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1166 巻 2000 年 17-26

JACOBI–TRUDI-TYPE IDENTITIES FOR IDEAL-TABLEAUX

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1. Introduction

The present article is concerned with the generating functions of certain tableaux consisting of order ideals of finite odd-ary trees. Here, a tree is a connected digraph without undirected cycles, which is identified with an ordered set in this way: $x \rightarrow y$ (an edge from x to y exists) $\Longleftrightarrow$ x covers y ($x > y$ and $x > \nexists z > y$). On the analogy of "binary tree", an odd-ary tree is defined to be a tree with vertices of degree $1, 2, 4, 6, \ldots$, where the degree of a vertex is the number of the edges incident into or from the vertex. The main result of the paper is a superdeterminantal formula for the above-mentioned generating function, which includes Wachs, Okada and Asai's extension of the Jacobi–Trudi identity [Wac85, Oka90, Asa98]. A superdeterminant is a natural extension of a determinant defined for even dimensional square arrays. Our result is the consequence of analogous Lindström's theorem [Lin73] and the Gessel–Viennot lattice paths [GV85, GV]. In the last section, we study the summation of the weights of (partially) unbounded tree-g-paths by a superpfaffian, which corresponds to Stembridge's prominent technique to enumerate unbounded ordinary g-paths [Ste90]. It has a strong connection with the minor-summation formula of an arbitrary matrix [Oka89].

We begin with elementary definitions. Let $D = (V, E) = (V(D), E(D))$ be a digraph. The number of edges from [resp. to] a vertex v is outdegree [resp. indegree] of v. If D has no multiedges or loops, the edge from x to y is often written as xy. For a given vertex set V, the (vertex-)induced subdigraph of D induced by V is the maximum digraph with the vertex set V. Similarly, given an edge set E, the edge-induced subdigraph of D induced by E is the minimum subdigraph with the edge set E. We assume that a path in a digraph is directed and has no vertex repetitions. An undirected path is called a semipath.

A digraph F is called *irreducible* when it includes no isolated vertices and no vertices of indegree = outdegree = 1. A reduction of D is a composition of the operations of deleting an isolated vertex simply; or deleting a vertex x of degree 2 such that $y \xrightarrow{e} x \xrightarrow{f} z$, together with the edge f, and attaching e to z so that we may have $y \xrightarrow{e} z$. The digraph F obtained by a reduction of D is called a reduced digraph of D, and if F is irreducible, it is called the factor of D.

1991 *Mathematics Subject Classification.* Primary 05E05, 05E10, 05E15.

Key words and phrases. ideal-tableaux, lattice paths, Lindström's theorem, Jacobi–Trudi identity, superdeterminants.

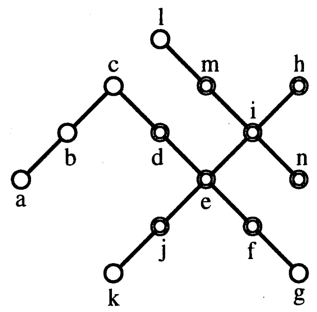

FIGURE 1. Odd-ary tree O.

Let O be a finite odd-ary tree with edges E, and $K = (K_1, \dots, K_r)$ be connected induced subdigraphs of O, including all the edges incident into/from ramification vertices (vertices of degree > 2) of O. As we note at the beginning, directed trees are identified with ordered sets, and so we can consider the order ideals of K_i. Here, an order ideal I of an ordered set S is defined as a subset of S such that, if $x \in I$ and $x > y$, then $y \in I$. Let $J(K_i)$ denote the ordered set of all order ideals of K_i ordered by inclusion. For $I \in J(K_i)$ and $I' \in J(K_{i'})$, we define a (non-order) relation $\preceq_{ii'}$ by $I \preceq_{ii'} I' \iff I \cap K_{i'} \subset I' \cap K_i$. Consider a tableau T with r rows and infinitely many columns, whose (i,j)-entry T_{ij} is an element of $J(K_i)$. Suppose that

T1: T_{ij} increases weakly as j increases $\quad (i \in [1,r])$,
T2: $T_{ij} \preceq_{i,i+1+l} T_{i+1+l,j+l} \quad (l \in [0, r-i-1],\ i \in [1, r-1],\ j \in \mathbb{Z})$.

(If $K_i = O$ $(i = 1, \dots, r)$, then (T2) is simply "T_{ij} increases weakly as i increases".) We call the tableau T *an ideal-tableau of* K.

The end vertices $\mathbf{end}(D)$ of a digraph D are defined to be the vertices of degree $= 1$. Let a map B_i from $\mathbf{end}(K_i)$ to $\mathbb{Z}$ be fixed. Also take a map α from the edges of O to the intervals of $\mathbb{N}$ (the set of nonnegative integers). Let $T_i(x)$ $(x \in V(K_i))$ denote $\min\{j \in \mathbb{Z}; x \in T_{ij}\} - i$. Set $E_i = E(K_i)$, $E_{ii'} = E_i \cap E_{i'}$. Define

$$\text{(1)} \qquad \begin{aligned} &\mathrm{Tab}(K,B,\alpha) = \{T : \text{ideal-tableaux of } K;\ T_i(x) = B_i(x) \\ &(x \in \mathbf{end}(K_i),\ i \in [1,r]),\ T_i(x) - T_i(y) \in \alpha(xy)\ (xy \in E_i,\ i \in [1,r])\}. \end{aligned}$$

Let the weight $w(T)$ of T be the following polynomial in the variables $Y = (Y^e_{ij})$, $t = (t_e)$ $(i,j \in \mathbb{Z},\ i - j \in \alpha(e),\ e \in E)$.

$$\text{(2)} \qquad w(T) = \prod_{(i,j)} w_i(T_{ij}) \cdot \prod_{xy \in E} |Y^{xy}_{T_i(x),T_{i'}(y)}|_{E_{ii'} \ni xy}, \quad w_i(I) = \prod_{\substack{xy \in E_i \\ x \notin I \ni y}} t_{xy}.$$

Here (i,j) runs over $[1,r] \times \mathbb{Z}$, and for $i - j \notin \alpha(e)$, $Y^e_{ij} := 0$. Also, the determinant of the empty matrix is defined as 1. The first factor of $w(T)$, denoted by t^T, is called the power weight of T, and the second one, denoted by $Y(T)$, the determinantal weight of T. We consider the ideal-tableau-generating function $g(K,B,\alpha)$ given by

$$\text{(3)} \qquad g(K,B,\alpha) = \sum_{T \in \mathrm{Tab}(K,B,\alpha)} w(T).$$

$\cdots$	-1	0	1	2	3	4	5	$\cdots$
$\cdots$	$\emptyset$	$\emptyset$	$\{gk\}$	$\{fgjk\}$	$\{adefgjkn\}$	$\{abdefgijkn\}$	K_1	$\cdots$
$\cdots$	$\emptyset$	$\emptyset$	$\{j\}$	$\{fj\}$	$\{defijn\}$	K_2	K_2	$\cdots$
$\cdots$	$\emptyset$	$\{k\}$	$\{jk\}$	$\{efgjk\}$	$\{defgijkn\}$	$\{adefghijkmn\}$	K_3	$\cdots$

Let $O = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$ be the odd-ary tree depicted in Figure 1. Let $K_1 = K_3 = O$, $K_2 = \{d, e, f, h, i, j, m, n\}$ be subtrees. Let T be the ideal-tableau of K displayed above. Set $\alpha(e) = \mathbb{N}$ for all $e \in E$. Then the weight $w(T) = t^T Y(T)$ is what follows.

$$Y(T) = \Delta^{32}_{21}(ba)\Delta^{42}_{32}(cb)\Delta^{42}_{20}(cd)\Delta^{210}_{21-1}(de)\Delta^{21-1}_{10-1}(ef)\Delta^{1-1}_{0-1}(fg)\Delta^{421}_{310}(hi)$$
$$\cdot \Delta^{310}_{21-1}(ie)\Delta^{21-1}_{1-1-2}(ej)\Delta^{1-2}_{0-3}(jk)\Delta^{42}_{41}(lm)\Delta^{421}_{310}(mi)\Delta^{310}_{210}(in),$$

where $\Delta^{pq}_{rs}(xy) := \begin{vmatrix} Y^{xy}_{pr} & Y^{xy}_{ps} \\ Y^{xy}_{qr} & Y^{xy}_{qs} \end{vmatrix}$, etc., and $t^T = t^2_{ba} t_{cb} t^4_{cd} t_{de} t^2_{ef} t_{fg} t^3_{hi} t^2_{ie} t^4_{ej} t^2_{jk} t_{lm} t^3_{mi} t_{in}$.

Let the end vertices of O be a^k $(k = 1, \ldots, s)$. As O is odd-ary, s is even. There exists a unique end vertex a^k_i of K_i that can be linked to a^k by semipath in O passing through no ramification vertices. For any sequence $i_1, \ldots, i_s$ in $[1, r]$, there exists the one and only connected induced subdigraph (tree) of O with end vertices $a^k_{i_k}$ $(k = 1, \ldots, s)$. Let us denote it by $K_{i_1 \ldots i_s}$. Let $\tilde{B}_{i_1 \ldots i_s}$ be a map $\mathbf{end}(K_{i_1 \ldots i_s}) \longrightarrow \mathbb{Z}$ such that $\tilde{B}_{i_1 \ldots i_s}(a^k_{i_k}) = B_{i_k}(a^k_{i_k})$. Consider the totality $\tilde{P}_{i_1 \ldots i_s}$ of the maps $p : V(K_{i_1 \ldots i_s}) \longrightarrow \mathbb{Z}$ satisfying $p(a^k_{i_k}) = \tilde{B}_{i_1 \ldots i_s}(a^k_{i_k})$ $(k = 1, \ldots, s)$ and $p(x) - p(y) \in \alpha(xy)$ $(xy \in E(K_{i_1 \ldots i_s}))$. Now define

$$P(K_{i_1 \ldots i_s}, \tilde{B}_{i_1 \ldots i_s}, \alpha) = \sum_{p \in \tilde{P}_{i_1 \ldots i_s}} \prod_{xy \in E(K_{i_1 \ldots i_s})} Y^{xy}_{p(x), p(y)} t^{p(x) - p(y)}_{xy}. \tag{4}$$

Let S_r denote the set of all permutations of $\{1, \ldots, r\}$. We introduce an *s-determinant* (*superdeterminant*) by the formula:

$$|M_{i_1 \ldots i_s}|_{s,r} := \frac{1}{r!} \sum_{\sigma_1, \ldots, \sigma_s \in S_r} \operatorname{sgn}(\sigma_1 \ldots \sigma_s) \prod_{i=1}^{r} M_{\sigma_1(i), \ldots, \sigma_s(i)}. \tag{5}$$

It is easy to show that, for odd s and $r \geq 2$, $|M_{i_1 \ldots i_s}|_{s,r} = 0$. Next we assume that the maps α and E satisfy the following.

Assumption 1. Let $\alpha(e) = [m_e, n_e]$ and $x_0, \ldots, x_c$ be the semipath (edges omitted) from a^k_i to $a^k_{i'}$ in O. Then, for all $1 \leq i < i' \leq r$, $k = 1, \ldots, s$,

$$B_i(a^k_i) - B_{i'}(a^k_{i'}) \geq \sum_{\substack{0 \leq j \leq c-1 \\ x_j > x_{j+1}}} n_{x_j x_{j+1}} - \sum_{\substack{0 \leq j \leq c-1 \\ x_j < x_{j+1}}} m_{x_{j+1} x_j}. \tag{6}$$

Note that this assumption is equivalent to the seemingly weaker one: "(6) holds for all $(i, i') = (1, 2), (2, 3), \ldots, (r-1, r)$ and $k = 1, \ldots, s$". Finally, we can state our main result.

Theorem 1. (Jacobi–Trudi-type identity) *It holds that*

(7) $$g(K, B, \alpha) = \left|P(K_{i_1\ldots i_s}, \tilde{B}_{i_1\ldots i_s}, \alpha)\right|_{s,r} = \left|g(K_{i_1\ldots i_s}, \tilde{B}_{i_1\ldots i_s}, \alpha)\right|_{s,r}.$$

Remark. The tool "s-determinant" is considered as a tensor invariant. Indeed, let $s = 2m$ and M be the transformation on a tensor space $E^{\otimes m}$ of an r-dimensional linear space E. For the basis $(e_1, \ldots, e_r)$, let $M(e_{i_1} \otimes \cdots \otimes e_{i_m}) = M^{j_1\ldots j_m}_{i_1\ldots i_m} e_{j_1} \otimes \cdots \otimes e_{j_m}$ with Einstein's convention. Then the s-determinant of $[M^{j_1\ldots j_m}_{i_1\ldots i_m}]$ depends only on M but not the choice of the basis.

2. Tree-r-paths and Lindström's theorem

Here we show that there exists an odd-ary-tree-path-analogue of Lindström-Gessel–Viennot method [Lin73, GV85, GV] on which the main theorem is based.

While F-paths are dealt with for $F =$ an odd-ary tree, the difficulty does not increase in giving general definition. If F is a digraph $\circ\!\longrightarrow\!\circ$, then an F-path is an ordinary path. In general, F should be irreducible.

An F-path in D is defined to be a pair of maps $p = (p^\bullet, \overline{p})$; $p^\bullet : V(F) \longrightarrow V(D)$, $\overline{p} : E(F) \longrightarrow \{\text{paths in } D\}$, such that $\overline{p}(xy)$ is a path from $p^\bullet(x)$ to $p^\bullet(y)$. For $e \in E(F)$, the *e-section* of p is the path $\overline{p}(e)$. Note that a section could be a path of length 0, that is, a vertex. The union of all underlying vertices of all sections of p is denoted by $\mathrm{v}(p)$. An element of the set $\{(x, e) \in V(D) \times E(F);\ \overline{p}(e)$ passes through x, x is not an end of $\overline{p}(e)\} \cup \{(p^\bullet(v), v);\ v \in V(F)\}$ is called a vertex of p. In this sense, an F-path has no vertex repetitions. For convenience, the vertex (x, v) is also written as (x, e), where e is incident with v. As in the case of ordinary paths, if one needs a *bounded* F-path, i.e. need to specify the end vertices of an F-path, one may designate the boundary map $\tau = p^\bullet|_{\mathsf{end}(F)}$. The vertices $\tau(\mathsf{end}(F))$ are called the boundary of p. An (F, r)-path is an r-tuple $(p_1, \ldots, p_r)$ of F-paths. In this case, the boundary map (if needed) is an r-tuple $(\tau_1, \ldots, \tau_r)$. A $(\circ\!\longrightarrow\!\circ, r)$-path is nothing but an r-path. An (F, r)-path is called *locally disjoint* (loc. disj.) if, for all $1 \le i < j \le r$ and $e \in E(F)$, the e-sections of p_i and p_j have no common vertices. "An F-path *locally intersects* another" means that they are not locally disjoint. A *disjoint* (F, r)-path is defined to have the disjoint sets $\mathrm{v}(p_1), \ldots, \mathrm{v}(p_r)$.

Let F be a finite irreducible odd-ary tree and $\mathsf{end}(F) = \{a^1, \ldots, a^s\}$. Let (b^k_i) $(i \in [1, r],\ k \in [1, s])$ be vertices of D such that $b^k_i \ne b^k_j$ for all k and distinct i, j. We denote by $\mathtt{PATH}_{i_1\ldots i_s}$ the totality of F-paths in D with the boundary map $\tau_{i_1\ldots i_s} : a^k \mapsto b^k_{i_k}$ $(k \in [1, s])$. Now define, for $\sigma = (\sigma_1, \ldots, \sigma_{s-1}) \in S_r^{s-1}$, using abbreviation $\sigma(i) = (\sigma_1(i), \ldots, \sigma_{s-1}(i))$,

(8) $$\mathtt{PATH}(\sigma) = \left\{(F, r)\text{-paths in } D \text{ with the boundary map } (\tau_{1,\sigma(1)}, \ldots, \tau_{r,\sigma(r)})\right\},$$

and denote by $\mathtt{PATH}^\circ(\sigma)$ [resp. $\mathtt{PATH}^\times(\sigma)$] the subset composed of all locally disjoint [resp. non locally disjoint] elements.

Assume D is acyclic and has finitely many bounded F-paths for each boundary map. Assign a weight $w(e)$ to each edge of D. Let the weight of an F-path be the product of those of all the underlying edges and the weight of (F, r)-path the product of those of the components. The weight of a set Q of (F, r)-paths is defined to be the sum of those of all elements, which is considered as the generating function for Q denoted by $\mathrm{g}[\,Q\,]$. For $\sigma \in S_r^{s-1}$, $\mathrm{sgn}(\sigma)$ is defined to be the signature of the product of the components of σ. The following is an analogue of Lindström's theorem.

Theorem 2. *The signed generating function of loc. disj. paths is evaluated by*

$$\sum_{\sigma \in S_r^{s-1}} \mathrm{sgn}(\sigma)\, \mathrm{g}[\,\mathtt{PATH}^{\circ}(\sigma)\,] = |\,\mathrm{g}[\,\mathtt{PATH}_{i_1 \dots i_s}\,]|_{s,r}. \tag{9}$$

Proof. By definition, the right-hand side is written as $\sum_\sigma \mathrm{sgn}(\sigma)\,\mathrm{g}[\,\mathtt{PATH}(\sigma)\,]$, thus it suffices to construct a weight-preserving involution $* : \mathtt{PATH}^{\times} \longrightarrow \mathtt{PATH}^{\times}$, where $\mathtt{PATH}^{\times} = \coprod_\sigma \mathtt{PATH}^{\times}(\sigma)$, such that if $p \in \mathtt{PATH}^{\times}(\sigma)$ and $p^* \in \mathtt{PATH}^{\times}(\rho)$, then $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\rho)$. For each F-path q, we can construct the unique order $<_q$ on the vertices of q as follows.

(i) The maximum element is $(q^{\bullet}(a^1), a^1)$.
(ii) The cover relation exists only between the vertices $(x, e), (y, e)$ such that x, y are adjacent in $\overline{q}(e)$.
(iii) The vertices with the fixed second component e are totally ordered.

Next, we fix an arbitrary total order on $V(D) \times E(F)$ and $\Omega = \{(i,j) \in [1,r] \times [1,r]; i < j\}$. For given $p \in \mathtt{PATH}^{\times}(\sigma)$, we can take the least local intersection $(v,e) \in V(D) \times E(F)$. (If a local intersection of two F-paths has several distinct expressions, we promise to use the least one.) Then choose 2 components (p_i, p_j) intersecting at (v,e) with the least pair $(i,j) \in \Omega$. Now define $p^* \in \mathtt{PATH}^{\times}(\rho)$ as follows: (i) $p_k^* = p_k$ for all $k \neq i, j$; (ii) the vertices of p_i^* consist of the vertices of p_i greater or equal to (v,e) in the order $<_{p_i}$ and the vertices of p_j less than (v,e) in the order $<_{p_j}$; (iii) the vertices of p_j^* consist of the vertices of p_j greater or equal to (v,e) in the order $<_{p_j}$ and the vertices of p_i less than (v,e) in the order $<_{p_i}$. Let us certify $*$ satisfies the condition. Since D is acyclic, the components of p^* have no self-intersecting sections, and so p^* is certainly an (F,r)-path contained in $\mathtt{PATH}^{\times}(\rho)$. This ensures that the set of intersection vertices in each section are preserved under the operation $*$, and therefore $*$ is an involution. The rest is (#) : $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\rho)$. By the effect of $*$, the end vertices of p_i, p_j corresponding to the identical a^k are replaced each other whenever a^k is opposite to a^1 with respect to the edge e. Thus $\mathrm{sgn}(\sigma_k) = -\mathrm{sgn}(\rho_k)$. As F is odd-ary, the number of those k's is always odd. Hence (#) holds. □

3. The lattice path method for Theorem 1

While O has already been regarded as an ordered set, we define O' by reordering with an order $<'$, which is similar to $<_q$. Let the vertex a^1 be the maximum element, and give cover relation between two vertices iff they are adjacent, that determines the order uniquely. The O' is naturally regarded as a digraph. Let F be the factor of O'. By the assumption for K_i, the F is also isomorphic to the factor of K_i' made of K_i with the order $<'$. To give a proof of Theorem 1, we construct a bijection between $\mathrm{Tab}(K, B, \alpha)$ and a set of bounded (F,r)-paths in a certain acyclic digraph D without multiedges. Now define D by

$$
\begin{aligned}
(10) \qquad & V(D) = V(O') \times \mathbb{Z}, \\
& E(D) = \{(x,i)(y,j);\ xy \in E(O'),\ i-j \in \alpha(xy)\ (xy \in E), \\
& \qquad j-i \in \alpha(yx)\ (yx \in E)\}.
\end{aligned}
$$

Next let $b_i^k = (a_i^k, B_i(a_i^k))$ $(k \in [1,s],\ i \in [1,r])$. Take the boundary map $\tau = (\tau_1, \dots, \tau_r)$, $\tau_i : \mathtt{end}(F) \longrightarrow V(D)$, defined by $\tau_i(a^k) = b_i^k$. Since K_i' is a tree, one sees that a bounded F-path p_i in D with τ_i is nothing else than the map $(p_i) : V(K_i') \longrightarrow \mathbb{Z}$ defined by $(x, (p_i)(x)) \in \mathrm{v}(p_i)$. We denote by $\mathtt{PATH}_\tau^\square$ the totality of bounded (F,r)-paths $p = (p_1, \dots, p_r)$ in D with τ such that, for all $i < i'$, $(x,j) \in \mathrm{v}(p_i)$ and $(x,j') \in \mathrm{v}(p_{i'})$ imply $j > j'$, which means intuitively that they are assumed to be disjoint and have no edge-intersection.

Lemma 1. *There exists a bijection* $\phi : \mathrm{Tab}(K,B,\alpha) \longrightarrow \mathtt{PATH}_\tau^\square : T \longmapsto p$ *defined by* $(p_i)(x) = T_i(x)$ $(x \in V(K_i),\ i \in [1,r])$.

Proof. We may give the inverse $\phi^{-1} : p \longmapsto T$ by $T_{ij} = \{x \in V(K_i); (p_i)(x) + i < j\}$ $((i,j) \in [1,r] \times \mathbb{Z})$. By definition (10), we see that this T_{ij} is an order ideal of K_i. Now what should be proved is (i): $\phi(\mathrm{Tab}(K,B,\alpha)) \subset \mathtt{PATH}_\tau^\square$ and (ii): $\phi^{-1}(\mathtt{PATH}_\tau^\square) \subset \mathrm{Tab}(K,B,\alpha)$. In a proof of (i), the rest of (a): "For all $i < i'$, $(x,j) \in \mathrm{v}(p_i)$ and $(x,j') \in \mathrm{v}(p_{i'})$ imply $j > j'$" is clear. Similarly, to show (ii), we only need to see (b): $T_{ij} \cap K_{i'} \subset T_{i',j+i'-i-1} \cap K_i$ $(i < i')$. They are deduced from the equivalence:

$$
\begin{aligned}
(a) &\iff T_i(x) - 1 \geq T_{i'}(x) \quad (i < i',\ x \in V(K_i) \cap V(K_{i'})) \\
&\iff \min\{j; x \in T_{ij}\} \geq \min\{j; x \in T_{i'j}\} - i' + i + 1 \iff (b). \qquad \square
\end{aligned}
$$

Proof of Theorem 1. Let the weight $Y_{ij}^{xy} t_{xy}^{i-j}$ be given to each edge $(x,i)(y,j)$ of D. Apply Theorem 2 for the above-mentioned F, D and the boundary maps $\tau_{i_1 \dots i_s} : a^k \longmapsto b_{i_k}^k$ $(k \in [1,s])$. From the property of D and (b_i^k), it follows that $\mathtt{PATH}^\circ(\sigma)$ on the left-hand side of (9) may be replaced with the subset $\mathtt{PATH}^\diamond(\sigma)$ consisting of all disjoint (F,r)-paths. Then we call this (9)'.

For each element $p \in \mathtt{PATH}_\tau^\square$ and $x \in V(O')$, let $p^+(x)$ [resp. $p^-(x)$] denote the sequence $(x,(p_1)(x)), \dots, (x,(p_r)(x))$, where the ith terms with $x \notin V(K_i) - \{a_i^2, \dots, a_i^r\}$ [resp. $x \notin V(K_i) - a_i^1$] are omitted. Note that, for $xy \in E(O')$, $|p^+(x)| = |p^-(y)|$. The cardinality is denoted by $\kappa(xy)$. For $\rho \in S_t$, and vertices $(x_1, \dots, x_t)$, set $\rho(x_1, \dots, x_t) = (x_{\rho(1)}, \dots, x_{\rho(t)})$. Define $D^{xy}(p) =$ the induced subdigraph of D with the vertices $p^+(x) \coprod p^-(y)$, and $\mathtt{PATH}^\diamond(xy, \rho, p) =$ the set of all vertex-disjoint $\kappa(xy)$-paths from $p^+(x)$ to $\rho(p^-(y))$ in $D^{xy}(p)$. By the definition of the boundary map τ, Assumption 1 assures that for all $p \in \mathtt{PATH}^\diamond(\sigma)$, $i'' < i < i'$ and k, we have $(p_{i''})(a_i^k) > (p_i)(a_i^k) > (p_{i'})(a_i^k)$ (for the defined left and/or right-hand side). This enables us to have the weight preserving bijection:

$$
(11) \qquad b : \coprod_{\sigma \in S_r^{s-1}} \mathtt{PATH}^\diamond(\sigma) \longrightarrow \coprod_{p \in \mathtt{PATH}_\tau^\square} \prod_{e \in E(O')} \coprod_{\rho_e \in S_{\kappa(e)}} \mathtt{PATH}^\diamond(e, \rho_e, p).
$$

Since F is odd-ary, the signs of the corresponding terms on both sides of (11) coincide. Thus, taking the weights with signs of both sides and combine it with (9)', we obtain Theorem 1. $\square$

4. Specialization of the weights

In Theorem 1, rather complicated determinantal weights creep into the formula, while most Jacobi–Trudi identities are more simple. The reason is that Theorem 1 never imposes strong conditions such as "row-strict", "column-strict", etc. on the ideal-tableaux. Here we intend to simplify the formula. First of all, we define the *e-shape* of an ideal-tableau T for each $e \in E$. Set $j(e) = \{i \in [1,r]; e \in E(K)_i\}_<$. For $e = v_+v_-$, define $(T_i(v_\pm))_{i \in j(e)} = (T_i^{e\pm} - i)_{i \in [1,|j(e)|]}$.

By Lemma 1, for $i < j$ such that $V(K_i), V(K_j) \ni x$, $T_i(x) > T_j(x)$. So $(T_i^{e\pm})$ decrease weakly, and one sees $T_i^{e+} \geq T_i^{e-}$. Now let T^e denote the diagram in $[1,r] \times \mathbb{Z}$: $\{(i,j); T_i^{e-} < j \leq T_i^{e+}\}$. It is called the e-shape of T. If we drag it along the j-axis until it enters the right-hand side of i-axis, it becomes a skew diagram $\lambda \setminus \mu$. Then we use the notations $s(T^e)$ and $s(T^{e\prime})$ for the skew S-functions $s_{\lambda/\mu}$ and $s_{\lambda'/\mu'}$, respectively.

Returning to Theorem 1, divide E into L, M. Suppose $\alpha(e) = [0,n]$ for all $e \in L$ and $\alpha(e) = \mathbb{N}$ for all $e \in M$. Let e_d and h_d denote the elementary and the complete symmetric functions, respectively. Now set $Y_{ij}^e = e_{i-j}(x_1, \ldots, x_n)$ when $e \in L$, and $Y_{ij}^e = h_{i-j}(x_1, \ldots, x_n)$, otherwise. Then we immediately see that the determinantal weight of T is written as $\prod_{e \in L} s(T^{e\prime})(x) \cdot \prod_{e \in M} s(T^e)(x)$. Next we define a set of (L, M)-semistandard ideal-tableaux of trees and a certain function of $t = (t_e)_{e \in E}$.

$$\begin{aligned}\mathrm{SST}_{LM}(K,B) = \big\{T : &\text{ ideal-tableaux of } K;\ T_i(x) = B_i(x)\ (x \in \mathtt{end}(K_i), \\ &i \in [1,r]),\ T^e : \text{vertical [resp. horizontal] strip } (e \in L \text{ [resp. } M])\big\},\end{aligned} \tag{12}$$

$$P_{LM}(K_{i_1 \ldots i_s}, \tilde{B}_{i_1 \ldots i_s})(t) = \Big[P(K_{i_1 \ldots i_s}, \tilde{B}_{i_1 \ldots i_s}, \alpha)\Big]_{Y_{ij}^e = \epsilon(i,j,e)}. \tag{13}$$

Here, $\epsilon(i,j,e)$ is defined to be 1 whenever $e \in M$or $i - j \in [0,1]$, and to be 0, otherwise.

By putting $x_1 = 1$ and $x_2 = x_3 = \cdots = 0$, (7) becomes a simple formula, which turns into the one for (L, M)-partially strict tableaux with bounded entries in each row, when K is an r-tuple of chains [Oka90, Wac85].

Corollary 1. *The power weight sum of semistandard ideal-tableaux of trees is expressed as*

$$\sum_{T \in \mathrm{SST}_{LM}(K,B)} t^T = \Big|P_{LM}(K_{i_1 \ldots i_s}, \tilde{B}_{i_1 \ldots i_s})(t)\Big|_{s,r}. \tag{14}$$

5. Superpfaffians for locally disjoint tree-g-paths

Okada gave a remarkable pfaffian formula for the minor sum of a matrix [Oka89], and Stembridge developed a useful technique for calculation of the weights of (partially) unbounded vertex-disjoint r-paths with pfaffians [Ste90]. Lindström's theorem shows a strong connection between them. It is also known that a symmetric analogue of Okada's result exists. In this section, we generalize those theories on tree-g-paths.

We introduce (λ, n)-*pfaffians* (*superpfaffians*). Let g, n be positive integers and $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of g. The multiplicity of the i-parts in λ is denoted by $m(i)$, say, $\lambda = (1^{m(1)}, 2^{m(2)}, \dots)$ in increasing order. We set $g_i = \lambda_1 + \cdots + \lambda_i$ for all $i = 1, \dots, r$, and $g_0 = 0$. Let $\mathcal{G}_\lambda$ denote the set of permutations ρ of $\{1, \dots, g\}$ satisfying $\rho(g_{i-1}+1) < \rho(g_{i-1}+2) < \cdots < \rho(g_i)$ $(i = 1, \dots, r)$, and $\mathcal{F}_\lambda$ denote the subset of $\mathcal{G}_\lambda$ consisting of ρ such that $\rho(g_{i-1}+1) < \rho(g_i+1)$ whenever $\lambda_i = \lambda_{i+1}$. Define

$$\text{(15)}\quad \begin{aligned} &\mathrm{pf}_{\lambda,n}\left[[M_{i_1 \dots i_{pn}}]_{\substack{1 \le i_{pk+1} < \cdots < i_{pk+p} \le g \\ (k=0,\dots,n-1)}} \right]_{p \in \{\lambda_1, \dots, \lambda_r\}} = \frac{1}{m!} \sum_{\sigma \in \mathcal{G}_\lambda^n} \mathrm{sgn}(\sigma) P_\sigma \\ &P_{(\sigma_1, \dots, \sigma_n)} = \prod_{1 \le i \le r} M_{\sigma_1(g_{i-1}+1), \dots, \sigma_1(g_i), \dots, \sigma_n(g_{i-1}+1), \dots, \sigma_n(g_i)}, \end{aligned}$$

where $m! = m(1)!\, m(2)! \dots$. Let d be the number of distinct parts of λ. By definition, the array on the left-hand side is a d-tuple of different dimensional arrays. For $\lambda = (2^r)$, $n = 1$, the above expression is led to an ordinary pfaffian for $2r \times 2r$ skew-symmetric matrix; while for $\lambda = (1^r)$, $n = s$ — an s-determinant. Furthermore, for odd n and λ such that $m(i) > 1$ for some odd i, that vanishes.

For example, take $\lambda = (2, 1)$ and $n = 2$. We have

$$\begin{aligned} &\mathrm{pf}_{(2,1),2}\left[[M_{i_1 \dots i_4}]_{\substack{1 \le i_1 < i_2 \le 3 \\ 1 \le i_3 < i_4 \le 3}}, [M_{jk}]_{1 \le j,k \le 3} \right] \\ &= M_{1212}M_{33} - M_{1213}M_{32} + M_{1223}M_{31} - M_{1312}M_{23} \\ &+ M_{1313}M_{22} - M_{1323}M_{21} + M_{2312}M_{13} - M_{2313}M_{12} + M_{2323}M_{11}. \end{aligned}$$

As in §2, we assume that F is a finite irreducible odd-ary tree with the end vertices $\{a^1, \dots, a^s\}$ (s: even), and D is an acyclic digraph with finitely many F-paths for each boundary map. Next let λ be chosen so that $m(i) \le 1$ for all odd i. Let V^1 be an arbitrary finite set of at least g vertices of D; and $V^2, \dots, V^s$ ones of g vertices. Assume for each $k \in [1, s]$, that V^k is totally ordered irrespective of the structure of D and the other V^l.

Assumption 2. All F-paths p, q satisfying that $p^\bullet(a^k) < q^\bullet(a^k)$ in V^k and $p^\bullet(a^l) > q^\bullet(a^l)$ in V^l for some $k, l \in [1, s]$ intersect locally.

Let us fix a set $\mathcal{A}$ of subsets of V^1 which contains at least $m(i)$ disjoint i-subsets whenever $m(i) > 0$, and no i-subsets otherwise. Let I be a subset of V^1. For every $k \in [2, s]$, denote by $v^k = (v^k_1, \dots, v^k_g)$, an arbitrary arrangement of all elements of V^k. Now define

$$\text{(16)}\quad \begin{aligned} \mathtt{PATH}_g(I, v^2, \dots, v^s) = \{p : (F, g)\text{-paths in } D;\ \{p^\bullet_1(a^1), \dots, p^\bullet_g(a^1)\} = I, \\ p^\bullet_i(a^k) = v^k_i\ ((i, k) \in [1, g] \times [2, s])\}, \end{aligned}$$

and $\mathtt{PATH}^\circ_g(I, v) = \mathtt{PATH}^\circ_g(I, v^2, \dots, v^s)$ to be the subset which contains exactly all locally disjoint elements as usual. For $\rho \in S_g$, set $\rho(v^k) = (v^k_{\rho(1)}, \dots, v^k_{\rho(g)})$. Let $I = \{v^1_1, \dots, v^1_g\}_<$ and assume that v^k is ordered increasingly for each k. The λ-generating function $\mathrm{g}_\lambda[\,\mathtt{PATH}^\circ_g(I, v)\,]$ for $\mathtt{PATH}^\circ_g(I, v)$ is defined as the product: $\epsilon(I) \cdot \mathrm{g}[\,\mathtt{PATH}^\circ_g(I, v)\,]$,

$\epsilon(I) = \sum \mathrm{sgn}(\rho)$; where the summation runs over all $\rho \in \mathcal{F}_\lambda$ such that, for every $i \in [1,r]$, $\{v^1_{\rho(j)}\}_{j\in[g_{i-1}+1,g_i]}$ belongs to $\mathcal{A}$.

Next, we set $\mathtt{PATH}_g(v^2,\dots,v^s) = \coprod_{I\subset V^1} \mathtt{PATH}_g(I,v)$ and consider the subset consisting of all locally disjoint elements: $\mathtt{PATH}^\circ_g(v^2,\dots,v^s) = \coprod_{I\subset V^1} \mathtt{PATH}^\circ_g(I,v)$. Define $\mathrm{g}_\lambda[\,\mathtt{PATH}^\circ_g(v^2,\dots,v^s)\,] = \sum_I \mathrm{g}_\lambda[\,\mathtt{PATH}^\circ_g(I,v)\,]$.

Theorem 3. *The λ-generating function is expressed by a superpfaffian, say,*

$$(17)\quad \mathrm{g}_\lambda[\,\mathtt{PATH}^\circ_g(v^2,\dots,v^s)\,] =$$

$$\mathrm{pf}_{\lambda,s-1}\left[\ \mathrm{g}_{(p)}\Big[\mathtt{PATH}^\circ_p((v^2_{i_1},\dots,v^2_{i_p}),\dots,(v^s_{l_1},\dots,v^s_{l_p}))\Big]_{\substack{1\le i_1<\cdots<i_p\le g,\dots,\\ 1\le l_1<\cdots<l_p\le g}}\ \right]_{p\in\{\lambda_1,\dots,\lambda_r\}}.$$

Proof. For $\sigma = (\sigma_2,\dots,\sigma_s) \in \mathcal{G}^{s-1}_\lambda$, we use the notation: $\sigma(v) = (\sigma_2(v^2),\dots,\sigma_s(v^s))$. We put $\mathtt{PATH}^\times_g(\sigma(v)) = \mathtt{PATH}_g(\sigma(v)) - \mathtt{PATH}^\circ_g(\sigma(v))$ and set

$$(18)\qquad \begin{aligned} \mathtt{PATH}^\times_\lambda(\sigma(v)) = \{p \in \mathtt{PATH}^\times_g(\sigma(v));\ &(p_{g_{i-1}+1},\dots,p_{g_i}) \text{ is locally}\\ &\text{disjoint and } \{p^\bullet_{g_{i-1}+1}(a^1),\dots,p^\bullet_{g_i}(a^1)\} \in \mathcal{A} \text{ for all } i \in [1,r]\}. \end{aligned}$$

By (15), we may translate the pfaffian (multiplied by $m!$) on the right-hand side of (17) to the signed weight of (F,g)-paths p such that (i): for all $i \in [1,r]$, $\tilde p_i = (p_{g_{i-1}+1},\dots,p_{g_i})$ is locally disjoint, (ii): the components of $\tilde p_i$ are arranged so that the boundaries corresponding to a^k are increasing for each $k \in [1,s]$, (iii): the boundaries of $\tilde p_i$ corresponding to a^1 form an element of $\mathcal{A}$, and (iv): the boundaries of p corresponding to a^k form V^k for all $k \in [2,s]$. If p is locally disjoint, Assumption 2 implies that there exists $\nu \in \mathcal{G}_\lambda$ such that for every $k \in [1,s]$, $p^\bullet_{\nu^{-1}(1)}(a^k) < \cdots < p^\bullet_{\nu^{-1}(g)}(a^k)$. Thus, the same weights, except signs, are arising from (F,g)-paths $\{(q_{\rho(1)},\dots,q_{\rho(g)})\}$ where $q = \nu^{-1}(p)$ and ρ runs over all permutations in $\mathcal{G}_\lambda$ such that $\{q^\bullet_{\rho(g_{i-1}+1)}(a^1),\dots,q^\bullet_{\rho(g_i)}(a^1)\} \in \mathcal{A}$ for all $i \in [1,r]$. From this, it follows that the weight of locally disjoint (F,g)-paths appearing in the pfaffian is equal to the left-hand side of (17). Therefore, dividing by $m!$, the right-hand side of (17) is written as

$$(19)\qquad \mathrm{g}_\lambda[\,\mathtt{PATH}^\circ_g(v^2,\dots,v^s)\,] + \frac{1}{m!}\sum_{\sigma\in\mathcal{G}^{s-1}_\lambda} \mathrm{sgn}(\sigma)\,\mathrm{g}[\,\mathtt{PATH}^\times_\lambda(\sigma(v))\,].$$

So we prove that the second term of (19) vanishes. To do this, as in the proof of Theorem 2, we take an involution $*$ on $\mathtt{PATH}^\times_\lambda = \coprod_{\sigma\in\mathcal{G}^{s-1}_\lambda} \mathtt{PATH}^\times_\lambda(\sigma(v))$ such that $w(p^*) = w(p)$, $\mathrm{sgn}(\rho) = -\mathrm{sgn}(\sigma)$ ($p \in \mathtt{PATH}^\times_\lambda(\sigma(v))$, $p^* \in \mathtt{PATH}^\times_\lambda(\rho(v))$). For this involution, we can use a slight deformation of $*$ in the proof of Theorem 2. The modified point is to choose the least local intersection $(v,e) \in V(D)\times E(F)$ such that each component p_i of p with local intersection (v,e) has no local intersection less than (v,e) with respect to the order $<_{p_i}$. In virtue of this, locally disjointness of $\tilde p_i$ (i) is preserved by this deformed $*$, and therefore (ii) is also satisfied (Assumption 2). The rest (iii),(iv) are preserved clearly. Hence $\mathtt{PATH}^\times_\lambda$ is $*$-invariant. We can confirm the other properties of $*$ as in the proof of Theorem 2. $\square$

Remarks. Depending on the structure of $\mathcal{A}$, Theorem 3 gives various weight-sums of loc. disj. tree-g-paths. For example, let $v_1^1 < \cdots < v_{2n}^1$ be all vertices in V^1 and set $\mathcal{A} = \{\{v_1^1, v_{2n}^1\}, \{v_2^1, v_{2n-1}^1\}, \ldots, \{v_n^1, v_{n+1}^1\}\}$. Let $\lambda = (2^r)$, $g = 2r$. The left-hand side of (17) becomes the "symmetric" sum: $\sum_I \mathrm{g}[\,\mathtt{PATH}_g^\circ(I, v)\,]$, where I runs over all g-subsets of V^1 such that $v_k^1 \in I \Rightarrow v_{g-k+1}^1 \in I$. Similarly, for a given λ in Theorem 3, let $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{r}}) = (1^{\tilde{m}(1)}, 2^{\tilde{m}(2)}, \ldots)$ be a partition of $n = |V^1|$ such that, for all nonzero $\tilde{m}(i)$, $\tilde{m}(i) \geq m(i) \geq 1$. Set $\tilde{g}_i = \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_i$. Let $\mathcal{A}$ be a partition of $V^1 = \{v_1^1, \ldots, v_n^1\}_<$ of type $\tilde{\lambda}$ consisting of the cells $\{v_{\tilde{g}_{i-1}+1}^1, \ldots, v_{\tilde{g}_i}^1\}$ $(i = 1, \ldots, \tilde{r})$. Now Theorem 3 gives the weight-sum of loc. disj. tree-g-paths p with coefficients $\epsilon(I) = 1$, where the set of boundaries $\{p_1^\bullet(a^1), \ldots, p_g^\bullet(a^1)\}$ corresponds to the collection of the cells of $\mathcal{A}$ consisting of $m(i)$ i-cells.

Another example is an ordinary summation formula, which is the most natural. Let $\lambda = (2^r)$ and $\mathcal{A} = \{$all 2-subsets of $V^1\}$. This case enumerate the sum of all weights $\sum_I \mathrm{g}[\,\mathtt{PATH}_g^\circ(I, v)\,]$ with coefficients$= 1$. In general, let λ be a partition with no odd parts, and $\mathcal{A} = \{$all i-subsets of $V^1; m(i) \geq 1\}$. In that case we can show by induction that $\epsilon(I) = \frac{(g/2)!}{m(2)!(2m(4))!(3m(6))!\ldots} \prod_{i:\text{even}} \prod_{j=1}^{m(i)} \binom{(i/2)j-1}{i/2-1}$ irrespective of I. Thus, the case also gives the weight-sum of all loc. disj. tree-g-paths.

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