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# PARAMETRIC VARIATIONAL PRINCIPLES IN BANACH SPACES AND SELECTION THEOREMS

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## 0. INTRODUCTION.

We present parametric variational principles of Ekeland's and Borwein-Preiss' type, stating that under suitable assumptions, the minimum point of the perturbed function is a measurable (resp. continuous, Carathéodory ) function of the parameters. As applications, new proofs are obtained of: Ky Fan's inequality, Sion's minimax theorem, Michael's selection theorem and a theorem for existence of Carathéodory selections.

## 1. MEASURABLE SELECTIONS AND BORWEIN-PREISS VARIATIONAL PRINCIPLE

Ekeland's variational principle and its smooth analogues are useful tools in the study of non-linear problems in various areas of mathematics (see for instance [E1], [E2], [B-P], [D-G-Z1], [D-G-Z2]).

Firstly we present a random version of the smooth variational principle of Borwein-Preiss [B-P]. Namely, we prove that a suitable perturbed function of a given one admits, as in [B-P], a minimum point, which in our setting is a measurable function of a random variable.

We give two applications of our random smooth variational principle. The first one is about weak Hadamard differentiability of some convex integral functionals in the Lebesgue-Bochner space $L^1(T,\mu;E)$.

Borwein and Fitzpatrick [B-F] have shown that in $L^1(T,\mu)$, where $\mu$ is sigma finite, there exists an equivalent weak Hadamard differentiable norm, hence using Preiss-Phelps-Namioka's theorem [P-P-N], they establish that $L^1(T,\mu)$ is a weak Hadamard Asplund space. Examination of their proofs will convince the reader that the results of Borwein-Fitzpatrick remain valid in the space $L^1(T,\mu;E)$, provided $E$ is a reflexive Banach space and $\mu$ is finite, since the Dunford-Pettis criterium for weak compactness in $L^1$ is crucial in their proofs.

Here we prove that if $E$ is a separable Banach space with a uniformly Fréchet differentiable norm, then any convex continuous integral functional on $L^1(T,\mu;E)$ from a certain class (in particular the usual $L^1$ norm), is weak Hadamard differentiable on a subset whose complement is $\sigma$-very porous. The proof of this result is direct and, unlike in [B-F], does not relay on the deep theorem in [P-P-N].

As a second application, a random version of Caristi's fixed point theorem for multifunctions is obtained.

Let $(E, \|.\|)$ be a Banach space with dual $E^*$ and let $S = \{x \in E : \|x\| = 1\}$.

The norm $\|.\|$ of $E$ is said to be *uniformly Fréchet differentiable*, if for every $x \in S$ there exists an element $\nabla\|x\| \in E^*$ such that the following condition holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that :

$$\frac{\|x + th\| - \|x\|}{t} - \langle \nabla\|x\|, h \rangle < \varepsilon \text{ for every } x \in S, h \in S, t \in (0, \delta).$$

The norm $\|.\|$ of $E$ is said to be *weak Hadamard differentiable* at $x \in E$ if there exists an element $\nabla\|x\| \in E^*$ such that for every weakly compact subset $W$ of $E$ the following holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|x + th\| - \|x\|}{t} - \langle \nabla\|x\|, h \rangle < \varepsilon \text{ for every } h \in W \text{ and } t \in (0, \delta).$$

About measurability we retain notation and terminology of Himmelberg [H].

The following theorem is a random version of the Borwein-Preiss smooth variational principle.

**Theorem 1** *([D-G]) Suppose that $(E, \|.\|)$ is a separable Banach space and $(T, \mathcal{A}, \mu)$ is a measurable space with a complete $\sigma$-finite measure $\mu$. Let $F : T \to 2^E$ be a measurable multifunction with non-empty closed values and $f : T \times E \to \mathbf{R}$ be a function with the following properties:*
(1) $\inf_{x \in F(t)} f(t, x) > -\infty$, *for every $t \in T$;*
(2) *$f(., x)$ is measurable, for every $x \in E$;*
(3) *$f(t, .)$ is continuous for every $t \in T$.*

*Let $x_0 : T \to E$ be a measurable single-valued selection of $F$ such that*

$$f(t, x_0(t)) < \inf_{x \in F(t)} f(t, x) + \varepsilon_0(t) \quad \textit{for every} \quad t \in T,$$

*where $\varepsilon_0 : T \to (0, +\infty)$ is a given measurable function . Let $p \geq 1$ be given and let $\varepsilon : T \to (0, +\infty)$ and $\lambda : T \to (0, +\infty)$ be measurable functions with $\varepsilon(t) > \varepsilon_0(t)$, $t \in T$.*

*Then there exist measurable selections of $F$, say $x_n, v : T \to E$ and measurable functions $\mu_n : T \to (0, 1)$ such that, for every $t \in T$, we have: $\sum_{n=0}^{\infty} \mu_n(t) = 1$, and*
(4) *$x_n(t) \to v(t)$ as $n \to \infty$;*
(5) *$f(t, v(t)) + \Delta(t, v(t)) \leq f(t, x) + \Delta(t, x)$, for every $x \in F(t)$, where*
(6) *$\Delta(t, x) = \frac{\varepsilon(t)}{\lambda(t)^p} \sum_{n=0}^{\infty} \mu_n(t) \|x - x_n(t)\|^p$;*
(7) *$\|x_n(t) - v(t)\| < \lambda(t)$ for every $n = 0, 1, 2, \ldots$ .*

**Proof.** Set $q(t) = \frac{1}{2}\left[\frac{\varepsilon(t)-\varepsilon_0(t)}{\varepsilon(t)+\varepsilon_0(t)}\right]^{1/2}$. For $n = 0, 1, ...$ and $t \in T, x \in E$, define inductively:

$$G_n(t) = \{x \in F(t) : f_n(t,x) \leq \inf_{z \in F(t)} f_n(t,z) + \varepsilon_n(t)\}, \tag{8}$$

where

$$f_{n+1}(t,x) = f_n(t,x) + \frac{\varepsilon(t)}{\lambda(t)^p}\mu_n(t)\|x - x_n(t)\|^p \ , f_0(t,x) = f(t,x) \tag{9}$$

$$\varepsilon_n(t) = \varepsilon_0(t)q(t)^{2n}, \quad \mu_n(t) = (1 - q(t))q(t)^n,$$

and $x_n : T \to E$ is a measurable function satisfying

$$x_n(t) \in G_n(t) \tag{10}$$

for every $t \in T$.

We shall prove by induction that the definitions of $x_n$ and $f_n$ are correct. For $n = 0$ this is true by assumption. Suppose that $x_{n-1}$ and $f_{n-1}$ are defined. Now, by (9), $f_n$ is well defined. By [H, Theorem 6.5] the multifunction $t \mapsto f_n(t, F(t))$ is weakly measurable (in fact it is measurabe, by [H, Theorem 3.5 (iii)] ) and by [H, Theorem 6.6] the function $t \mapsto \inf_{z \in F(t)} f_n(t,z)$ is measurable. By [H, Theorem 6.4] and [H, Theorem 3.5 (iii)] the multifunction $t \mapsto G_n(t)$ is measurable and Kuratowski and Ryll-Nardzewski's theorem [K-RN] (see also [H, Theorem 5.1]) produces a measurable function $x_n$ satisfying (10), completing the induction.

We shall prove that $\{x_n(t)\}$ is a fundamental sequence for every $t \in T$. By (9) and (10) we have:

$$\begin{aligned}
&\tfrac{\varepsilon(t)}{\lambda(t)^p}\mu_n(t)\|x_{n+1}(t) - x_n(t)\|^p \\
&= f_{n+1}(t, x_{n+1}(t)) - f_n(t, x_{n+1}(t)) \\
&= f_{n+1}(t, x_{n+1}(t)) - f_{n+1}(t, x_n(t)) + f_n(t, x_n(t)) - f_n(t, x_{n+1}(t)) \\
&\leq \varepsilon_{n+1}(t) + \varepsilon_n(t) \\
&= \varepsilon_0(t)q(t)^{2n}(q(t)^2 + 1).
\end{aligned}$$

Hence for $m > n$ we obtain:

$$\|x_m(t) - x_n(t)\| \leq \lambda(t)\Big(\frac{\varepsilon_0(t)}{\varepsilon(t)}.\frac{1 + q(t)^2}{1 - q(t)^2}\Big)^{\frac{1}{p}} q(t)^{\frac{n}{p}} < \lambda(t)q(t)^{\frac{n}{p}}. \tag{11}$$

Therefore $\{x_n(t)\}_{n=0}^{\infty}$ is a fundamental sequence and so converges to $v(t)$ and clearly $v$ is measurable. From (11), letting $m \to +\infty$, we obtain $\|v(t) - x_n(t)\| < \lambda(t)$, which is (7).

To establish (5), let $\gamma > 0$ be given. As $(f + \Delta)(t, .)$ is continuous, there exists $\delta(t) > 0$ such that

$$f(t, v(t)) + \Delta(t, v(t)) < f(t, x) + \Delta(t, x) + \gamma/3, \tag{12}$$

whenever $\|x - v(t)\| < \delta(t)$. For fixed $t \in T$, choose $n$ sufficiently large so that the following inequalities hold: $\varepsilon_n(t) < \gamma/3$, $\|v(t) - x_k(t)\| < \delta(t)$ for every $k \geq n$, and $\frac{\varepsilon(t)}{\lambda(t)^p} \sum_{k=n}^{\infty} \mu_k(t) \|x_n(t) - x_k(t)\|^p < \gamma/3$.

For every $x \in F(t)$, using (12), (8), (9) and (10), we can write

$$\begin{aligned} & f(t, v(t)) + \Delta(t, v(t)) \\ & < f(t, x_n(t)) + \Delta(t, x_n(t)) + \gamma/3 \\ & = f_n(t, x_n(t)) + \tfrac{\varepsilon(t)}{\lambda(t)^p} \textstyle\sum_{k=n}^{\infty} \mu_k(t) \|x_n(t) - x_k(t)\|^p + \gamma/3 \\ & < f_n(t, x) + \varepsilon_n(t) + \gamma/3 + \gamma/3 \\ & < f(t, x) + \Delta(t, x) + \gamma \end{aligned}$$

and (5) is proved. ∎

**Remark .** It is clear that if the conditions of Theorem 1 are satisfied for a.e. $t \in T$, then its conclusions are satisfied for a.e. $t \in T$ too. From Theorem 1 it follows a random version of Ekeland's variational principle.

Now we recall the following definition.

A subset $P$ of $E$ is said to be *very porous*, if there exists $\alpha > 0$ for which the following holds: for every $x \in E$ and every $r \in (0, \alpha)$ there is $y \in E$ such that

$$B(y, \alpha r) \subset B(x, r) \cap (E \setminus P).$$

$P$ is called *$\sigma$-very porous*, if it is a countable union of very porous sets.

In the sequel $L^1(T, \mu; E)$ will denote the usual Lebesgue-Bochner space, i.e. the set of all (equivalence classes of) $\mu$-Bochner integrable functions $f : T \to E$ with the norm $\|f\|_{L^1} = \int_T \|f(t)\| d\mu(t)$.

**Theorem 2** *Let $(E, \|.\|)$ be a separable superreflexive Banach space and let $(T, \mathcal{A}, \mu)$ be a measurable space, with a complete $\sigma$-finite measure $\mu$, $\int_T d\mu(t) = 1$. Suppose that $f : T \times E \to \mathbf{R}$ is a function satisfying conditions (1), (2), (3) of Theorem 1, and*

*a) $f(t, .)$ is convex for every $t \in T$;*

*b) there exists a function $L \in L^\infty(T, \mu; \mathbf{R})$ such that for all $t \in T$*

$$|f(t, x_1) - f(t, x_2)| \leq L(t) \|x_1 - x_2\| \quad \text{for every } x_1, x_2 \in E; \tag{13}$$

*c) $f(., 0) \in L^1(T, \mu; \mathbf{R})$.*

*Define the function $g : L^1(T, \mu; E) \to \mathbf{R}$ by $g(x) = \displaystyle\int_T f(t, x(t)) d\mu(t)$. Then $g$ is weak Hadamard differentiable on a subset $X_0$ of $L^1(T, \mu; E)$, whose complement is $\sigma$-very porous.*

**Proof.** Since every suprreflexive Banach space has an equivalent uniformly Fréchet differentiable norm (see [D-G-Z2, Corollary 4.6, page 152]), we may suppose without loss of generality that the norm $||.||$ is uniformly Fréchet differentiable.

In view of b) and c), the definition of $g$ makes sense. The absolute continuity of the Lebesgue integral (see [K-F, Theorem V.5.5]) allows us to define, for every $x, h \in L^1(T, \mu; E)$,

$$\begin{aligned}\gamma_n(x,h) = \sup\Big\{ \quad & \gamma > 0 : \int_\Gamma \Big(f(t,x(t)+h(t)) + f(t,x(t)-h(t)) \\ & -2f(t,x(t))\Big)d\mu(t) \le \tfrac{1}{n} \\ & \text{for every } \ \Gamma \subset T \ \text{ with } \ \mu(\Gamma) \le \gamma ||h||_{L^1} \Big\}.\end{aligned} \tag{14}$$

Put

$$H_{n,m}(x) = \{h \in L^1(T,\mu;E) : \gamma_n(x,h) \ge \frac{1}{m}\} \tag{15}$$

and

$$\Gamma_\gamma(h) = \{t \in T : ||h(t)|| \ge \frac{1}{\gamma}\}. \tag{16}$$

Obviously

$$\mu(\Gamma_\gamma(h)) \le \gamma \int_T ||h(t)||d\mu(t) = \gamma ||h||_{L^1} \quad \text{for every } \ h \in L^1(T,\mu;E). \tag{17}$$

Since the norm $||.||$ is uniformly Fréchet differentiable, it is easy to see that there exists $s_{n,m} \in (0, 1/m)$ such that

$$\frac{||x + s_{n,m}h||^2 - ||x||^2}{s_{n,m}} - \langle \nabla ||x||^2, h\rangle < \frac{1}{n} \quad \text{whenever } \ ||x|| < 1, ||h|| \le m. \tag{18}$$

Define the set:

$$\begin{aligned} X_{n,m} \;=\; & \{x \in L^1(T,\mu;E) : \ \text{there exists } s \in (0,1/m) \ \text{ such that} \\ & \frac{g(x+sh) + g(x-sh) - 2g(x)}{s} < 14/n, \ \text{ for every } h \in H_{n,m}(x)\}.\end{aligned}$$

**Claim .** *$L^1(T,\mu;E) \setminus X_{n,m}$ is very porous for every integer $n, m$.*

Assume the contrary. Then for some integer $n, m$, for $\alpha \in (0, s_{n,m}/2n)$, there exist $x_0 \in L^1(T,\mu,E)$ and $r \in (0,\alpha)$ such that

$$B(v;\alpha r) \cap (L^1(T,\mu;E) \setminus X_{n,m}) \ne \emptyset \qquad \text{for every } \ v \in B(x_0;r). \tag{19}$$

By Himmelberg [H] it follows that the multivalued mapping $F : t \mapsto B[x_0(t); r]$ is measurable.

Without loss of generality we may assume that $\|L\|_\infty < 1$. In view of (13) we have

$$\|f(t,x) - f(t,x_0(t)\| \leq r \qquad \text{for a.e. } t \in T \text{ and for every } x \in F(t)\ t \in T.$$

By Theorem 1, with the above $F$ and $\varepsilon_0 = r = \varepsilon/2, \lambda = r/2, p = 2$, we obtain measurable selections $v, x_n : T \to E$ of $F$, and measurable functions $\mu_n : T \to (0,1)$ such that for the function $\Delta(t,x)$ given by (6) (with the above constructed $x_n$), we have

(20)
$$f(t,x) - f(t,v(t)) \leq \Delta(t,x) - \Delta(t,v(t)) \quad \text{for a.e. } t \in T \text{ and for every } x \in F(t),$$

and

$$\|v(t) - x_0(t)\| < \lambda \qquad \text{for a.e. } t \in T. \tag{21}$$

So by (19) we can find $z \in B(v; \alpha r) \cap (L^1(T,\mu;E) \setminus X_{n,m})$.

Hence, by definition of $X_{n,m}$, for $s = s_{n,m}r/2$, there exists $h \in H_{n,m}(z)$ such that , setting $\Gamma = \Gamma_{\gamma_n(z,h)}(h)$, we have $\mu(\Gamma) \leq \gamma_n(z,h)\|h\|_{L^1}$ (from (17)) and

$$\begin{aligned}
14/n &\leq \frac{g(z+sh) + g(z-sh) - 2g(z)}{s} \\
&\leq \int_T \frac{f(t,v(t)+sh(t)) + f(t,v(t)-sh(t)) - 2f(t,v(t))}{s} d\mu(t) + 4/n \\
&\qquad \text{(by (13) and by the choice of } \alpha \text{ and } s) \\
&< \int_{T\setminus\Gamma} \frac{f(t,v(t)+sh(t)) + f(t,v(t)-sh(t)) - 2f(t,v(t))}{s} d\mu(t) + 5/n \\
&\qquad \text{(by (14) and monotonicity of the differential quotient of a convex} \\
&\qquad \text{function )} \\
&\leq \int_{T\setminus\Gamma} \frac{1}{s}\Big(\Delta(t,v(t)+sh(t)) + \Delta(t,v(t)-sh(t)) - 2\Delta(t,v(t))\Big) d\mu(t) + 5/n \\
&\qquad \text{(by (20), since (16) and (15) implies that } h(t) \leq m \text{ for } t \notin \Gamma \\
&\qquad \text{and (21) implies that } v(t) + sh(t) \in F(t)) \\
&= \frac{\varepsilon}{\lambda} \int_{T\setminus\Gamma} \sum_{n=0}^{\infty} \mu_n \Big[ \frac{\|\frac{v(t)-x_n(t)}{\lambda} + \frac{s}{\lambda}h(t)\|^2 - \|\frac{v(t)-x_n(t)}{\lambda}\|^2}{\frac{s}{\lambda}}
\end{aligned}$$

$$+\frac{\|\frac{v(t)-x_n(t)}{\lambda}-\frac{s}{\lambda}h(t)\|^2-\|\frac{v(t)-x_n(t)}{\lambda}\|^2}{\frac{s}{\lambda}}]d\mu(t)+5/n$$

$$< \ 13/n \quad (\text{by (18), since } \frac{s}{\lambda}\le s_{n,m}),$$

which is a contradiction.

Therefore the set $L^1(T,\mu;E)\setminus X_{n,m}$ is very porous.

We need the following.

**Proposition 3** *If $K$ is a weakly compact subset in $L^1(T,\mu;E)$, then for every integer $n$ and every $v\in E$ there exists an integer $m$ such that $K\subset H_{n,m}(v)$.*

**Proof.** Assuming the contrary, there exist an integer $n$ and a $v\in E$ such that for every $m$ there exists $h_m\in K\setminus H_{n,m}(v)$, i.e. there exist $\Gamma_m$ such that $\mu(\Gamma_m)=\frac{1}{m}\|h_m\|$ and

$$(22)\qquad \int_{\Gamma_m}(f(t,v(t)+h_m(t))+f(t,v(t)-h_m(t))-2f(t,v(t))d\mu(t)\ge\frac{1}{n}.$$

Since $K$ is weakly compact, it is bounded, so $\mu(\Gamma_m)\to 0$. Now by Dunford-Pettis' theorem of weak compactness in $L^1(T,\mu;E)$ (see [D-U], page 105), $K$ is uniformly integrable, i.e. $\lim_{\mu(E)\to 0}\int_E\|h(t)\|d\mu(t)=0$ uniformly for $h\in K$. So, by (22) and (13), when $m$ tends to infinity, we obtain a contradiction. ∎

The proof of the following proposition is the same as that of [Ph, Proposition 1.23] (which concerns Fréchet differentiability).

**Proposition 4** *A convex continuous function $f:E\to\mathbf{R}$ is weak Hadamard differentiable at $x\in E$ if and only if for every $\varepsilon>0$ and every weakly compact subset $W\subset E$ there exists $t>0$ such that*

$$\frac{f(x+th)+f(x-th)-2f(x)}{t}<\varepsilon\quad \textit{for every}\ \ h\in W.$$

Now we can complete the proof of Theorem 3. By Proposition 4 and Proposition 5, it follows that $g$ is weak Hadamard differentiable on the set $X_0=\cap_{n,m=1}^{\infty}X_{n,m}$. By the Claim, it follows that the set $L^1(T,\mu;E)\setminus X_0$ is $\sigma$-very porous and the theorem is proved. ∎

Comparing the Fréchet and the weak Hadamard differentiability of the usual norm $\|.\|_{L^1}$ of $L^1(T,\mu;E)$, as an interesting corollary of Theorem 3, we obtain that $\|.\|_{L^1}$ is weak Hadamard differentiable on a complement of $\sigma$-very porous subset of $L^1(T,\mu;E)$, while $\|.\|_{L^1}$ is nowhere Fréchet differentiable (see [Ph]).

The following theorem is a random version of Caristi's fixed point theorem [A-E, Theorem 14, Ch.5 Sec.1].

**Theorem 5** *(Multivalued Caristi's random fixed point theorem). Suppose that $(E, \|.\|)$ is a separable Banach space and $(T, \mathcal{A}, \mu)$ is a measurable space with a complete $\sigma$-finite measure $\mu$. Let $X$ be a closed subset of $E$, $F : T \times X \to X$ be a multivalued mapping. Assume that there exists a function $f : T \times X \to \mathbf{R}$ such that $f(., x)$ is measurable, $f(t, .)$ is continuous, and for every $x \in X$ and every $t \in T$ there exists $y_{t,x} \in F(t, x)$ such that*

$$f(t, y_{t,x}) + \|x - y_{t,x}\| \leq f(t, x) \ . \tag{23}$$

*Then there exists a measurable mapping $v : T \to X$ such that $v(t) \in F(t, v(t))$, for every $t \in T$.*

**Proof.** Apply Theorem 1 with $\varepsilon_0 < \varepsilon < 1, \lambda = 1, p = 1$ and obtain a measurable mapping $v : T \to X$ such that

(24) $f(t, v(t)) < f(t, x) + \varepsilon\|v(t) - x\|$ for every $t \in T$ and $x \in X, x \neq v(t)$.

By (23), with $x = v(t)$, and by (24), with $x = y_{t,v(t)}$, we obtain:

$$\|v(t) - y_{t,v(t)}\| \leq f(t, v(t)) - f(t, y_{t,v(t)}) \leq \varepsilon\|v(t) - y_{t,v(t)}\|$$

for every $t \in T$. Hence $v(t) = y_{t,v(t)}$ and the theorem is proved. ∎

## 2. CONTINUOUS ANS CARATHÉODORY SELECTIONS AND EKELAND'S VARIATIONAL PRINCIPLE.

The following lemma is simple, but very useful, since it implies Ky Fan's inequality and Sion's minimax theorem.

**Lemma 1.** *Suppose that $X$ is a paracompact topological space, $E$ is a Banach space, $Y \subset E$ is closed, convex and nonempty, $F : X \to 2^Y$ is lower semicontinuous with convex images and the functions $f : X \times Y \to \mathbf{R}$, $g : X \to \mathbf{R}$ satisfy the conditions:*

*(i) the function $f(x, .)$ is quasiconvex for every $x \in X$;*

*(ii) the function $f(., y)$ is upper semicontinuous for every $y \in Y$;*

*(iii) $g$ is lower semicontinuous and $g(x) \geq \inf_{y \in F(x)} f(x, y)$ $\forall x \in X$.*

*Then*

*(a) for every $\varepsilon > 0$ there exists a continuous selection $\varphi_\varepsilon : X \to Y$ of $F_\varepsilon$ (i.e. $\varphi_\varepsilon(x) \in F_\varepsilon(x)$ for every $x \in X$), where $F_\varepsilon(x) = F(x) + \varepsilon B$, $B$ is the open unit ball, and*

$$f(x, \varphi_\varepsilon(x)) < g(x) + \varepsilon \quad \forall x \in X. \tag{1}$$

*(b) If $F(x)$ is open for every $x \in X$, then there exists a continuous selection $\varphi_\varepsilon$ of $F$ satisfying (1).*

**Proof.** Define:

$$S_y = \{x \in X : y \in F_\varepsilon(x), f(x, y) < g(x) + \varepsilon\}.$$

By lower semicontinuity of $F$, by (ii) and (iii), we obtain that this set is open. Obviously $\cup_{y\in Y} S_y = X$. Since $X$ is paracompact, there exists a locally finite refinement $\{U_j\}_{j\in J}$ of the cover $\{S_y\}_{y\in Y}$. Let $\{p_j\}_{j\in J}$ be a continuous partition of unity, subordinate to this cover. For each $j \in J$ choose $y_j \in Y$ such that $supp(p_j) \subset S_{y_j}$. Define the function

$$\varphi(x) = \sum_{j\in J} p_j(x) y_j.$$

By (i) we obtain (1). ∎

**Theorem 3 (Ky Fan).** Let $X$ be convex, compact and nonempty subset of a topological vector space $E$, $f : X \times X \to \mathbf{R}$ be a function such that

a) $f(.,y)$ is lower semicontinuous for every $y \in X$;

b) $f(x,.)$ is concave for every $x \in X$.

Then there exists $x_0 \in X$ such that

$$\sup_{y\in X} f(x_0, y) \le \sup_{y\in X} f(y,y). \tag{2}$$

**Proof.** The essential part of the proof is to establish (2) when $E$ is finite dimensional, since it is easy to extend it using compactness (finite intersection property) to arbitrary topological vector spaces.

Apply Lemma 1 when $G(x) = Y$ to the function $-f$ and the constant function

$$g(z) = \sup_{x\in X} \inf_{y\in X} (-f)(x,y).$$

So we obtain a continuous functinon $\varphi_n : X \to X$ such that

$$-f(x, \varphi_n(x)) \le g(x) + 1/n \quad \forall x \in X. \tag{3}$$

By Brouwer's fixed point theorem there exists a fixed point $x_n = \varphi_n(x_n)$ of $\varphi_n$. Passing to limit in (3) and using a), we obtain the result. ∎

Next, we shall show that Sion's minimax theorem follows from Lemma 1 and something more, in the setting of Sion's theorem, we shall prove that for every $\varepsilon > 0$ there exists an $\varepsilon$-saddle point.

**Theorem 3.** *Let $X$ be a nonempty convex subset of topological vector space, $Y$ a nonempty compact convex subset of topological vector space, and $f : X \times Y \to \mathbf{R}$ quasiconcave and upper semicontinuous in its first variable and quasiconvex and lower semicontinuous in its second variable. Then for every $\varepsilon > 0$ there exists an $\varepsilon$-saddle point, that is*

$$f(x, y_\varepsilon) - \varepsilon \le f(x_\varepsilon, y_\varepsilon) \le f(x_\varepsilon, y) + \varepsilon. \quad falx \in X, y \in Y \tag{4}$$

From (4) we obtain Sion's minimax theorem:

$$\min_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \min_{y\in Y\in Y} f(x,y).$$

**Proof.** The essential part of the proof is to establish (4) in finite dimensional spaces, since its extention to arbitrary topological vector spaces is straightforward (see, for instance, the proof of Theorem 1.4 in [Simon].)

Applying Lemma 1, we obtain: there exists continuous functions $\varphi : X \to Y, \psi : Y \to X$ such that

$$f(x, \varphi(x)) \le f(x, y) + \varepsilon \quad \forall x \in X, y \in Y,$$

$$-f(\psi(y), y) \le -f(x, y) + \varepsilon \quad \forall x \in X, y \in Y.$$

The composition $\varphi(\psi(.)) : Y \to Y$ is continuous, and by Brouwer's fixed point theorem, there exists a fixed point $y_\varepsilon = \varphi(\psi(y_\varepsilon))$. Denoting $x_\varepsilon = \psi(y_\varepsilon))$, we obtain (4). ∎

**Theorem 4 (Continuous parametrization of Ekeland's variational principle).** *Suppose that $X$ is a paracompact topological space, $Y$ is a convex closed and nonempty subset of a Banach space and the function $f : X \times Y \to \mathbf{R}$ satisfies the conditions:*

*(i) the function $f(x, .)$ is quasiconvex for every $x \in X$,*

*(ii) the function $f(., y)$ is upper semicontinuous for every $y \in Y$,*

*(iii) the function $f(.,.)$ is lower semicontinuous.*

*Given $\varepsilon > 0$, $\lambda > 0$, let $\varphi_0 : X \to Y$ be a continuous mapping, such that*

$$f(x, \varphi_0(x)) \le \inf f(x, Y) + \varepsilon, \quad \forall x \in X.$$

*Then there exist a continuous mapping $v : X \to Y$, such that*

$$\|v(x) - \varphi_0(x)\| \le \lambda, \quad \forall x \in X, \tag{5}$$

$$f(x, v(x)) \le f(x, y) + \frac{\varepsilon}{\lambda}\|y - v(x)\| \quad \forall x \in X, \ \forall y \in Y. \tag{6}$$

**Proof.** Put $g_n(x) = \inf_{y \in F_n(x)} f(x, y)$ and define by induction sequences of lower semicontinuous multifunctions $F_n : X \times Y \to 2^Y$ by

$$F_n(x) = \{y \in Y : f(x, y) + \frac{\varepsilon}{\lambda}\|\varphi_n(x) - y\| < f(x, \varphi_n(x))\}$$

and continuous selections (by Lemma 1 (b)) $\varphi_{n+1} : X \to Y$ of $F_n$ satisfying

$$f(x, \varphi_{n+1}(x)) < g_n(x) + 2^{-n},$$

starting at $\varphi_0$. Assuming that $\varphi_n$ is continuous (by induction hypothesys), we shall prove that the conditions of Lemma 1 are satisfied. Indeed, by (ii) and (iii) it follows that $F_n$ is lower semicontinuous with open images and by Proposition 3.1 of [H-P], the function $\inf_{y \in F_n(.)} f(., y)$ is lower semicontinuous, which is condition (iii) of Lemma 1. This completes the induction.

It is clear that the diameter $\sup_{y,z\in F_n(x)} \|y-z\|$ of $F_n(x)$ is not grater than $\frac{2\lambda}{\varepsilon}(f(x,\varphi_n(x)) - g_n(x))$. Since $F_{n+1}(x) \subset F_n(x)$ for every $x \in X$, we have

$$g_n(x) \le g_{n+1}(x) \quad \forall x \in X.$$

On the other hand, the inequality $g_n(x) \le f(x,\varphi_n(x))$ implies that

$$g_{n+1}(x) \le f(x,\varphi_{n+1}(x)) \le g_n(x) + 2^{-n} \le g_{n+1}(x) + 2^{-n}$$

hence

$$0 \le f(x,\varphi_{n+1}(x)) - g_{n+1}(x) \le 2^{-n}.$$

Therefore the diameters of $F_n(x)$ converges uniformly to 0. Therefore

$$\bigcap_{n\ge 0} \overline{F_n(x)} = \{v(x)\} \quad \forall x \in X$$

and $v : X \to Y$ is a continuous mapping. This implies that

$$v(x) \in \overline{F_0(x)} \quad \forall x \in X, \tag{7}$$

which proves inequality (5). It is easy to see that

$$y \in \overline{F_n(x)} \Rightarrow \overline{F_n(y)} \subset \overline{F_n(x)}. \tag{8}$$

From (7) and (8) it follows that $\overline{F_n(x)} = \{v(x)\}$. Consequently, if $y \ne v(x)$, then $y \notin \overline{F_n(x)}$, which implies (6). ∎

**Corollary (Michael's selection theorem [M]).** *Every lower semicontinuous multifunction $F : X \to 2^E$ with closed convex images from a paracompact space $X$ into a Banach space $E$ has a continuous selection.*

**Proof.** Apply Theorem 4 for $f(x,y) = \inf_{z\in F(x)} \|y-z\|$ with $\varepsilon < 1$ and $\lambda = 1$. ∎

The proof of the next theorem is analogous to that one of Lemma 1, but it uses Carathéodory partition of unity, instead of continuous partition of unity.

**Theorem 5.** *Suppose that $(T, \mathcal{A}, \mu)$ is a measurable space, with a complete $\sigma$-finite measure $\mu$, $X$ is a Polish space, $E$ is a separable Banach space, $Y \subset E$ is closed, convex, nonempty. Let $F : T \times X \to 2^Y$ be a multifunction, such that $F(.,.)$ is measurable, $F(t,.)$ is lower semicontinuous for every $t \in T$. Suppose that the functions $f : T \times X \times Y \to \mathbf{R}$, $g : T \times X \to \mathbf{R}$ satisfy the conditions:*

*(i) the function $f(t,x,.)$ is quasiconvex for every $t \in T$ and $x \in X$;*

*(ii) the function $f(t,.,y)$ is upper semicontinuous for every $t \in T$ and $y \in Y$;*

*(iii) $f(.,.,y)$ is measurable for every $y \in Y$,*

*(iv)* $g(.,.)$ *measurable,* $g(t,.)$ *is lower semicontinuous and* $g(t,x) \geq \inf_{y \in F(t,x)} f(t,x,y)$.
*Then*

*(a) for every* $\varepsilon > 0$ *there exists a Carathéodory selection* $\varphi_\varepsilon : T \times X \to Y$ *of* $F_\varepsilon$ *(i.e.* $\varphi_\varepsilon(t,x) \in F_\varepsilon(t,x)$ *for every* $y \in T, x \in X$*), where* $F_\varepsilon(t,x) = F(t,x) + \varepsilon B$*,* $B$ *is the open unit ball, and*

$$f(t,x,\varphi_\varepsilon(x)) < g(t,x) + \varepsilon \quad \forall t \in T, x \in X. \tag{9}$$

*(b) If* $F(t,x)$ *is open for every* $x \in X$*, then there exists a Carathéodory selection* $\varphi_\varepsilon$ *of* $F$ *satisfying (9).*

*The proof of the next theorem is similar to that one of Theorem 4.*

**Theorem 6 (Carathéodory parametrization of Ekeland's variational principle).** *Suppose that* $(T, \mathcal{A}, \mu)$ *is a measurable space, with a complete* $\sigma$*-finite measure* $\mu$*,* $X$ *is a Polish space,* $E$ *is a separable Banach space,* $Y \subset E$ *is closed, convex, nonempty. Suppose that the function* $f : T \times X \times Y \to \mathbf{R}$ *satisfies the conditions:*

*(i)* *the function* $f(t,x,.)$ *is quasiconvex for every* $t \in T$ *and* $x \in X$*;*

*(ii)* *the function* $f(t,.,y)$ *is upper semicontinuous for every* $t \in T$ *and* $y \in Y$*;*

*(iii)* $f(.,.,y)$ *is measurable for every* $y \in Y$*,*

*(iv)* $f(t,.,.)$ *is lower semicontinuous for every* $t \in T$*.*

*Given* $\varepsilon > 0$*,* $\lambda > 0$*, let* $y_0 : T \times X \to Y$ *be a Carathéodory mapping, such that*

$$f(t,x,y_0(t,x)) \leq \inf f(t,x,Y) + \varepsilon, \quad \forall t \in T, \forall x \in X.$$

*Then there exist a Carathéodory mapping* $v : T \times X \to Y$*, such that*

$$\|v(t,x) - y_0(t,x)\| < \lambda, \quad \forall t \in T, \forall x \in X,$$

$$f(t,x,v(t,x)) \leq f(t,x,y) + \frac{\varepsilon}{\lambda}\|y - v(t,x)\| \quad \forall t \in T, \forall x \in X, \ \forall y \in Y.$$

**Corollary (Carathéodory selections, [H-P, Theorem 7.23]).** *Suppose that* $(T, \mathcal{A}, \mu)$ *is a measurable space, with a complete* $\sigma$*-finite measure* $\mu$*,* $X$ *is a Polish space,* $E$ *is a separable Banach space,* $Y \subset E$ *is closed, convex, nonempty. Let* $F : T \times X \to 2^Y$ *be a multifunction, such that* $F(.,.)$ *is measurable,* $F(t,.)$ *is lower semicontinuous for every* $t \in T$*. Then* $F$ *admits a Carathéodory selection.*

**Proof.** Apply Theorem 4, for $f(t,x,y) = \inf_{z \in F(t,x)} \|y - z\|$ with $\varepsilon < 1$ and $\lambda = 1$. ∎

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