JACOBI–TRUDI-TYPE IDENTITIES FOR IDEAL-TABLEAUX

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1. INTRODUCTION

The present article is concerned with the generating functions of certain tableaux consisting of order ideals of finite odd-ary trees. Here, a tree is a connected digraph without undirected cycles, which is identified with an ordered set in this way: $x \rightarrow y$ (an edge from $x$ to $y$ exists) $\iff x$ covers $y$ $(x > y$ and $x \not> \exists z > y)$. On the analogy of “binary tree”, an odd-ary tree is defined to be a tree with vertices of degree $1, 2, 4, 6, \ldots$, where the degree of a vertex is the number of the edges incident into or from the vertex. The main result of the paper is a superdeterminantal formula for the above-mentioned generating function, which includes Wachs, Okada and Asai’s extension of the Jacobi–Trudi identity [Wac85, Oka90, Asa98]. A superdeterminant is a natural extension of a determinant defined for even dimensional square arrays. Our result is the consequence of analogous Lindström’s theorem [Lin73] and the Gessel–Viennot lattice paths [GV85, GV]. In the last section, we study the summation of the weights of (partially) unbounded tree-$g$-paths by a superpfaffian, which corresponds to Stembridge’s prominent technique to enumerate unbounded ordinary $g$-paths [Ste90]. It has a strong connection with the minor-summation formula of an arbitrary matrix [Oka89].

We begin with elementary definitions. Let $D = (V, E) = (V(D), E(D))$ be a digraph. The number of edges from [resp. to] a vertex $v$ is outdegree [resp. indegree] of $v$. If $D$ has no multiedges or loops, the edge from $x$ to $y$ is often written as $xy$. For a given vertex set $V$, the (vertex-)induced subdigraph of $D$ induced by $V$ is the maximum digraph with the vertex set $V$. Similarly, given an edge set $E$, the edge-induced subdigraph of $D$ induced by $E$ is the minimum subdigraph with the edge set $E$. We assume that a path in a digraph is directed and has no vertex repetitions. An undirected path is called a semipath.

A digraph $F$ is called irreducible when it includes no isolated vertices and no vertices of indegree = outdegree = 1. A reduction of $D$ is a composition of the operations of deleting an isolated vertex simply; or deleting a vertex $x$ of degree 2 such that $y \overset{e}{\rightarrow} x \overset{f}{\rightarrow} z$, together with the edge $f$, and attaching $e$ to $z$ so that we may have $y \overset{e}{\rightarrow} z$. The digraph $F$ obtained by a reduction of $D$ is called a reduced digraph of $D$, and if $F$ is irreducible, it is called the factor of $D$.

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Let $O$ be a finite odd-ary tree with edges $E$, and $K = (K_1, \ldots, K_r)$ be connected induced subdigraphs of $O$, including all the edges incident into/from ramification vertices (vertices of degree $> 2$) of $O$. As we note at the beginning, directed trees are identified with ordered sets, and so we can consider the order ideals of $K_i$. Here, an order ideal $I$ of an ordered set $S$ is defined as a subset of $S$ such that, if $x \in I$ and $x > y$, then $y \in I$. Let $J(K_i)$ denote the ordered set of all order ideals of $K_i$ ordered by inclusion. For $I \in J(K_i)$ and $I' \in J(K_i')$, we define a (non-order) relation $\preceq_{ii'}$ by $I \preceq_{ii'} I' \iff I \cap K_i' \subset I' \cap K_i$. Consider a tableau $T$ with $r$ rows and infinitely many columns, whose $(i, j)$-entry $T_{ij}$ is an element of $J(K_i)$. Suppose that

T1: $T_{ij}$ increases weakly as $j$ increases $(i \in [1, r])$,
T2: $T_{ij} \preceq_{i,i+1+t} T_{i+1+t,j+l}$ $(l \in [0, r-i-1], i \in [1, r-1], j \in \mathbb{Z})$.

(If $K_i = O$ $(i = 1, \ldots, r)$, then (T2) is simply “$T_{ij}$ increases weakly as $i$ increases”.) We call the tableau $T$ an ideal-tableau of $K$.

The end vertices $\text{end}(D)$ of a digraph $D$ are defined to be the vertices of degree $= 1$. Let a map $B_i$ from $\text{end}(K_i)$ to $\mathbb{Z}$ be fixed. Also take a map $\alpha$ from the edges of $O$ to the intervals of $\mathbb{N}$ (the set of nonnegative integers). Let $T_i(x)$ $(x \in V(K_i))$ denote $\min\{j \in \mathbb{Z}; x \in T_{ij}\} - i$. Set $E_i = E(K_i)$, $E_{ii'} = E_i \cap E_{i'}$. Define

\[
\text{Tab}(K, B, \alpha) = \{T: \text{ideal-tableaux of } K; T_i(x) = B_i(x) \\
(x \in \text{end}(K_i), i \in [1, r]), T_i(x) - T_i(y) \in \alpha(xy) \ (xy \in E_i, i \in [1, r])\}.
\]

Let the weight $w(T)$ of $T$ be the following polynomial in the variables $Y = (Y_{ij}^e)$, $t = (t_e)$ $(i, j \in \mathbb{Z}, i - j \in \alpha(e), e \in E)$.

\[
w(T) = \prod_{(i,j)} w_i(T_{ij}) \cdot \prod_{xy \in E} |Y_{T_i(x),T_i(y)}^{xy}|_{E_{ii'} \ni xy}, \quad w_i(I) = \prod_{xy \in E_i, x \notin I \ni y} t_{xy}.
\]

Here $(i, j)$ runs over $[1, r] \times \mathbb{Z}$, and for $i - j \notin \alpha(e)$, $Y_{ij}^e := 0$. Also, the determinant of the empty matrix is defined as $1$. The first factor of $w(T)$, denoted by $t^T$, is called the power weight of $T$, and the second one, denoted by $Y(T)$, the determinantal weight of $T$. We consider the ideal-tableau-generating function $g(K, B, \alpha)$ given by

\[
g(K, B, \alpha) = \sum_{T \in \text{Tab}(K, B, \alpha)} w(T).
\]
Let $O = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$ be the odd-ary tree depicted in Figure 1. Let $K_1 = K_3 = O$, $K_2 = \{d, e, f, h, i, j, m, n\}$ be subtrees. Let $T$ be the ideal-tableau of $K$ displayed above. Set $\alpha(e) = \mathbb{N}$ for all $e \in E$. Then the weight $w(T) = t^TY(T)$ is what follows.

$$Y(T) = \Delta_{21}^{32}(ba)\Delta_{32}^{42}(cb)\Delta_{20}^{42}(cd)\Delta_{21-1}^{21}(de)\Delta_{10-1}^{21}(ef)\Delta_{0-1}^{1-1}(fg)\Delta_{310}^{421}(hi),$$

where $\Delta_{pq}^{rs}(xy) := \begin{vmatrix} Y_{xy}^{xy} & Y_{xy}^{xy} & Y_{xy}^{xy} \\ Y_{xy}^{xy} & Y_{xy}^{xy} & Y_{xy}^{xy} \\ Y_{xy}^{xy} & Y_{xy}^{xy} & Y_{xy}^{xy} \end{vmatrix}$, etc., and $t^T = t_1^T t_2^T t_3^T t_4^T t_{ef}^T t_{f}^T t_{g}^T t_{h}^T t_{i}^T t_{j}^T t_{k}^T t_{l}^T t_{m}^T t_{n}^T t_{in}^T$.

Let the end vertices of $O$ be $a_k^k$ ($k = 1, \ldots, s$). As $O$ is odd-ary, $s$ is even. There exists a unique end vertex $a^k_k$ of $K_i$ that can be linked to $a^k_k$ by semipath in $O$ passing through no ramification vertices. For any sequence $i_1, \ldots, i_s$ in $[1, r]$, there exists the one and only connected induced subdigraph (tree) of $O$ with end vertices $a_k^k (k = 1, \ldots, s)$. Let us denote it by $K_{i_1\ldots i_s}$. Let $\tilde{B}_{i_1\ldots i_s}$ be a map $\text{end}(K_{i_1\ldots i_s}) \rightarrow \mathbb{Z}$ such that $\tilde{B}_{i_1\ldots i_s}(a^k_k) = B_{i_1\ldots i_s}(a^k_k)$. Consider the totality $\tilde{P}_{i_1\ldots i_s}$ of the maps $p : V(K_{i_1\ldots i_s}) \rightarrow \mathbb{Z}$ satisfying $p(a^k_k) = \tilde{B}_{i_1\ldots i_s}(a^k_k)$ ($k = 1, \ldots, s$) and $p(x) - p(y) \in \alpha(xy)$ $(xy \in E(K_{i_1\ldots i_s}))$. Now define

$$P(K_{i_1\ldots i_s}, \tilde{B}_{i_1\ldots i_s}, \alpha) = \sum_{p \in \tilde{P}_{i_1\ldots i_s}} \prod_{xy \in E(K_{i_1\ldots i_s})} Y_{p(xy)}^{xy} t^p(x) - p(y).$$

Let $S_r$ denote the set of all permutations of $\{1, \ldots, r\}$. We introduce an $s$-determinant (superdeterminant) by the formula:

$$|M_{i_1\ldots i_s}|_{s,r} := \frac{1}{r!} \sum_{\sigma_1, \ldots, \sigma_s \in S_r} \text{sgn}(\sigma_1 \ldots \sigma_s) \prod_{i=1}^r M_{\sigma_1(i), \ldots, \sigma_s(i)}.$$

It is easy to show that, for odd $s$ and $r \geq 2$, $|M_{i_1\ldots i_s}|_{s,r} = 0$. Next we assume that the maps $\alpha$ and $E$ satisfy the following.

**Assumption 1.** Let $\alpha(e) = [m_e, n_e]$ and $x_0, \ldots, x_c$ be the semipath (edges omitted) from $a^k_k$ to $a^k_k$ in $O$. Then, for all $1 \leq i < i' \leq r$, $k = 1, \ldots, s$,

$$B_i(a^k_k) - B_{i'}(a^k_k) \geq \sum_{0 \leq j \leq c-1} n_{x_j} x_{j+1} - \sum_{0 \leq j \leq c-1} m_{x_j+1} x_j.$$  

Note that this assumption is equivalent to the seemingly weaker one: "(6) holds for all $(i, i') = (1, 2), (2, 3), \ldots, (r - 1, r)$ and $k = 1, \ldots, s". Finally, we can state our main result.
Theorem 1. (Jacobi–Trudi-type identity) It holds that

\[(7) \quad g(K, B, \alpha) = |P(K_{i_1 \ldots i_s}, B_{i_1 \ldots i_s}, \alpha)|_{s,r} = |g(K_{i_1 \ldots i_s}, B_{i_1 \ldots i_s}, \alpha)|_{s,r}.\]

Remark. The tool “s-determinant” is considered as a tensor invariant. Indeed, let \(s = 2m\) and \(M\) be the transformation on a tensor space \(E^\otimes m\) of an \(r\)-dimensional linear space \(E\). For the basis \((e_1, \ldots, e_r)\), let \(M(e_{i_1} \otimes \cdots \otimes e_{i_m}) = M_{i_1 \ldots i_m}^{j_1 \ldots j_m} e_{j_1} \otimes \cdots \otimes e_{j_m}\) with Einstein’s convention. Then the \(s\)-determinant of \([M_{i_1 \ldots i_m}^{j_1 \ldots j_m}]\) depends only on \(M\) but not the choice of the basis.

2. TREE-\(r\)-PATHS AND LINDSTRÖM’S THEOREM

Here we show that there exists an odd-ary-tree-path-analogue of Lindström-Gessel–Viennot method [Lin73, GV85, GV] on which the main theorem is based.

While \(F\)-paths are dealt with for \(F = \) an odd-ary tree, the difficulty does not increase in giving general definition. If \(F\) is a digraph \(\circ\rightarrow\circ\), then an \(F\)-path is an ordinary path. In general, \(F\) should be irreducible.

An \(F\)-path in \(D\) is defined to be a pair of maps \(p = (p^*, \overline{p}); p^* : V(F) \rightarrow V(D), \overline{p} : E(F) \rightarrow \{\text{paths in } D\}\), such that \(\overline{p}(xy)\) is a path from \(p^*(x)\) to \(p^*(y)\). For \(e \in E(F)\), the \(e\)-section of \(p\) is the path \(\overline{p}(e)\). Note that a section could be a path of length \(0\), that is, a vertex. The union of all underlying vertices of all sections of \(p\) is denoted by \(v(p)\). An element of the set \(\{(x, e) \in V(D) \times E(F); \overline{p}(e) \text{ passes through } x, x \text{ is not an end of } \overline{p}(e)\} \cup \{(p^*(v), v); v \in V(F)\}\) is called a vertex of \(p\). In this sense, an \(F\)-path has no vertex repetitions. For convenience, the vertex \((x, v)\) is also written as \((x, e)\), where \(e\) is incident with \(v\). As in the case of ordinary paths, if one needs a bounded \(F\)-path, i.e. need to specify the end vertices of an \(F\)-path, one may designate the boundary map \(\tau = p^*|_{\text{end}(F)}\). The vertices \(\tau(\text{end}(F))\) are called the boundary of \(p\).

An \((F, r)\)-path is an \(r\)-tuple \((p_1, \ldots, p_r)\) of \(F\)-paths. In this case, the boundary map (if needed) is an \(r\)-tuple \((\tau_1, \ldots, \tau_r)\). A \((\circ\rightarrow\circ, r)\)-path is nothing but an \(r\)-path. An \((F, r)\)-path is called locally disjoint (loc. disj.) if, for all \(1 \leq i < j \leq r\) and \(e \in E(F)\), the \(e\)-sections of \(p_i\) and \(p_j\) have no common vertices. “An \(F\)-path locally intersects another” means that they are not locally disjoint. A disjoint \((F, r)\)-path is defined to have the disjoint sets \(v(p_1), \ldots, v(p_r)\).

Let \(F\) be a finite irreducible odd-ary tree and \(\text{end}(F) = \{a^1, \ldots, a^s\}\). Let \((b^k_i)\) \((i \in [1, r], k \in [1, s])\) be vertices of \(D\) such that \(b^k_i \neq b^k_j\) for all \(k\) and distinct \(i, j\). We denote by \(\text{PATH}_{i_1 \ldots i_s}\) the totality of \(F\)-paths in \(D\) with the boundary map \(\tau_{i_1 \ldots i_s} : a^k \mapsto b^k_{i_k} (k \in [1, s])\). Now define, for \(\sigma = (\sigma_1, \ldots, \sigma_{s-1}) \in S_s^{s-1}\), using abbreviation \(\sigma(i) = (\sigma_1(i), \ldots, \sigma_{s-1}(i))\),

\[(8) \quad \text{PATH}(\sigma) = \{(F, r)\text{-paths in } D \text{ with the boundary map } (\tau_{1,\sigma(1)}, \ldots, \tau_{r,\sigma(r)})\} ,\]

and denote by \(\text{PATH}^e(\sigma) [\text{resp. } \text{PATH}^x(\sigma)]\) the subset composed of all locally disjoint [resp. non locally disjoint] elements.

Assume \(D\) is acyclic and has finitely many bounded \(F\)-paths for each boundary map. Assign a weight \(w(e)\) to each edge of \(D\). Let the weight of an \(F\)-path be the product of those of all the underlying edges and the weight of \((F, r)\)-path the product of those of the components. The weight of a set \(Q\) of \((F, r)\)-paths is defined to be the sum of those
of all elements, which is considered as the generating function for \( Q \) denoted by \( g[Q] \).
For \( \sigma \in S_r^{s-1} \), \( \text{sgn}(\sigma) \) is defined to be the signature of the product of the components of \( \sigma \). The following is an analogue of Lindström's theorem.

**Theorem 2.** The signed generating function of loc. disj. paths is evaluated by

\[
\sum_{\sigma \in S_r^{s-1}} \text{sgn}(\sigma) g[\text{PATH}^o(\sigma)] = |g[\text{PATH}_{i_1 \ldots i_s}]|_{s,r}.
\]

**Proof.** By definition, the right-hand side is written as \( \sum_{\sigma} \text{sgn}(\sigma) g[\text{PATH}(\sigma)] \), thus it suffices to construct a weight-preserving involution \( * : \text{PATH}^x \rightarrow \text{PATH}^x \), where \( \text{PATH}^x = \coprod_{\sigma} \text{PATH}^x(\sigma) \), such that if \( p \in \text{PATH}^x(\sigma) \) and \( p^* \in \text{PATH}^x(\rho) \), then \( \text{sgn}(\sigma) = -\text{sgn}(\rho) \). For each \( F \)-path \( q \), we can construct the unique order \( <_q \) on the vertices of \( q \) as follows.

(i) The maximum element is \( (q^*_{(a^1)}, a^1) \).
(ii) The cover relation exists only between the vertices \( (x, e), (y, e) \) such that \( x, y \) are adjacent in \( \overline{q}(e) \).
(iii) The vertices with the fixed second component \( e \) are totally ordered.

Next, we fix an arbitrary total order on \( V(D) \times E(F) \) and \( \Omega = \{(i, j) \in [1, r] \times [1, r] ; i < j \} \). For given \( p \in \text{PATH}^x(\sigma) \), we can take the least local intersection \( (v, e) \in V(D) \times E(F) \). (If a local intersection of two \( F \)-paths has several distinct expressions, we promise to use the least one.) Then choose 2 components \((p_i, p_j)\) intersecting at \((v, e)\) with the least pair \((i, j) \in \Omega \). Now define \( p^* \in \text{PATH}^x(\rho) \) as follows: (i) \( p_k^* = p_k \) for all \( k \neq i, j \); (ii) the vertices of \( p_i^* \) consist of the vertices of \( p_i \) greater or equal to \((v, e)\) in the order \( <_{p_i} \) and the vertices of \( p_j \) less than \((v, e)\) in the order \( <_{p_j} \); (iii) the vertices of \( p_j^* \) consist of the vertices of \( p_j \) greater or equal to \((v, e)\) in the order \( <_{p_j} \) and the vertices of \( p_i \) less than \((v, e)\) in the order \( <_{p_i} \). Let us certify * satisfies the condition. Since \( D \) is acyclic, the components of \( p^* \) have no self-intersecting sections, and so \( p^* \) is certainly an \((F, r)\)-path contained in \( \text{PATH}^x(\rho) \). This ensures that the set of intersection vertices in each section are preserved under the operation *, and therefore * is an involution. The rest is \((#) : \text{sgn}(\sigma) = -\text{sgn}(\rho) \). By the effect of *, the end vertices of \( p_i, p_j \) corresponding to the identical \( a^k \) are replaced each other whenever \( a^k \) is opposite to \( a^1 \) with respect to the edge \( e \). Thus \( \text{sgn}(\sigma_k) = -\text{sgn}(\rho_k) \). As \( F \) is odd-ary, the number of those \( k \)'s is always odd. Hence \((#) \) holds. ⊠

### 3. The Lattice Path Method for Theorem 1

While \( O \) has already been regarded as an ordered set, we define \( O' \) by reordering with an order \( <' \), which is similar to \( <_q \). Let the vertex \( a^1 \) be the maximum element, and give cover relation between two vertices iff they are adjacent, that determines the order uniquely. The \( O' \) is naturally regarded as a digraph. Let \( F \) be the factor of \( O' \). By the assumption for \( K_i \), the \( F \) is also isomorphic to the factor of \( K'_i \) made of \( K_i \) with the order \( <' \). To give a proof of Theorem 1, we construct a bijection between \( \text{Tab}(K, B, \alpha) \) and a set of bounded \((F, r)\)-paths in a certain acyclic digraph \( D \) without
multiedges. Now define \( D \) by
\[
V(D) = V(O') \times \mathbb{Z},
\]
\[
E(D) = \{(x, i)(y, j); \ xy \in E(O'), \ i - j \in \alpha(xy) \ (xy \in E), \ j - i \in \alpha(yx) \ (yx \in E)\}.
\]

Next let \( b^k_i = (a^k_i, B_i(a^k_i)) \ (k \in [1, s], \ i \in [1, r]) \). Take the boundary map \( \tau = (\tau_1, \ldots, \tau_r), \ \tau_i : \text{end}(F) \to V(D) \), defined by \( \tau_i(a^k_i) = b^k_i \). Since \( K'_i \) is a tree, one sees that a bounded \( F \)-path \( p_i \) in \( D \) with \( \tau_i \) is nothing else than the map \( (p_i) : V(K'_i) \to \mathbb{Z} \) defined by \( (x, (p_i)(x)) \in v(p_i) \). We denote by \( \text{PATH}_r^\tau \) the totality of bounded \( (F, r) \)-paths \( p = (p_1, \ldots, p_r) \) in \( D \) with \( \tau \) such that, for all \( i < i' \), \( (x, j) \in v(p_i) \) and \( (x, j') \in v(p_{i'}) \) imply \( j > j' \), which means intuitively that they are assumed to be disjoint and have no edge-intersection.

**Lemma 1.** There exists a bijection \( \phi : \text{Tab}(K, B, \alpha) \to \text{PATH}^\tau_r : T \mapsto p \) defined by \((p_i)(x) = T_i(x) \ (x \in V(K_i), \ i \in [1, r])\).

**Proof.** We may give the inverse \( \phi^{-1} : p \mapsto T \) by \( T_{ij} = \{x \in V(K_i); (p_i)(x) + i < j\} \ ((i, j) \in [1, r] \times \mathbb{Z}) \). By definition (10), we see that this \( T_{ij} \) is an order ideal of \( K_i \). Now what should be proved is (i): \( \phi(\text{Tab}(K, B, \alpha)) \subseteq \text{PATH}^\tau_r \) and (ii): \( \phi^{-1}(\text{PATH}^\tau_r) \subseteq \text{Tab}(K, B, \alpha) \). In a proof of (i), the rest of (a): “For all \( i < i' \), \( (x, j) \in v(p_i) \) and \( (x, j') \in v(p_{i'}) \) imply \( j > j' \)” is clear. Similarly, to show (ii), we only need to see (b): \( T_{ij} \cap K_{i'} \subseteq T_{i', j + i' - i - 1} \cap K_i \ (i < i') \). The are deduced from the equivalence:

\[
(a) \iff T_i(x) - 1 \geq T_{i'}(x) \ (i < i', \ x \in V(K_i) \cap V(K_{i'}))
\]

\[
\iff \min\{j; x \in T_{ij}\} \geq \min\{j; x \in T_{i'j}\} - i' + i + 1 \iff (b) \quad \square
\]

**Proof of Theorem 1.** Let the weight \( Y_{ij}^{xy} t_{xy}^{i-j} \) be given to each edge \((x, i)(y, j)\) of \( D \). Apply Theorem 2 for the above-mentioned \( F, D \) and the boundary maps \( \tau_{i_1 \ldots i_s} : a^k \mapsto b^k_i \ (k \in [1, s]) \). From the property of \( D \) and \( (b^k_i) \), it follows that \( \text{PATH}^\sigma(\sigma) \) on the left-hand side of (9) may be replaced with the subset \( \text{PATH}^\sigma(\sigma) \) consisting of all disjoint \( (F, r) \)-paths. Then we call this (9)'.

For each element \( p \in \text{PATH}^\tau_r \) and \( x \in V(O') \), let \( p^+(x) \) [resp. \( p^-(x) \)] denote the sequence \((x, (p_1)(x)), \ldots, (x, (p_r)(x))\), where the \( i \)th terms with \( x \notin V(K_i) - \{a_i^2, \ldots, a_i^s\} \) are omitted. Note that, for \( xy \in E(O') \), \( |p^+(x)| = |p^-(y)| \). The cardinality is denoted by \( \kappa(xy) \). For \( \rho \in S_t \), and vertices \( x_1, \ldots, x_t \), set \( \rho(x_1, \ldots, x_t) = (x_{\rho(1)}, \ldots, x_{\rho(t)}) \). Define \( D^\rho(p) = \) the induced subdigraph of \( D \) with the vertices \( p^+(x) \coprod p^-(y) \), and \( \text{PATH}^\sigma(xy, \rho, p) = \) the set of all vertex-disjoint \( \kappa(xy) \)-paths from \( p^+(x) \) to \( p^-(y) \) in \( D^\rho(p) \). By the definition of the boundary map \( \tau \), Assumption 1 assures that for all \( p \in \text{PATH}^\sigma(\sigma), \ i'' < i < i' \) and \( k \), we have \((p_{i''})(a^k_i) > (p_i)(a^k_i) > (p_{i'})(a^k_i) \) (for the defined left and/or right-hand side). This enables us to have the weight preserving bijection:

\[
b : \coprod_{\sigma \in S_t}^{\sigma} \text{PATH}^\sigma(\sigma) \to \coprod_{p \in \text{PATH}^\tau_r} \coprod_{e \in E(O')} \coprod_{\rho \in S} \text{PATH}^\sigma(e, \rho, p).
\]
Since $F$ is odd-ary, the signs of the corresponding terms on both sides of (11) coincide. Thus, taking the weights with signs of both sides and combine it with (9)', we obtain Theorem 1.

4. Specialization of the weights

In Theorem 1, rather complicated determinantal weights creep into the formula, while most Jacobi–Trudi identities are more simple. The reason is that Theorem 1 never imposes strong conditions such as "row-strict", "column-strict", etc. on the ideal-tableaux. Here we intend to simplify the formula. First of all, we define the $e$-shape of an ideal-tableau $T$ for each $e \in E$. Set $j(e) = \{i \in [1, r] ; e \in E(K)_i\}_\prec$. For $e = v_+v_-$, define $(T_i(v_\pm))_{i \in j(e)} = (T^{e\pm}_i - i), i \in [1, j(e)]$.

By Lemma 1, for $i < j$ such that $V(K_i), V(K_j) \ni x$, $T_i(x) > T_j(x)$. So $(T^{e\pm}_i)$ decrease weakly, and one sees $T^{e\pm}_i \geq T^{e-}_i$. Now let $T^e$ denote the diagram in $[1, r] \times \mathbb{Z}$: $\{(i, j) ; T^{e-}_i < j \leq T^{e+}_i\}$. It is called the $e$-shape of $T$. If we drag it along the $j$-axis until it enters the right-hand side of $i$-axis, it becomes a skew diagram $\lambda \setminus \mu$. Then we use the notations $s(T^e)$ and $s(T'^e)$ for the skew S-functions $s_{\lambda/\mu}$ and $s_{\lambda'/\mu'}$, respectively.

Returning to Theorem 1, divide $E$ into $L, M$. Suppose $\alpha(e) = [0, n]$ for all $e \in L$ and $\alpha(e) = N$ for all $e \in M$. Let $e_d$ and $h_d$ denote the elementary and the complete symmetric functions, respectively. Now set $Y_j^e = e_{i-j}(x_1, \ldots, x_n)$ when $e \in L$, and $Y_j^e = h_{i-j}(x_1, \ldots, x_n)$, otherwise. Then we immediately see that the determinantal weight of $T$ is written as $\prod_{e \in L} s(T^e)(x) \cdot \prod_{e \in M} s(T^e)(x)$. Next we define a set of $(L, M)$-semistandard ideal-tableaux of trees and a certain function of $t = (t_e)_{e \in E}$.

\begin{equation}
\text{SST}_{LM}(K, B) = \{ T : \text{ideal-tableaux of } K ; T_i(x) = B_i(x) (x \in \text{end}(K_i)), \ i \in [1, r], \ T^e : \text{vertical [resp. horizontal] strip } (e \in L \ [\text{resp. } M]) \},
\end{equation}

\begin{equation}
P_{LM}(K_{i_1 \ldots i_s}, \bar{B}_{i_1 \ldots i_s})(t) = \left[ P(K_{i_1 \ldots i_s}, \bar{B}_{i_1 \ldots i_s}, \alpha) \right]_{Y_j^e = \epsilon(i, j, e)}.
\end{equation}

Here, $\epsilon(i, j, e)$ is defined to be 1 whenever $e \in M$ or $i - j \in [0, 1]$, and to be 0, otherwise.

By putting $x_1 = 1$ and $x_2 = x_3 = \cdots = 0$, (7) becomes a simple formula, which turns into the one for $(L, M)$-partially strict tableaux with bounded entries in each row, when $K$ is an $r$-tuple of chains [Oka90, Wac85].

Corollary 1. The power weight sum of semistandard ideal-tableaux of trees is expressed as

\begin{equation}
\sum_{T \in \text{SST}_{LM}(K, B)} t^T = \left| P_{LM}(K_{i_1 \ldots i_s}, \bar{B}_{i_1 \ldots i_s})(t) \right|_{s, r}.
\end{equation}

5. Superpfaffians for locally disjoint tree-\(g\)-paths

Okada gave a remarkable pfaffian formula for the minor sum of a matrix [Oka89], and Stembridge developed a useful technique for calculation of the weights of (partially) unbounded vertex-disjoint $r$-paths with pfaffians [Ste90]. Lindström's theorem shows a strong connection between them. It is also known that a symmetric analogue of Okada's result exists. In this section, we generalize those theories on tree-\(g\)-paths.
We introduce $(\lambda, n)$-pfaffians (superpfaffians). Let $g, n$ be positive integers and $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $g$. The multiplicity of the $i$-parts in $\lambda$ is denoted by $m(i)$, say, $\lambda = (1^{m(1)}, 2^{m(2)}, \ldots)$ in increasing order. We set $g_i = \lambda_1 + \cdots + \lambda_i$ for all $i = 1, \ldots, r$, and $g_0 = 0$. Let $G_\lambda$ denote the set of permutations $\rho$ of $\{1, \ldots, g\}$ satisfying $\rho(g_{i-1} + 1) < \rho(g_{i-1} + 2) < \cdots < \rho(g_i)$ ($i = 1, \ldots, r$), and $F_\lambda$ denote the subset of $G_\lambda$ consisting of $\rho$ such that $\rho(g_{i-1} + 1) < \rho(g_i + 1)$ whenever $\lambda_i = \lambda_{i+1}$. Define

$$
\text{pf}_{\lambda, n} \left[ M_{i_1 \ldots i_{pn}}|_{1 \leq i_{p+1} < \cdots < i_{p+q} \leq g} \right]_{(k=0, \ldots, n-1)} = \frac{1}{m!} \sum_{\sigma \in G_\lambda} \text{sgn}(\sigma) P_{\sigma}
$$

where $m! = m(1)!m(2)\ldots$. Let $d$ be the number of distinct parts of $\lambda$. By definition, the array on the left-hand side is a $d$-tuple of different dimensional arrays. For $\lambda = (2^r)$, $n = 1$, the above expression is led to an ordinary pfaffian for $2r \times 2r$ skew-symmetric matrix; while for $\lambda = (1^r)$, $n = s$ — an $s$-determinant. Furthermore, for odd $n$ and $\lambda$ such that $m(i) > 1$ for some odd $i$, that vanishes.

For example, take $\lambda = (2, 1)$ and $n = 2$. We have

$$
\text{pf}_{(2,1),2} \left[ M_{i_1 \ldots i_4}|_{1 \leq i_1 < i_2 \leq 3, 1 \leq i_3 < i_4 \leq 3} \right] = M_{1212}M_{33} - M_{1213}M_{32} + M_{1223}M_{31} - M_{1312}M_{23} + M_{1313}M_{22} - M_{1323}M_{21} + M_{2313}M_{12} - M_{2313}M_{12} + M_{2333}M_{11}.
$$

As in §2, we assume that $F$ is a finite irreducible odd-ary tree with the end vertices $\{a^1, \ldots, a^s\}$ (s: even), and $D$ is an acyclic digraph with finitely many $F$-paths for each boundary map. Next let $\lambda$ be chosen so that $m(i) \leq 1$ for all odd $i$. Let $V^1$ be an arbitrary finite set of at least $g$ vertices of $D$; and $V^2, \ldots, V^s$ ones of $g$ vertices. Assume for each $k \in [1, s]$, that $V^k$ is totally ordered irrespective of the structure of $D$ and the other $V^l$.

**Assumption 2.** All $F$-paths $p, q$ satisfying that $p^*(a^k) < q^*(a^k)$ in $V^k$ and $p^*(a^l) > q^*(a^l)$ in $V^l$ for some $k, l \in [1, s]$ intersect locally.

Let us fix a set $A$ of subsets of $V^1$ which contains at least $m(i)$ disjoint $i$-subsets whenever $m(i) > 0$, and no $i$-subsets otherwise. Let $I$ be a subset of $V^1$. For every $k \in [2, s]$, denote by $v^k = (v^k_1, \ldots, v^k_g)$, an arbitrary arrangement of all elements of $V^k$.

Now define

$$
\text{PATH}^g(I, v^2, \ldots, v^s) = \{ p : (F, g) \text{-paths in } D; \{ p^*_1(a^1), \ldots, p^*_g(a^1) \} = I, \}
$$

$$
p^*_i(a^k) = v^k_i \quad ((i, k) \in [1, g] \times [2, s]),
$$

and $\text{PATH}^g_\lambda(I, v) = \text{PATH}^g_\lambda(I, v^2, \ldots, v^s)$ to be the subset which contains exactly all locally disjoint elements as usual. For $\rho \in S_g$, set $\rho(v^k) = (v^k_{\rho(1)}, \ldots, v^k_{\rho(g)})$. Let $I = \{v^1_1, \ldots, v^1_g\}$ and assume that $v^k$ is ordered increasingly for each $k$. The $\lambda$-generating function $\epsilon(I) \cdot g[\text{PATH}^g_\lambda(I, v)]$ for $\text{PATH}^g_\lambda(I, v)$ is defined as the product: $\epsilon(I) \cdot g[\text{PATH}^g_\lambda(I, v)]$, 

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\[\epsilon(I) = \sum \text{sgn}(\rho)\]; where the summation runs over all \(\rho \in \mathcal{F}_\lambda\) such that, for every \(i \in [1, r]\), \(\{v^1_{\rho(i)}\}_{j \in [g_{i-1}+1, g_i]} \) belongs to \(\mathcal{A}\).

Next, we set \(\text{PATH}_g(v^2, \ldots, v^s) = \prod_{I \subset V^1} \text{PATH}_g(I, v)\) and consider the subset consisting of all locally disjoint elements: \(\text{PATH}_g^\circ(v^2, \ldots, v^s) = \prod_{I \subset V^1} \text{PATH}_g^\circ(I, v)\). Define \(g_\lambda[\text{PATH}_g^\circ(v^2, \ldots, v^s)] = \sum_I g_\lambda[\text{PATH}_g^\circ(I, v)]\).

**Theorem 3.** The \(\lambda\)-generating function is expressed by a superpfaffian, say,

\[
(17) \quad g_\lambda[\text{PATH}_g^\circ(v^2, \ldots, v^s)] = \left[ \prod_{i=1}^{\lambda} \text{pf}_g^\lambda \left( \text{PATH}_g^\circ((v^2_{I_1}, \ldots, v^2_{I_i}), \ldots, (v^s_{I_1}, \ldots, v^s_{I_s})) \right) \right]_{1 \leq i_1 < \ldots < i_p \leq g, \ldots, 1 \leq i_1 < \ldots < i_p \leq g, \ldots, p \in \{1, \ldots, \lambda \}}.
\]

**Proof.** For \(\sigma = (\sigma_2, \ldots, \sigma_s) \in G^{s-1}_\lambda\), we use the notation: \(\text{PATH}_g^\circ(\sigma(v)) = (\sigma_2(v^2), \ldots, \sigma_s(v^s))\). We put \(\text{PATH}_g^\lambda(\sigma(v)) = \text{PATH}_g(\sigma(v)) - \text{PATH}_g^\circ(\sigma(v))\) and set

\[
(18) \quad \text{PATH}_g^\lambda(\sigma(v)) = \{ p \in \text{PATH}_g(\sigma(v)) \mid (p_{g_{i-1}+1}, \ldots, p_{g_i}) \text{ is locally disjoint and } \{p^\circ_{g_{i-1}+1}(a^1), \ldots, p^\circ_{g_i}(a^1)\} \in \mathcal{A} \text{ for all } i \in [1, r] \}.
\]

By (15), we may translate the pfaffian (multiplied by \(m!\)) on the right-hand side of (17) to the signed weight of \((F, g)\)-paths \(p\) such that (i): for all \(i \in [1, r]\), \(\tilde{p}_i = (p_{g_{i-1}+1}, \ldots, p_{g_i})\) is locally disjoint, (ii): the components of \(\tilde{p}_i\) are arranged so that the boundaries corresponding to \(a^k\) are increasing for each \(k \in [1, s]\), (iii): the boundaries of \(\tilde{p}_i\) corresponding to \(a^1\) form an element of \(\mathcal{A}\), and (iv): the boundaries of \(p\) corresponding to \(a^k\) form \(V^k\) for all \(k \in [2, s]\). If \(p\) is locally disjoint, Assumption 2 implies that there exists \(\nu \in G_\lambda\) such that for every \(k \in [1, s]\), \(p^\circ_{\nu^{-1}(1)}(a^k) < \cdots < p^\circ_{\nu^{-1}(s)}(a^k)\). Thus, the same weights, except signs, are arising from \((F, g)\)-paths \(\{(q^\circ_{\rho(1)}, \ldots, q^\circ_{\rho(s)})\} \) where \(q = \nu^{-1}(p)\) and \(\rho\) runs over all permutations in \(G_\lambda\) such that \(\{q^\circ_{\rho(g_{i-1}+1)}(a^1), \ldots, q^\circ_{\rho(g_i)}(a^1)\} \in \mathcal{A}\) for all \(i \in [1, r]\). From this, it follows that the weight of locally disjoint \((F, g)\)-paths appearing in the pfaffian is equal to the left-hand side of (17). Therefore, dividing by \(m!\), the right-hand side of (17) is written as

\[
(19) \quad g_\lambda[\text{PATH}_g^\circ(v^2, \ldots, v^s)] = \frac{1}{m!} \sum_{\sigma \in G^{s-1}_\lambda} \text{sgn}(\sigma) g[\text{PATH}_g^\lambda(\sigma(v))].
\]

So we prove that the second term of (19) vanishes. To do this, as in the proof of Theorem 2, we take an involution \(*\) on \(\text{PATH}_g^\lambda = \prod_{\sigma \in G^{s-1}_\lambda} \text{PATH}_g^\lambda(\sigma(v))\) such that \(w(p^*) = w(p), \text{sgn}(\rho) = -\text{sgn}(\sigma) (p \in \text{PATH}_g^\lambda(\sigma(v)), p^* \in \text{PATH}_g^\lambda(\rho(v)))\). For this involution, we can use a slight deformation of \(*\) in the proof of Theorem 2. The modified point is to choose the least local intersection \((v, e) \in V(D) \times E(F)\) such that each component \(p_i\) of \(p\) with local intersection \((v, e)\) has no local intersection less than \((v, e)\) with respect to the order \(<_{p_i}\). In virtue of this, locally disjointness of \(\tilde{p}_i\) (i) is preserved by this deformed \(*\), and therefore (ii) is also satisfied (Assumption 2). The rest (iii),(iv) are preserved clearly. Hence \(\text{PATH}_g^\lambda\) is \(*\)-invariant. We can confirm the other properties of \(*\) as in the proof of Theorem 2. \(\Box\)
Remarks. Depending on the structure of $\mathcal{A}$, Theorem 3 gives various weight-sums of loc. disj. tree-$g$-paths. For example, let $v_1^1 < \cdots < v_{2n}^1$ be all vertices in $V^1$ and set $\mathcal{A} = \{\{v_1^1, v_{2n}^1\}, \{v_2^1, v_{2n-1}^1\}, \ldots, \{v_n^1, v_{n+1}^1\}\}$. Let $\lambda = (2^r)$, $g = 2r$. The left-hand side of (17) becomes the "symmetric" sum: $\sum_I g[PATH_g^0(I, v)]$, where $I$ runs over all $g$-subsets of $V^1$ such that $v_k^1 \in I \Rightarrow v_{g-k+1}^1 \in I$. Similarly, for a given $\lambda$ in Theorem 3, let $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_r) = (1^{m(1)}, 2^{m(2)}, \ldots)$ be a partition of $n = |V^1|$ such that, for all nonzero $\tilde{m}(i)$, $\tilde{m}(i) \geq m(i) \geq 1$. Set $\tilde{g}_i = \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_i$. Let $\mathcal{A}$ be a partition of $V^1 = \{v_1^1, \ldots, v_n^1\}$ of type $\tilde{\lambda}$ consisting of the cells $\{v_{\tilde{g}_i-1+1}^1, \ldots, v_{\tilde{g}_i}^1\}$ ($i = 1, \ldots, \tilde{r}$). Now Theorem 3 gives the weight-sum of loc. disj. tree-$g$-paths $p$ with coefficients $\epsilon(I) = 1$, where the set of boundaries $\{p_1^1(a^1), \ldots, p_g^1(a^1)\}$ corresponds to the collection of the cells of $\mathcal{A}$ consisting of $m(i)$ $i$-cells.

Another example is an ordinary summation formula, which is the most natural. Let $\lambda = (2^r)$ and $\mathcal{A} = \{\text{all 2-subsets of } V^1\}$. This case enumerate the sum of all weights $\sum_I g[PATH_g^0(I, v)]$ with coefficients $= 1$. In general, let $\lambda$ be a partition with no odd parts, and $\mathcal{A} = \{\text{all } i\text{-subsets of } V^1; m(i) \geq 1\}$. In that case we can show by induction that $\epsilon(I) = \frac{(g/2)!}{(m(2))(2m(4))(3m(6))\ldots}\prod_{i: \text{even}}\prod_{j=1}^{m(4)}(\frac{(i/2)j-1}{i/2-1})$ irrespective of $I$. Thus, the case also gives the weight-sum of all loc. disj. tree-$g$-paths.

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