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Harmonic Relations between Green's Functions and Green's Matrices for Boundary Value Problems

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1 Introduction

A principal purpose of numerical analysis is to project continuous problems onto finite dimensional spaces and attack the original problems by analyzing the discretized systems. Although infinite dimensional spaces are quite different from finite dimensional spaces, there is a beautiful harmony between a continuous problem and the discretized one.

The purpose of this paper is to show this through boundary value problems of Sturm-Liouville type and their discretizations by finite difference methods (FDMs). It is also shown that the FDMs work well even for the case where local truncation errors diverge as the mesh-size h tends to zero.

First, in §2, we present some new results on inversion of tridiagonal matrices. Next, in §3, relations between Green's functions and Green's matrices are described for continuous and discrete boundary value problems. Furthermore, in §4, an example is given such that even if the truncation error does not converge to zero as h approaches to zero, the centered finite difference method does converge to the true solution as $h \to 0$. This suggests us a robustness of FDM. Finally, on the basis of the result in §4, a mesh refinement technique for the Shortley-Weller (S-W) method is proposed in §5, which improves the results obtained by the centered three-point formula. In fact, an application of our technique to the example in §4 yields numerical results with $\mathrm{O}(h^2 \log \frac{1}{h})$ accuracy, while the usual centered formula gives $\mathrm{O}(h^{\frac{1}{2}})$ accuracy.

2 Inversion of Tridiagonal Matrices

We consider an $n \times n$ nonsingular tridiagonal matrix A of the form

$$A = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix}, \quad a_i, c_i \neq 0 \quad \forall i.$$

Concerning the inverse of A, the following is known:

Theorem 2.1 (Yamamoto-Ikebe [13]). Define the two sequence $\{u_m\}$, $\{v_m\}$ as follows:

$$u_0 = 0,\ u_1 = h_1,\ u_m = -\frac{1}{c_{m-1}}(a_{m-1}u_{m-2} + b_{m-1}u_{m-1}) \quad (m \geq 2), \quad (2.1)$$

$$v_{n+1} = 0,\ v_n = h_2,\ v_m = -\frac{1}{a_{m+1}}(b_{m+1}v_{m+1} + c_{m+1}v_{m+2}) \quad (m \leq n-1), \quad (2.2)$$

where h_1, h_2, a_1 and c_n may be chosen arbitrarily, but may not be zero. Then $A^{-1} = (\alpha_{ij})$ is given by

$$\alpha_{ij} = \begin{cases} -\frac{u_i v_j}{a_1 h_1 v_0} \prod_{k=2}^{\prime j} \frac{c_{k-1}}{a_k} & (i \leq j) \\ -\frac{u_j v_i}{a_1 h_1 v_0} \prod_{k=2}^{\prime j} \frac{c_{k-1}}{a_k} & (i \geq j), \end{cases} \quad (2.3)$$

where $\prod'$ is understood to be

$$\prod_{k=2}^{j}{}' \frac{c_{k-1}}{a_k} = \begin{cases} \prod_{k=2}^{j} \frac{c_{k-1}}{a_k} & (j \geq 2) \\ 1 & (j = 1). \end{cases}$$

In [13], this theorem has been proved as a corollary of the more general inversion formula for a nonsingular (q,p)-type band matrix $A = (a_{ij})$ with $a_{ij} = 0$ if $i > j + q$ or $j > i + p$ and $a_{ij} \neq 0$ if $i - j = q$ or $j - i = p$.

We can also prove Theorem 2.1 by directly verifying

$$\begin{gathered} b_1\alpha_{1j} + c_1\alpha_{2j} = \delta_{1j}, \\ a_i\alpha_{i-1j} + b_i\alpha_{ij} + c_i\alpha_{i+1j} = \delta_{ij}, \quad 2 \leq i \leq n-1 \\ a_n\alpha_{n-1j} + b_n\alpha_{nj} = \delta_{nj}. \end{gathered}$$

Furthermore, Theorem 2.1 includes the following results as a special case:

Theorem 2.2 (Gantmakher-Krein [4], Bukhberger-Emel'yanenko [3]). If A is a nonsingular symmetric and irreducible tridiagonal matrix, then there exist two column vectors $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ such that

$$A^{-1} = \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & & \vdots \\ u_1v_n & u_2v_n & \cdots & u_nv_n \end{pmatrix} = \left(u_{\min(i,j)}v_{\max(i,j)}\right). \tag{2.4}$$

Conversely, if $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ are given and a matrix $B = \left(u_{\min(i,j)}v_{\max(i,j)}\right)$ is nonsingular, then B^{-1} is a symmetric tridiagonal matrix with nonzero superdiagonal elements.

Theorem 2.3 (Ikebe [5]). Under the assumptions of Theorem 2.1, four column vectors $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i)$, $\boldsymbol{x} = (x_i)$ and $\boldsymbol{y} = (y_i)$, $i = 1, 2, \ldots, n$ exist such that $A^{-1} = (\alpha_{ij})$ is given by

$$\alpha_{ij} = \begin{cases} u_iv_j & (i \le j) \\ x_jy_i & (i \ge j) \end{cases}, \quad u_iv_i = x_iy_i$$

Proof. In Theorem 2.1, put

$$U_i = u_i, \quad V_j = -\frac{1}{a_1h_1v_0}\prod_{k=2}^{j}{}' \frac{c_{k-1}}{a_k},$$

$$X_j = -\frac{u_j}{a_1h_1v_0}\prod_{k=2}^{j}{}' \frac{c_{k-1}}{a_k}, \quad Y_i = v_i.$$

Then $\boldsymbol{U} = (U_i)$, $\boldsymbol{V} = (V_i)$, $\boldsymbol{X} = (X_i)$ and $\boldsymbol{Y} = (Y_i)$ satisfy the conditions for u, v, x and y in Theorem 2.3. $\square$

On the basis of Theorem 2.1, we now state the following result which appears to be new.

Theorem 2.4. Under the assumptions of Theorem 2.1, there exist three column vectors $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i)$ and $\boldsymbol{d} = (d_i)$ such that

$$A^{-1} = \begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & & \vdots \\ u_1v_n & u_2v_n & \cdots & u_nv_n \end{pmatrix} \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = GD, \tag{2.5}$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $G = (g_{ij})$ with $g_{ij} = u_{\min(i,j)}v_{\max(i,j)}$. If we add a normalization condition $d_1 = 1$, then G and D are uniquely determined and A is symmetric if and only if $D = I$.

Proof. A proof of this result will be given in [11]. □

Remark 2.1. A matrix $R = (r_{ij})$ is called a Green's matrix (cf. [2]) if there exist number $a_1, \dots, a_n$, $b_1, \dots, b_n$ such that

$$r_{ij} = a_{\min(i,j)} b_{\max(i,j)} = \begin{cases} a_i b_j & (i \le j) \\ b_i a_j & (i \ge j). \end{cases}$$

Hence the matrix G in (2.5) is a Green's matrix. We shall call the expression (2.5) a GD decomposition of the inverse of a nonsingular irreducible tridiagonal matrix A. The expression (2.5) with $d_1 = 1$ may be called the normalized GD decomposition of the inverse of a nonsingular irreducible tridiagonal matrix A.

Now, for certain tridiagonal matrices, we can derive the explicit formula for A^{-1}. Only two results are presented here without proofs.

Theorem 2.5. Let

$$A = \begin{pmatrix} a_1 + b_1 & -b_1 & & & \\ -a_2 & a_2 + b_2 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -b_{n-1} \\ & & & -a_n & a_n + b_n \end{pmatrix}, \quad a_i, b_i \ne 0 \quad \forall i. \tag{2.6}$$

Then $A^{-1} = (\alpha_{ij})$ is given by

$$\alpha_{ij} = \begin{cases} \frac{1}{w_j} \sum_{\lambda=1}^{i} c_\lambda^{-1} \sum_{\mu=j+1}^{n+1} c_\mu^{-1} & (i \le j) \\ \frac{1}{w_j} \sum_{\lambda=1}^{j} c_\lambda^{-1} \sum_{\mu=i+1}^{n+1} c_\mu^{-1} & (i \ge j), \end{cases} \tag{2.7}$$

where

$$c_1 = a_1, \quad c_\lambda = a_1 \prod_{i=1}^{\lambda-1} \frac{b_i}{a_i} = b_{\lambda-1} \prod_{i=2}^{\lambda-1}{}' \frac{b_{i-1}}{a_i} \ (\lambda \ge 2), \quad w_j = \frac{a_j}{c_j} \sum_{\lambda=1}^{n+1} c_\lambda^{-1}.$$

Theorem 2.6. Let

$$A = \begin{pmatrix} a_1 + b_1 & -b_1 & & & \\ -a_2 & a_2 + b_2 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_n & a_n + b_n & -b_n \\ & & & -(a_{n+1} + b_{n+1}) & a_{n+1} + b_{n+1} \end{pmatrix}, \quad a_i, b_i \ne 0 \quad \forall i, \tag{2.8}$$

and c_λ, w_j be defined as in Theorem 2.5. Then

$$A^{-1} = \begin{pmatrix} z_1 & z_1 & \cdots & \cdots & z_1 \\ z_1 & z_2 & \cdots & \cdots & z_2 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & z_n & z_n \\ z_1 & z_2 & \cdots & z_n & z_{n+1} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \frac{z_{n+1}}{w_2} & & & \\ & & \ddots & & \\ & & & \frac{z_{n+1}}{w_n} & \\ & & & & \frac{b_n}{a_{n+1}+b_{n+1}} \frac{z_{n+1}}{w_n} \end{pmatrix}, \tag{2.9}$$

where

$$z_i = \sum_{\lambda}^{i} c_\lambda^{-1}, \quad i = 1, 2, \ldots, n+1.$$

The right-hand side of (2.9) gives a normalized GD decomposition of A^{-1}.

Proofs of Theorems 2.5 and 2.6 will be given in [11].

3 Green's Functions and Green's Matrices

Consider the Sturm-Liouville equation

$$L[u] = -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x) \quad a < x < b, \tag{3.1}$$

subject to the boundary conditions

$$B_a[u] = \alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{3.2}$$
$$B_b[u] = \beta_1 u(b) + \beta_2 u'(b) = 0, \tag{3.3}$$

where $p \in C^1[a,b]$, $p(x) > 0$, $q, f \in C[a,b]$, $q(x) \geq 0$ and, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants satisfying $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$. Then, as is well known, the Green function $G(x,\xi)$ for the operator

$$-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) : \mathscr{D} = \{u \in C^2[a,b] \mid B_a[u] = B_b[u] = 0\} \to C[a,b]$$

is constructed as follows: Let $\bar{u}(x)$ and $\bar{v}(x)$ be the solutions of the initial value problems

$$L[u] = 0; \quad u(a) = \alpha_2, \ u'(a) = -\alpha_1, \tag{3.4}$$

and

$$L[v] = 0; \quad v(b) = \beta_2, \ v'(b) = -\beta_1, \tag{3.5}$$

respectively. Then $\bar{u}(x)$ and $\bar{v}(x)$ are linearly independent over $[a, b]$ and

$$G(x,\xi) = \begin{cases} -\frac{\bar{u}(x)\bar{v}(\xi)}{p(\xi)W(\xi;\bar{u},\bar{v})} & (a \le x \le \xi) \\ -\frac{\bar{v}(x)\bar{u}(\xi)}{p(\xi)W(\xi;\bar{u},\bar{v})} & (\xi \le x \le b) \end{cases}$$
$$= \begin{cases} \frac{1}{c}\bar{u}(x)\bar{v}(\xi) & (a \le x \le \xi) \\ \frac{1}{c}\bar{v}(x)\bar{u}(\xi) & (\xi \le x \le b) \end{cases}$$

where $W(\xi;\bar{u},\bar{v})$ denotes the Wronskian determinant of $\bar{u}$ and $\bar{v}$ and $c = -p(\xi)W(\xi;\bar{u},\bar{v})$. Observe that c is a nonzero constant independent of ξ. If we discretize (3.1)–(3.3) by the usual difference formula:

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b,$$
$$x_{i-\frac{1}{2}} = \frac{1}{2}(x_{i-1} + x_i), \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n+1,$$

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)_{x=x_j} \fallingdotseq -\left[p_{i+\frac{1}{2}}\frac{(u_{i+1} - u_i)}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{(u_i - u_{i-1})}{h_i}\right] \Big/ \left(\frac{h_i + h_{i+1}}{2}\right), \tag{3.6}$$

where $p_{i+\frac{1}{2}} = p(x_{i+\frac{1}{2}})$, then the resulting matrix is not necessarily symmetric tridiagonal. The iterations (2.1) and (2.2), essentially due to Bukhberger-Emel'yanenko [3], correspond to (3.4) and (3.5), respectively. In particular, if we consider the case $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 0$, then the discretized system is

$$(DA + Q)\boldsymbol{U} = \boldsymbol{f},$$

where

$$A = \begin{pmatrix} b_1 + b_2 & -b_2 & & & \\ -b_2 & b_2 + b_3 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -b_n \\ & & & -b_n & b_n + b_{n+1} \end{pmatrix}, \quad b_i = \frac{p_{i-\frac{1}{2}}}{h_i}$$

$$D = \begin{pmatrix} \frac{2}{h_1+h_2} & & \\ & \ddots & \\ & & \frac{2}{h_n+h_{n+1}} \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix},$$
$$\boldsymbol{U} = (U_1, \ldots, U_n)^t, \quad \boldsymbol{f} = (f_1, \ldots, f_n)^t.$$

Hence, if we denote by τ_i the local truncation error of the approximation

$$-\left[\frac{p_{i+\frac{1}{2}}\frac{(u_{i+1}-u_i)}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{(u_i-u_{i-1})}{h_i}}{\frac{h_i+h_{i+1}}{2}}\right] + q_i u_i = f_i$$

at x_i, then

$$(DA+Q)(\boldsymbol{u}-\boldsymbol{U}) = \boldsymbol{\tau}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^t$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)^t$. It follows that

$$|\boldsymbol{u}-\boldsymbol{U}| \le (DA+Q)^{-1}|\boldsymbol{\tau}| \le (DA)^{-1}|\boldsymbol{\tau}| = A^{-1}D^{-1}|\boldsymbol{\tau}|$$

since DA is an M-matrix and Q is nonnegative diagonal, where $|\boldsymbol{u}-\boldsymbol{U}| = (|u_1-U_1|, \ldots, |u_n-U_n|)^t$ and $|\boldsymbol{\tau}| = (|\tau_1|, \ldots, |\tau_n|)^t$. An application of Theorem 2.5 then yields

$$\begin{aligned}
A^{-1} &= (g_{ij}),\\
g_{ij} &= \begin{cases} \left(\sum_{\lambda=1}^{n+1} \frac{h_\lambda}{p_{\lambda-\frac{1}{2}}}\right)^{-1} \left(\sum_{\mu=1}^{i} \frac{h_\mu}{p_{\mu-\frac{1}{2}}}\right) \left(\sum_{\nu=j+1}^{n+1} \frac{h_\nu}{p_{\nu-\frac{1}{2}}}\right) & (i \le j) \\ \left(\sum_{\lambda=1}^{n+1} \frac{h_\lambda}{p_{\lambda-\frac{1}{2}}}\right)^{-1} \left(\sum_{\mu=1}^{j} \frac{h_\mu}{p_{\mu-\frac{1}{2}}}\right) \left(\sum_{\nu=i+1}^{n+1} \frac{h_\nu}{p_{\nu-\frac{1}{2}}}\right) & (i \ge j) \end{cases} \qquad (3.7)\\
&\fallingdotseq \begin{cases} \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \left(\int_a^{x_i} \frac{ds}{p(s)}\right) \left(\int_{x_j}^b \frac{ds}{p(s)}\right) & (i \le j) \\ \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \left(\int_a^{x_j} \frac{ds}{p(s)}\right) \left(\int_{x_i}^b \frac{ds}{p(s)}\right) & (i \ge j) \end{cases}\\
&= G(x_i, x_j),
\end{aligned}$$

where $G(x, \xi)$ is the Green function for

$$-(pu')' = f, \quad a < x < b, \quad u(a) = u(b) = 0.$$

Therefore, we obtain

$$|u_i - U_i| \le \sum_{j=1}^{n} g_{ij} \frac{h_j + h_{j+1}}{2} |\tau_j|.$$

On the other hand, if $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$ in (3.2)–(3.3), then, with the use of fictitious node x_{n+2} (cf. [1]), the difference equation at $x_{n+1} = b$ is given by

$$\frac{-2p_{n+\frac{1}{2}}}{h_{n+1}^2} U_n + \frac{2p_{n+\frac{1}{2}}}{h_{n+1}^2} U_{n+1} + q_{n+1} U_{n+1} = f_{n+1},$$

so that we again have

$$|\boldsymbol{u}-\boldsymbol{U}| \leq (\hat{D}\hat{A}+Q)^{-1}|\hat{\boldsymbol{\tau}}| \leq \hat{A}^{-1}\hat{D}^{-1}|\hat{\boldsymbol{\tau}}|,$$

since $\hat{D}\hat{A}$ is an M-matrix, where

$$\hat{A} = \begin{pmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & b_2+b_3 & -b_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_n & b_n+b_{n+1} & -b_{n+1} \\ & & & -b_{n+1} & b_{n+1} \end{pmatrix}, \quad b_i = \frac{p_{i-\frac{1}{2}}}{h_i}$$

$$\hat{D} = \begin{pmatrix} \frac{2}{h_1+h_2} & & & \\ & \ddots & & \\ & & \frac{2}{h_n+h_{n+1}} & \\ & & & \frac{2}{h_{n+1}} \end{pmatrix}, \quad \hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)^t$$

and $\hat{\tau}_i$ stands for the local truncation error at x_i. Then, an application of Theorem 2.6 (replacing $a_{n+1}+b_{n+1}$ by b_{n+1}) to $\hat{A}^{-1}$ yield

$$\hat{A}^{-1} = \begin{pmatrix} z_1 & z_1 & \cdots & \cdots & z_1 \\ z_1 & z_2 & \cdots & \cdots & z_2 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & z_n & z_n \\ z_1 & z_2 & \cdots & z_n & z_{n+1} \end{pmatrix}, \quad z_i = \sum_{\lambda=1}^{i} \frac{h_\lambda}{p_{\lambda-\frac{1}{2}}} \doteqdot \int_a^{x_i} \frac{ds}{p(s)}.$$

In this case, the Green function $\hat{G}(x,\xi)$ for the operator

$$-\frac{d^2}{dx^2} : \hat{\mathscr{D}} = \{u \in C^2[a,b] \mid u(a) = u'(b) = 0\} \to C[a,b]$$

is given by

$$\hat{G}(x,\xi) = \begin{cases} \displaystyle\int_a^x \frac{ds}{p(s)} & (x \leq \xi) \\ \displaystyle\int_a^\xi \frac{ds}{p(s)} & (x \geq \xi). \end{cases}$$

Hence $\hat{A}^{-1}$ is also a Green's matrix which approximates $\hat{G}(x,\xi)$.

4 Superconvergence and Nonsuperconvergence of the S-W Approximation

Consider the boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=f(x),\ a<x<b,\quad u(a)=\alpha,\ u(b)=\beta. \tag{4.1}$$

Then, as was already seen in §3, the FDM which employs (3.6) leads to the estimate

$$|u_i-U_i|\le (A^{-1}D^{-1}|\tau|)_i=\sum_{j=1}^{n} g_{ij}\frac{h_j+h_{j+1}}{2}|\tau_j|. \tag{4.2}$$

If $p(x)=1$, then it follows from (3.7) that

$$g_{ij}=\begin{cases}\dfrac{1}{b-a}(x_i-a)(b-x_j) & (i\le j)\\[2mm] \dfrac{1}{b-a}(x_j-a)(b-x_i) & (i\ge j)\end{cases}=G(x_i,x_j),$$

where $G(x,\xi)$ is the Green function for

$$-\frac{d^2}{dx^2}:\mathscr{D}=\{u\in C^2[a,b]\mid u(a)=u(b)=0\}\to C[a,b].$$

Therefore, if the solution $u(x)$ of (4.1) belongs to the class $C^4[a,b]$ and the nodes $\{x_i\}$ are equidistant, i.e., $h_i=h\ \forall i$, then $\sup_i|\tau_i|\le \mathrm{O}(h^2)$ and, from (4.2), the error of FDM solution $\{U_i\}$ yields $\mathrm{O}(h^2)$ at every x_i.

Even if $h_1\ne h$ or $h_{n+1}\ne h$ and $h_i=h$ for $i\ne 1,n+1$, this estimate is true. More precisely we have

$$u_i-U_i=\begin{cases}\mathrm{O}(h^3) & \text{at } x_i \text{ near } x=a \text{ or } x=b\\ \mathrm{O}(h^2) & \text{otherwise.}\end{cases} \tag{4.3}$$

This is a special case of recent results due to Yamamoto [9] and Matsunaga-Yamamoto [7], which holds for the Dirichlet problem

$$\Delta u=f(x,y,u)\quad \text{in }\Omega,\quad f_u\ge 0 \tag{4.4}$$
$$u=g(x,y)\quad \text{on }\Gamma=\partial\Omega \tag{4.5}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$. The property (4.3) is called "superconvergence". It is reported in [6] that the Collatz approximation does not have such a property. If $u \notin C^4[a,b]$, for example, if $u \in C[a,b] \cap C^4(a,b)$, then does "superconvergence" occur? It has been shown in Yamamoto-Fang-Chen [12] that different situations may happen. For example, if we apply the centered finite difference method with $h_i = h = \frac{1}{n+1}$, $i = 1, 2, \ldots, n+1$ to the problem

$$\begin{cases} -\frac{d^2}{dx^2}u = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{4.6}$$

and the exact solution u satisfies $u \in C[0,1] \cap C^4(0,1)$ and

$$\sup_{x \in (0,1)} \frac{x^4(1-x)^4|u^{(4)}(x)|}{x^\rho(1-x)^\sigma} < \infty,$$

with some constants $\rho, \sigma \in (0,2)$, $\rho \neq 1$, $\sigma \neq 1$ then, putting $\theta = \min\{\rho, \sigma\}$, we have

$$|u_i - U_i| = \mathrm{O}(h^\theta) \quad \forall i. \tag{4.7}$$

That is, the FDM solution $\{U_i\}$ converges to the exact solution u_i as $h \to 0$, although τ_1 and $\tau_n \to +\infty$ as $h \to 0$. But, superconvergence does not occur at any x_i. If the solution u satisfies $u \in C[0,1] \cap C^4(0,1)$ and

$$\sup_{x \in (0,1)} \frac{x^4|u^{(4)}(x)|}{x^\rho} < \infty$$

with some constant $\rho \in (0,2)$ and $\rho \neq 1$, then

$$|u_i - U_i| \leq \begin{cases} \mathrm{O}(h^{\rho+1}) & \text{at } x_i \text{ near } x = 1 \\ \mathrm{O}(h^\rho) & \text{otherwise.} \end{cases}$$

That is, superconvergence occurs near $x = 1$. These results can be extended to two dimensional problems like (4.4)–(4.5).

This suggests us that FDM is a quite simple and natural approximation method for solving boundary value problems in $\Omega \subset \mathbb{R}^m$, $1 \leq m \leq 3$ and might approximate the exact solution in many cases, even if the truncation error does not converges to zero at a point. The modern numerical analysis has not well grasped such a harmonic relation between the problem (4.1) and their discretization by FDM.

5 A Mesh Refinement Technique

Again consider the FDM applied to the problem (3.1)–(3.3), with a mesh spacing

$$a = x_0 < x_1 < \cdots < x_{n+1} = b, \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n+1.$$

Without loss of generality, we may assume $a = 0$ and $b = 1$. Then

$$\tau(x_i) \equiv -\left[p_{i+\frac{1}{2}}\frac{(u_{i+1}-u_i)}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{(u_i-u_{i-1})}{h_i}\right] / \left(\frac{h_i+h_{i+1}}{2}\right) + q_i u_i - f_i$$
$$= (h_{i+1}-h_i)\left(\frac{1}{2}p_i' u_i'' + \frac{1}{3}p_i u_i''\right) - \frac{1}{6}(h_{i+1}^2 - h_{i+1}h_i + h_i^2)p_i' u_i''' + \cdots .$$

If, we put $x = \varphi(t)$ with $\varphi \in C^2[0,1]$ and set

$$t_i = ih, \quad i = 0,1,2,\dots,n+1, \quad h = \frac{1}{n+1},$$

then

$$h_{i+1} - h_i = \varphi_{i+1} - 2\varphi_i + \varphi_{i-1} = h^2\varphi''(\zeta), \quad t_{i-1} < \zeta < t_{i+1}.$$

Therefore, if $u \in C^4[0,1]$ and $\{U_i\}$ denotes the FDM solution, then $u_i - U_i = \mathrm{O}(h^2)$ $\forall i$. The superconvergence property also holds near the end points $x = 0$ and $x = 1$. Even if $\tau(x_i) = \mathrm{O}(h^{-\kappa})$ for some constant $\kappa > 0$, the convergence of $\{U_i\}$ as $h \to 0$ may be expected. In fact, let

$$f(x) = \frac{1}{4}\{x(1-x)\}^{-\frac{3}{2}}, \quad 0 < x < 1.$$

Then $u = \sqrt{x(1-x)}$ is the solution of the problem

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0$$

and $\tau(x_i) = \mathrm{O}(h^{\frac{1}{2}-2})$ for x_i near $x = 0$ or $x = 1$. In this case, choosing $\varphi(t)$ so as to satisfy $\varphi'(t) = \{t(1-t)\}^3$, i.e., $\varphi(t) = t^4(\frac{1}{4} - \frac{3}{5}t + \frac{3}{2}t^2 - \frac{1}{7}t^3)$, we can prove that

$$|u_i - U_i| = \mathrm{O}(h^2 \log\frac{1}{h}) \ (= \mathrm{O}(h^{2-\varepsilon}) \ \forall\varepsilon > 0) \ \forall i.$$

This is a marked improvement over the estimate (4.7) with $\theta = \frac{1}{2}$.

It is now easy to apply this technique to refine finite difference net near finite numbers of grid points $a_1, \dots, a_m \in (0,1)$. Furthermore, it is also possible to extend the technique to 2 or 3-dimensional boundary value problem (4.4)–(4.5). The detail will be reported elsewhere, together with numerical examples.

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