean algebra consisting of all subsets of the set $\{0,1\}^k$.

1 Introduction

The decomposition of commutative asynchronous automata is studied in [1] and it is proved that every commutative asynchronous automaton can be embedded isomorphically into a suitable quasi-direct power of a two-state commutative asynchronous automaton. Moreover, the directable commutative asynchronous automata are also investigated in [1], and it is shown that the exact bound for the maximal length of minimum-length directing words

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of commutative asynchronous automata of \( n \) states is equal to \( n - 1 \), i.e., the exact bound is the same as in the commutative case (see eg. [3] or [4]). Surprisingly, the exact bound decreases drastically to \( \log_2(n) \) if we consider only such elements of this class which are generated by one element. Paper [2] deals with the decomposition of commutative asynchronous nondeterministic automata. Here, we study now the languages recognized by commutative asynchronous automata. It turns out that there are a few of them, and they constitute a Boolean algebra under a fixed alphabet.

## 2 Preliminaries

We recall here a few notions and notation necessary in the sequel. Let \( X \) be a nonempty alphabet with \( |X| = k \). Without loss of generality, we may assume that \( X = \{x_1, \ldots, x_k\} \). Throughout this paper we shall work under this fixed alphabet \( X \). The set of all finite words over \( X \) is denoted by \( X^* \). For the length of a word \( p \in X^* \), we use the notation \( |p| \). For any \( p \in X^* \), let us denote by \( \text{alph}(p) \) the set of the all letters occurring in the word \( p \). One can extend the function \( \text{alph} \) to languages in a natural way. The shuffle product of two words \( u, v \in X^* \) is the set

\[
\begin{equation}
\{w : w = u_1v_1 \ldots u_nv_n, u = u_1 \ldots u_n, v = v_1 \ldots v_n, u_i.v_j \in X^*\}.
\end{equation}
\]

The shuffle product can be extended to languages as well. We use the Parikh mapping denoted by \( \Psi \). For its definitions, let \( N = \{0, 1, 2, \ldots\} \), and let us define the mapping \( \Psi : X^* \to N^k \), by

\[
\Psi(u) = (\mu_{x_1}(u), \ldots, \mu_{x_k}(u)),
\]

where \( \mu_{x_j}(u) \) denotes the number of the occurrences of \( x_j \) in \( u \), for every \( j \), \( j = 1, \ldots, k \).

By automaton or \( X \)-automaton we mean a system \( A = (A, X) \), where \( A \) is the finite nonvoid set of states, \( X \) is the finite nonempty set of input signs, and every input sign \( x \in X \) is realized as a unary operation \( x^A : A \to A \). The automaton \( A = (A, X) \) is commutative if \( a(xy)^A = a(yx)^A \) is valid, for all \( a \in A \) and \( x, y \in X \). Another particular automata are the asynchronous ones. \( A \) is called asynchronous if \( ax^A = a(xx)^A \), for all \( a \in A \) and \( x \in X \).
Some particular commutative asynchronous automata introduced in [1] will be used in the following section.

For every $n \geq 1$, let us define the automaton $H_n = (\{0,1\}^n, \{x_1, \ldots, x_n\})$ in the following way. For all $(i_1, \ldots, i_n) \in \{0,1\}^n$ and $x_j \in \{x_1, \ldots, x_n\}$, let

$$(i_1, \ldots, i_n)x_j^{H_n} = \begin{cases} (i'_1, \ldots, i'_n) & \text{if } i_j = 0, \text{ where } i'_t = i_t, t = 1, \ldots, n, t \neq j, \\ (i_1, \ldots, i_n) & \text{otherwise.} \end{cases}$$

The automaton $H_n$ can be visualized as follows. Its states are the vertices of the $n$-dimensional hyper-cube and any input sign takes the automaton from a vertex into its neighbour or fixes the state given. Moreover, $x_j$ changes only the $j$th component. By the definition of $H_n$, it is easy to see that $H_n$ is commutative and asynchronous.

A recognizer or $X$-recognizer is a system $A = (A, a_0, F)$ which consists of an $X$-automaton $A$, an initial state $a_0 \in A$, and a set $F(\subseteq A)$ of final states. The language recognized by $A$ is

$$L(A) = \{w : w \in X^* \text{ and } a_0w^A \in F\}.$$ 

It is also said that $L(A)$ is recognizable by the automaton $A$.

### 3 Results

For every $k$ dimensional binary vector $i = (i_1, \ldots, i_k)$, a language $L_i$ over $X$ can be defined as follows. Let

$$L_i = \Psi^{-1}(i) \diamond (\text{alph}(\Psi^{-1}(i)))^*.$$ 

Moreover, if $B \subseteq \{0,1\}^k$, then we can define the language $L_B$ by

$$L_B = \cup_{i \in B} L_i.$$ 

The languages $L_B$, $B \subseteq \{0,1\}^k$ are strongly related to the languages recognizable by commutative asynchronous $X$-automata. This strong relationship is presented by the following statement.

**Proposition 1.** A language $L \subseteq X^*$ is recognized by a commutative asynchronous $X$-automaton if and only if $L = L_B$ for some $B \subseteq \{0,1\}^k$. 

Proof. Let $L \subseteq X^*$ be an arbitrary language and let us suppose that $L$ can be recognized by a recognizer $A = (A, a_0, F)$, where $A = (A, X)$ is a commutative asynchronous $X$-automaton. Let us observe that $ap^A = a(x_{i_1} \ldots x_{i_s})^A$, $1 \leq s \leq k$ is valid for every $p \in X^*$ with $\text{alph}(p) = \{x_{i_1}, \ldots, x_{i_s}\}$ since $A = (A, X)$ is commutative and asynchronous. By the commutativity, we may suppose that $i_1 < i_2 < \ldots < i_s$. Therefore, for every $p \in L$, there exists a uniquely determined word $x_{i_1} \ldots x_{i_s}$ such that $a_0 p^A = a_0 (x_{i_1} \ldots x_{i_s})^A$.

Now, let us denote by $K$ the subset of $L$ which consists of all words $q$ in $L$ for which $|q| = |\text{alph}(q)|$ and if $q = x_{i_1} \ldots x_{i_s}$, then $i_1 < i_2 < \ldots < i_s$. Then it is easy to see that

$$L = \bigcup_{q \in K} (\Psi^{-1}(\Psi(q))) \circ (\text{alph}(q))^*.$$  

On the other hand, by the definition of $K$, the mapping $\mu$ which is defined by $\mu : q \mapsto \Psi(q), q \in K$, is a one-to-one mapping of the language $K$ into $\{0, 1\}^k$. Consequently, if the image of $K$ under $\mu$ is denoted by $B$, then $B \subseteq \{0, 1\}^k$, moreover,

$$L = \bigcup_{q \in K} (\Psi^{-1}(\Psi(q))) \circ (\text{alph}(q))^* = \bigcup_{i \in B} (\Psi^{-1}(i)) \circ (\text{alph}(\Psi^{-1}(i))^* = \bigcup_{i \in B} L_i = L_B.$$  

and consequently, $L = L_B$. In particular, if $L = \emptyset$, then $B = \emptyset$.

Conversely, let $L = L_B = \bigcup_{i \in B} L_i$ for some $B \subseteq \{0, 1\}^k$. Then it is easy to prove that the commutative asynchronous automaton $H_k$ based on the $k$ dimensional hyper-cube recognizes $L$ by $(H_k, (0, 0, \ldots, 0), B)$, and thus, $L$ can be recognized by a commutative asynchronous $X$-automaton.

From the description of the languages over $X$, recognized by commutative asynchronous $X$-automata, it follows that these languages are closed under the union and intersection. What is more that is presented by the following assertion.
Proposition 2. The number of the languages over \( X = \{x_1, \ldots, x_k\} \), which can be recognized by commutative asynchronous \( X \)-automata, is equal to \( 2^{2^k} \), moreover, these languages constitute a Boolean algebra which is isomorphic to the Boolean algebra consisting of all the subsets of the set \( \{0,1\}^k \).

Proof. Let us denote by \( \mathcal{L}_X \) the set of languages, recognized by commutative asynchronous \( X \)-automata. Let \( L \in \mathcal{L}_X \) be an arbitrary language. By the proof of Proposition 1, there exists a \( B \subseteq \{0,1\}^k \) such that \( L = L_B \). Therefore, to every language \( L \in \mathcal{L}_X \), we can assign a subset \( B \) of \( \{0,1\}^k \). Let us denote this mapping by \( \varphi \). Then \( \varphi \) is a mapping of \( \mathcal{L}_X \) into \( \{0,1\}^k \).

On the other hand, in the proof of Proposition 1 it is shown that for every \( B \subseteq \{0,1\}^k \), there exists a language \( L \in \mathcal{L}_X \) such that \( L = L_B \), and therefore, \( \varphi \) is surjective. Finally, it is easy to see that if \( L_1 \neq L_2 \in \mathcal{L}_X \), then \( L_1 \varphi \neq L_2 \varphi \).

Consequently, \( \varphi \) is a one-to-one mapping of \( \mathcal{L}_X \) onto \( \{0,1\}^k \). Moreover, it is evident that \( (L_1 \cup L_2)\varphi = L_1\varphi \cup L_2\varphi \), \( (L_1 \cap L_2)\varphi = L_1\varphi \cap L_2\varphi \), and \( \overline{L_1} = \overline{L_1\varphi} \), for all \( L_1, L_2 \in \mathcal{L}_X \), where \( \overline{L} \) and \( \overline{L\varphi} \) denotes the corresponding complements, respectively. Consequently, \( \varphi \) is an isomorphism. This isomorphism provides that \( |\mathcal{L}_X| = 2^{2^k} \). This ends the proof of Proposition 2.

References


