数理解析研究所講究録
1174 巻 2000 年 83-95

Title:
Representation of Fuzzy Numbers and Fuzzy Differential Equations

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Abstract We introduce some representation of fuzzy numbers with bounded supports as well as we consider a Banach space including the set of fuzzy numbers, where the addition in the Banach space is the same one due to the extension principle but the difference and scalar products are not the same as those of the principle. In this article we treat initial value problems of fuzzy differential equations and give existence and uniqueness theorems and sufficient conditions for the continuous dependence with respect to initial conditions of solutions.

Keywords. analysis, fuzzy number, fuzzy differential equation, initial value problem

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1 Introduction

Let a set of fuzzy numbers with bounded supports be as follows (e.g. [3]):

$$\mathcal{F}_{\mathrm{b}} = \{\mu : \mathbf{R} \to I = [0,1] \text{ satisfying the following conditions } (i) - (iv)\}.$$

(i) The membership function μ has a unique point $m \in \mathbf{R}$ such that $\mu(m) = 1$;

(ii) $\mathrm{supp}(\mu)$ is a bounded set in $\mathbf{R}$;

(iii) μ is fuzzy convex on $\mathbf{R}$;

(iv) μ is upper semi-continuous on $\mathbf{R}$.

By the extension principle (e.g., the one due to Zadeh), the binary operation over the fuzzy numbers is nonlinear. For example it doesn't necessarily hold that $(k_1+k_2)x = k_1x + k_2x$ holds for $x \in \mathcal{F}_{\mathbf{b}}, k_i \in \mathbf{R}, i = 1, 2$.

In Section 2 we introduce some kind of representation to the fuzzy numbers so that we can easily calculate addition and difference between fuzzy numbers and scalar product as well as it seems that a set X including $\mathcal{F}_{\mathbf{b}}$ construct a Banach space with suitable addition, scalar product and norm.

In Section 3 we define differentiation and integration of fuzzy functions. In differentiation our representation of fuzzy numbers is enable to calculate addition , scalar product and difference without difficulties, but it is not easy to calculate the difference by the extension principle. Moreover we define the integral of fuzzy functions by calculating end-points of $\alpha-$cut sets.

In Section 4 we treat initial value problems of fuzzy differential equations in the type of $x' = f(t,x)$. We give existence and uniqueness theorems of the fuzzy differential equations. And also we show sufficient conditions for the continuous dependence with respect to initial conditions of solutions.

2 Representation of Fuzzy Numbers

Let $I = [0,1]$. Denote a fuzzy numbers $x \in \mathcal{F}_{\mathbf{b}}$ by $x = (a,b)$, where $a(\alpha) = \min x_\alpha, b(\alpha) = \max x_\alpha$ for $\alpha \in I$, where x_α is the $\alpha-$ cut set of x. In the case that $r \in \mathbf{R}$, we denote $r = (a,b) \in \mathcal{F}_{\mathbf{b}}$, where $a(\alpha) = b(\alpha) \equiv r$ for $0 \le \alpha \le 1$.

Define $x - y = (a-c, b-d)$ for $x = (a,b), y = (c,d) \in \mathcal{F}_{\mathbf{b}}$. Denote the set $\{x - y : x, y \in \mathcal{F}_{\mathbf{b}}\}$ by $\mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}}$. In the following definition we give ones of addition and scalar product etc.

Definition 2.1 *Let* $z = (a,b), z_1 = (a_1,b_1) \in \mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}}$.

(i) $z + z_1 = (a + a_1, b + b_1)$;

(ii) $\beta z = (\beta a, \beta b)$ *for* $\beta \in \mathbf{R}$;

(iii) Define $z = z_1$ *by* $(a(\alpha) = a_1(\alpha))$ *and* $(b(\alpha) = b_1(\alpha))$ *for* $\alpha \in I$;

(iv) The zero $0 = (a,b) \in \mathcal{F}_{\mathbf{b}}$, *where* $a(\alpha) = b(\alpha) \equiv 0$ *for* $\alpha \in I$;

(v) Let a norm $\| z \| = \sup_{\alpha \in I} \sqrt{|a(\alpha)|^2 + |b(\alpha)|^2}$.

It follows that $\mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}}$ constructs a normed space and the smallest linear space including $\mathcal{F}_{\mathbf{b}}$. Denote X by a completion of $\mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}}$.

We get properties of end-points of the $\alpha-$ cut sets of fuzzy numbers. Denote $x = (a,b) \in \mathcal{F}_{\mathbf{b}}$. The following properties(i)-(iv) hold:

(i) a is lower semi-continuous and b is upper semi-continuous on I;

(ii) a is non-decreasing with $\max a(\alpha) = a(1) = m$ and b is non-increasing with $\min b(\alpha) = b(1) = m$;

(iii) If $0 < \alpha < 1$, then it follows that $a(\alpha) < b(\alpha)$, or $a(\alpha) = b(\alpha)$;

(iv) The set $\{(a(\alpha), b(\alpha)) : \alpha \in I\} \subset \mathbf{R}^2$ is a bounded curve.

See Figure 1.

Theorem 2.1 *$\mathcal{F}_{\mathbf{b}}$ is a closed convex cone in X.*

Proof. It can be easily proved and it is omitted.

Let $X^n = \{(x_1, x_2, \cdots, x_n)^T : x_i \in X, i = 1, 2, \cdots, n\}$ and $\mathcal{F}_{\mathbf{b}}^n = \{(x_1, x_2, \cdots, x_n)^T : x_i \in \mathcal{F}_{\mathbf{b}}, i = 1, 2, \cdots, n\}$. The notation T means the transpose. Define $\| x \| = \max_{1 \le i \le n} \| x_i \|$ for $x \in X^n$. It's clear that X^n is a Banach space and that $\mathcal{F}_{\mathbf{b}}^n$ is a closed convex cone in X^n.

In [7] Puri and Ralescu introduce the following equivalence relation and norm. Let $(u, v), (u', v') \in \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{b}}$. Define an equivalence relation $\sim$ by

$$(u, v) \sim (u', v') \Longleftrightarrow u + v' = v + u'$$

so that the equivalence classes $\mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{b}} / \sim = \{\langle (u, v) \rangle : u, v \in \mathcal{F}_{\mathbf{b}}\}$ is a linear space with some addition and scalar product. Denote a norm $\| \cdot \|_{PR}$ in the linear space by $\| \langle (u, v) \rangle \| = \sup_{\alpha > 0} d_H(u_\alpha, v_\alpha)$, where $d_H(\cdot, \cdot)$ is the Hausdorff metric. Let $u = (a, b), v = (c, d), u' = (a', b'), v' = (c', d') \in \mathcal{F}_{\mathbf{b}}$. such that $\langle (u, v) \rangle = \langle (u', v') \rangle$, i.e., $a - c = a' - c'$ and $b - d = b' - d'$. Define $T(u - v) = \langle (u, v) \rangle$. Then we have

$$\begin{aligned} T(u - v) &= T((a, b) - (c, d)) \\ &= T((a - c, b - d)) \\ &= T(u' - v'), \end{aligned}$$

where $u + v' = u' + v$. Then we get the following theorem.

Theorem 2.2 *There exists a one-to-one linear mapping T such that*

$$\| Tz \|_{PR} \le \| z \| \le \sqrt{2} \| Tz \|_{PR}$$

for $z \in \mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}}$.

Proof. For $z = u - v$ we denote $T : \mathcal{F}_{\mathbf{b}} - \mathcal{F}_{\mathbf{b}} \to \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{b}} / \sim$ by $Tz = \langle (u, v) \rangle$. It follows that for $u = (a, b), v = (c, d)$

$$\begin{aligned} &\| Tz \|_{PR} \\ &= \sup_{\alpha \in I} \max\Big(\sup_{\xi_1 \in u_\alpha} \inf_{\xi_2 \in v_\alpha} \| \xi_1 - \xi_2 \|, \sup_{\xi_2 \in v_\alpha} \inf_{\xi_1 \in u_\alpha} \| \xi_1 - \xi_2 \| \Big) \\ &= \sup_{\alpha \in I} \max(|a(\alpha) - c(\alpha)|, |b(\alpha) - d(\alpha)|) \\ &\le \sup_{\alpha \in I} \sqrt{|a(\alpha) - c(\alpha)|^2 + |b(\alpha) - d(\alpha)|^2} \\ &= \| z \| \\ &\le \sqrt{2} \| Tz \|_{PR} . \end{aligned}$$

Q.E.D.

3 Fuzzy Differentiation and Fuzzy Integration

In what follows we consider a function $f : E \to Y$, where E is a subset in a normed space and Y is a normed space. In this section we give definitions of differentiation and integration of fuzzy functions.

Definition 3.1 *A function f is continuous at $p_0 \in E$, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $p \in E$ and $\| p - p_0 \| < \delta$ satisfy $\| f(p) - f(p_0) \| < \varepsilon$. It is called that f is continuous on E if f is continuous at any $p \in E$.*

Let J be an interval in $\mathbf{R}$. In what follows f is fuzzy function from J to $\mathcal{F}_{\mathbf{b}}$.

Definition 3.2 *It is called that f is differentiable at $t_0 \in J$ if there exists an $\eta \in \mathcal{F}_{\mathbf{b}}$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying*

$$\| \frac{f(t) - f(t_0)}{t - t_0} - \eta \| < \varepsilon$$

for $t \in J$ and $0 < |t - t_0| < \delta$. Denote $\eta = f'(t_0)$, $\frac{df}{dt}(t_0) = \eta$. f is differentiable on J if f is differentiable at any $t \in J$. In the similar way higher order derivatives of f are defined by $f^{(k)} = (f^{(k-1)})'$ for $k = 2, 3, \cdots$. In case that $f : J \to X$, the derivative of f is defined in the same way.

(Cf. [1, 4, 5, 8])

In [7] they define the embedding $j : \mathcal{F}_{\mathbf{b}} \to \mathcal{F}_{\mathbf{b}} \times \mathcal{F}_{\mathbf{b}} / \sim$ such that $j(u) = \langle (u, 0) \rangle$. The function $f : J \to \mathcal{F}_{\mathbf{b}}$ is said o be differentiable in the sense of Puri-Ralescu, if $j(f(\cdot))$ is differentiable. Suppose that f is differentiable at $t \in J$ in the above sense, denoted the differential $f'(t) \in \mathcal{F}_{\mathbf{b}}$. Then we have $\frac{d}{dt}(j(f(t))) = \langle (f'(t), 0) \rangle$, i.e., f is differentiable in the sense of Puri-Ralescu.

In [6, 7] H-difference and H-differentiation of f is treated as follows. Suppose that for $f(t+h), f(t) \in \mathcal{F}_{\mathbf{b}}$, there exists $g \in \mathcal{F}_{\mathbf{b}}$ such that $f(t+h) = f(t) + g$, then g is called to be the Hukuhara-difference, denoted $f(t+h) - f(t)$. The function f is said to be Hukuhara-differentiable at $t \in J$ if there exists an $\eta \in \mathcal{F}_{\mathbf{b}}$ such that both $\lim_{h \to +0} \frac{f(t+h) - f(t)}{h}$ and $\lim_{h \to +0} \frac{f(t) - f(t-h)}{h}$ exist and equal to η. If f is H-differentiable, then $f'(t) = \eta$.

Proposition 3.1 *If f is differentiable at t_0, then f is continuous at t_0.*

Proof. It is clear and the proof is omitted.

Theorem 3.1 *Suppose that f is differentiable at t_0, then it follows that there exist $\frac{\partial}{\partial t}(\min f(t)_\alpha)$, $\frac{\partial}{\partial t}(\max f(t)_\alpha)$ and that*

$$f'(t_0) = (\frac{\partial}{\partial t}(\min f(t)_\alpha)|_{t=t_0}, \frac{\partial}{\partial t}(\max f(t)_\alpha)|_{t=t_0})$$

for $\alpha \in I$, where $\min f(t)_\alpha$ and $\max f(t)_\alpha$ are left, right end-points of the α-cut set of $f(t)$, respectively.

Proof. In the same way in the proof of Theorem 2.2 in [5] it can be proved.

Theorem 3.2 *It follows that $f'(t) \equiv 0$ if and only if $f(t) \equiv const \in \mathcal{F}_{\mathbf{b}}$.*

Proof. Let $f'(t) \equiv 0$. Suppose that $f \neq const$. Therefore there exist $t_1 \neq t_2$ such that $f(t_1) - f(t_2) \neq 0$. By applying the Hahn-Banach extension theorem there exists a bounded linear functional $x^* \in X^*$ such that $\| x^* \| = 1$ and $x^*(f(t_1) - f(t_2)) = \| f(t_1) - f(t_2) \|$. Denote $\phi(t) = x^*(f(t) - f(t_1))$. Here $\phi : I \to \mathbf{R}$ is differentiable so that $\phi'(t) = x^*(f'(t)) \equiv 0$. Then we have $\phi(t_1) = x^*(f(t_1) - f(t_1)) = 0$. This contradicts with the above assumption. Thus we get $f = const$. In case that $f(t_1) - f(t_2) \notin \mathcal{F}_{\mathbf{b}}$, $f' \in X$. **Q.E.D.**

In the following definition we give one of integrals of fuzzy functions.

Definition 3.3 *Let $J = [a, b]$ and f be a mpping from J to X(or $\mathcal{F}_{\mathbf{b}}$). Divide the interval J such that $a = t_0 < t_1 < \cdots < t_n = b$ and $\tau_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \cdots, n$. It is called that f is integrable over J if there exists the limit $\lim_{|\Delta| \to 0} \sum_{i=1}^{n} f(\tau_i)\Delta_i$, where $\Delta_i = t_i - t_{i-1}, |\Delta| = \max_{1 \leq i \leq n} \Delta_i$. Define*

$$\int_a^b f(s)ds = \lim_{|\Delta| \to 0} \sum_{i=1}^{n} f(\tau_i)\Delta_i.$$

Proposition 3.2 *Let f be integrable over J. Then the following statements (i)-(ii) hold.*

(i) f is bounded on J, i.e., there exists an $M > 0$ such that $\| f(t) \| \leq M$ for $t \in J$.

(ii) If $f(t) \in \mathcal{F}_{\mathbf{b}}$ for $t \in J$, then $\int_a^t f(s)ds \in \mathcal{F}_{\mathbf{b}}$ for $t \in J$.

Proposition 3.3 *If f is continuous on $[a, b]$ then f is integrable over the interval.*

Theorem 3.3 *Let $f : J \to X$ with $f(t) = \{(c(t, \alpha), d(t, \alpha)) : \alpha \in I\}$ be integrable over $[a, b]$. Then it follows that*

$$\int_a^b f(s)ds = \{(\int_a^b c(s, \alpha)ds, \int_a^b d(s, \alpha)ds) : \alpha \in I\}.$$

Conversely, if c, d are continuous on $[a, b] \times I$, then f is integrable over $[a, b]$.

Proposition 3.4 *Let f be continuous on the interval $[a, b]$.*

Denote $F(t) = \int_a^t f(s)ds$. Then the following properties (i) and (ii) hold.

(i) F is differentiable on $[a, b]$ and $F' = f$;

(ii) For $t_1, t_2 \in [a, b]$ and $t_1 \leq t_2$, we have $\int_{t_1}^{t_2} f(s)ds = F(t_2) - F(t_1)$.

Proposition 3.5 *Let f is continuous on $[a,b]$. Then it follows that*

$$\| \int_a^b f(s)ds \| \leq \int_a^b \| f(s) \| ds.$$

Theorem 3.4 *Let $f : [a,b] \to \mathcal{F}_{\mathrm{b}}$ be continuous on $[a,b]$ and differentiable on (a,b), Then it follows that there exists a number $c \in (a,b)$ such that*

$$\| f(b) - f(a) \| \leq (b-a) \| f'(c) \| .$$

Proof. Suppose that $\| f(b) - f(a) \| \neq 0$ without loss of generality. From the Hanh-Banach extension theorem there exists a bounded linear functional $x^* \in X^*$ such that $\| x^* \| = 1$ and $x^*(f(b) - f(a)) = \| f(b) - f(a) \|$. Denote $\phi(t) = x^*(f(t))$, which is differentiable function from (a,b) to $\mathbf{R}$ and $\phi'(t) = x^*(f'(t))$. Then we have

$$\begin{aligned} \frac{x^*(f(b) - f(a))}{b-a} &= \frac{\phi(b) - \phi(a)}{b-a} \\ &= \phi'(c) = x^*(f'(c)) \end{aligned}$$

for $c \in (a,b)$. From $\| x^*(f'(c)) \| \leq \| f'(c) \|$, the conclusion holds.
Q.E.D.

Definition 3.4 *Let $f : J \to \mathcal{F}_{\mathrm{b}}^n$ such that $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T$. It is called that f is differentiable on J if each f_i is differentiable on J for $i = 1,2,\cdots,n$. Define the derivative $f'(t) = (f_1'(t), f_2'(t), \cdots, f_n'(t))^T$.*

Let $f : [a,b] \to X^n$ such that $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T$. It is called that f is integrable over $[a,b]$ if f_i is integrable over $[a,b]$ for $i = 1,2,\cdots,n$. Define the integral $\int_a^b f(s)ds = (\int_a^b f_1(s)ds, \int_a^b f_2(s)ds, \cdots, \int_a^b f_n(s)ds)^T$.

It is easily seen that similar theorems and propositions concerning to $\mathcal{F}_{\mathrm{b}}^n$−valued functions to ones in this section hold.

4 Fuzzy Differential Equations

In this section we consider the initial value problems of the following type of fuzzy differential equation

$$x'(t) = f(t, x(t)) \tag{4.1}$$

$$x(t_0) = x_0. \tag{4.2}$$

Here $f : \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \to \mathcal{F}_{\mathrm{b}}^n, t_0 \in \mathbf{R}, x_0 \in \mathcal{F}_{\mathrm{b}}^n$. We mean that a solution $x : J \to \mathcal{F}_{\mathrm{b}}^n$ satisfies the above equation and initial condition of ((4.1),(4.2)), where $J \subset \mathbf{R}$ is an interval.

We denote the initial value problem of higher order fuzzy differential equations by

$$x^{(n)} = f(t, x(t), x'(t), \cdots, x^{(n-1)}(t)) \tag{4.3}$$

$$x^{(k)}(t_0) = \xi_k, \quad k = 0,1,\cdots,n-1,$$

where $f : \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \to \mathcal{F}_{\mathrm{b}}, t_0 \in \mathbf{R}, \xi_k \in \mathcal{F}_{\mathrm{b}}$. We mean that a solution $x : J \to \mathcal{F}_{\mathrm{b}}$ satisfies the above equation and conditions for $t \in J$, where $J \subset \mathbf{R}$ is an interval. Define $x_1(t) = x(t), x_2(t) = x'(t), \cdots, x_n(t) = x^{(n-1)}(t)$ so that the above problem can be reduced to Problem ((4.1),(4.2)). In this section we show some kinds of conditions to solutions of ((4.1),(4.2)) for the existence, uniqueness and continuation.

Definition 4.1 *Define a norm* $\| p \| = \max(|t|, \| x \|)$ *for* $p = (t, x) \in \mathbf{R} \times X^n$. *Let* $p_0 \in \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$. *Denote a neighborhood of* p_0 *by* $U(p_0, \delta) = \{p \in \mathbf{R} \times X^n : \| p - p_0 \| < \delta\}$ *and a relative neighborhood of* p_0 *by* $V(p_0, \delta) = U(p_0, \delta) \cap (\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n)$ *for* $\delta > 0$. *Let* $V \subset \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$. *It is called that* V *is a relatively open subset in* $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$, *if for any* $p \in V$ *there exists a relative neighborhood* $V(p) \subset \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$ *such that* $V(p) \subset V$. *In the similar way we define relatively open subsets in* $\mathcal{F}_{\mathrm{b}}^n, \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}, \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}$.

Let a function $f : V \to \mathcal{F}_{\mathrm{b}}^n$, *where* V *is a relatively open subset in* $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$. *It is called that* f *satisfies a locally Lipschitz condition if for any* $p = (t_0, x_0) \in V$ *there exists a relative neighborhood* $V(p) \subset V$ *and a number* $L_p > 0$ *such that*

$$\| f(t, x_1) - f(t, x_2) \| \le L_p \| x_1 - x_2 \|$$

for $(t, x_1), (t, x_2) \in V(p)$.

Theorem 4.1 *Let* $f : V \to \mathcal{F}_{\mathrm{b}}^n$ *satisfy the locally Lipschitz condition and be continuous on* V. *Then there exists one and only one solution* x *of ((4.1),(4.2)) defined on* $[t_0, t_0 + r]$ *passing through* $p = (t_0, x_0) \in V$, *where* $r > 0$.

Proof. From the Lipschitz condition and continuity of f it follows that there exists an $M > 0$ such that $\| f(t, x) \| \le M$ for $(t, x) \in V(p)$, which is the relative neighborhood in Definition 4.1. Denote a subset

$$A = \{(t, x) \in \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n : t \in [t_0, t_0 + \rho], \| x - x_0 \| \le k\} \subset V(p),$$

where sufficiently small $\rho > 0, k > 0$. Let $r = \min(\rho, k/M, \frac{1}{2L_p})$.

Let $I_r = [t_0, t_0 + r]$. There exists a solution x of ((4.1),(4.2)), which has a continuous derivative x' for $t \in I_r$, if and only if there exists a continuous solution x of an integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$ for $t \in I_r$. We shall show the existence of solution of the integral equation. A set $C(I_r, X^n) = \{x : I_r \to X^n$ are continuous $\}$ is a Banach space with the norm $\| x \|_\infty = \sup_{t \in I_r} \| x(t) \|$. Denote $S_k = \{x \in C(I_r, \mathcal{F}_{\mathrm{b}}^n) : \| x - x_0 \|_\infty \le k\}$, which is a closed subset in $C(I_r, X^n)$. Then we have $(t, x(t)) \in A \subset V(p) \subset V$ for $x \in S_k, t \in I_r$ and there exists $f(t, x(t))$ on I_r. Define a mapping $T : S_k \to C(I_r, \mathcal{F}_{\mathrm{b}}^n)$ by

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds,$$

where $t \in I_r$. Then $\| Tx - x_0 \|_\infty \le k$ so that T is an into mapping on S_k. Moreover $\| Tx_1 - Tx_2 \|_\infty \le 2^{-1} \| x_1 - x_2 \|_\infty$ for $x_i \in S_k, i = 1, 2$. Thus T is a contraction mapping on S_k. There exists a unique point $x \in S_k$, which satisfies ((4.1),(4.2)). **Q.E.D.**

Theorem 4.2 *Suppose that the same conditions as Theorem 4.1 hold. Let functions $x, y : J \to \mathcal{F}_b^n$ be solutions of ((4.1),(4.2)), where $J = [t_0, T)$ and $T > t_0$. Then $x(t) = y(t)$ for $t \in J$.*

Proof. Suppose that there exists $t_1 \in J$ such that $x(t_1) \neq y(t_1)$. Denote $A = \{t \in I : x(t) \neq y(t)\}$ and $t_0^* = \inf A$. From Theorem 4.1 there exists a number $r > 0$ such that $x(t) = y(t)$ for $t \in [t_0^*, t_0^* + r]$. This leads to a contradiction. Thus the theorem holds.
Q.E.D.

Suppose that the same conditions of Theorem 4.1 hold. Denote an interval $\mathbf{J} = \{[t_0, T) \in \mathbf{R} :$ there exits a solution x of ((4.1),(4.2)) on $[t_0, T)\}$. For $J \in \mathbf{J}$ there exists a unique solution of ((4.1),(4.2)) on J. Denote $J(t_0, x_0) = \cup_{J \in \mathbf{J}} J$ and $x_f(t_0, x_0, t) = x_J(t)$ for $t \in J \in \mathbf{J}$. For $t \in J(t_0, x_0)$ there exists a unique value $x_J(t)$. The function $x_f : V \times J(t_0, x_0) \to \mathcal{F}_b^n$ is said to be the solution of ((4.1),(4.2)) with the maximal interval $J(t_0, x_0)$. Denote a mapping $x_f : \mathbf{R} \times \mathcal{F}_b^n \times \mathbf{R} \to \mathcal{F}_b^n$ defined on $D(f) = \{(t_0, x_0, t) : (t_0, x_0) \in V, t \in J(t_0, x_0)\}$. See [9].

Theorem 4.3 *Suppose that the same conditions of Theorem 4.1 hold. Let $J = [t_0, T] \subset J(t_0, x_0) \cap J(t_0, x_0')$, where $T > t_0$. Then there exists an $M > 0$ such that*

$$\| x_f(t_0, x_0', t) - x_f(t_0, x_0, t) \| \leq M \| x_0' - x_0 \|$$

for $t \in J$.

Proof. Let $\phi(t) = x_f(t_0, x_0, t), \psi(t) = x_f(t_0, x_0', t)$for $t \in J$. Then we have $\phi, \psi \in C(J, \mathcal{F}_b^n)$. From condition of f and compactness of J, there exists a number $L > 0$ such that

$$\| f(t, \psi(t)) - f(t, \phi(t)) \| \leq L \| \psi(t) - \phi(t) \|$$

for $t \in J$. So we have

$$\| \psi(t) - \phi(t) \| \leq \| x_0 - x_0' \| + L \| \psi - \phi \|_\infty (t - t_0)$$

for $t \in J$. In the same way we get

$$\begin{aligned} &\| \psi(t) - \phi(t) \| \\ &\leq \| x_0 - x_0' \| \sum_{k=0}^{n} \frac{(L(T - t_0))^k}{k!} + \frac{(L(T - t_0))^{n+1}}{(n+1)!} \| \psi - \phi \|_\infty . \end{aligned}$$

Put $M = e^{L(T - t_0)}$, then the above conclusion holds. **Q.E.D.**

Consider the following fuzzy differential equation

$$x'(t) = f(x(t)). \tag{4.4}$$

Corollary 4.1 *Let $f : V \to \mathcal{F}_b^n$ satisfy the locally Lipschitz condition on V, where $V \subset \mathcal{F}_b^n$ is a relatively open subset. Then there exists one and only one solution x of ((4.4),(4.2)) defined on $[t_0, t_0 + r]$ passing through $t_0 \in \mathbf{R}$ and $p = x_0 \in V$, where $r > 0$.*

From the similar discussion to ((4.4),(4.2)) the maximal interval $J(t_0.x_0)$ and the corresponding to solution x_f can be defined. (see [9]). It can be seen that

$$\begin{aligned} J(t_0, x_0) &= J(0, x_0) + t_0 \\ &= \{t + t_0 : t \in J(0, x_0)\} \end{aligned}$$

for $(t_0, x_0) \in \mathbf{R} \times V$. and for $t \in J(t_0, x_0)$ we get

$$x_f(t_0, x_0, t) = x_f(0, x_0, t - t_0).$$

Thus we denote $J(x_0) = J(0, x_0), x_f(x_0, t) = x_f(0, x_0, t)$ and $D_0(f) = \{(x_0, t) \in V \times J(x_0)\}$.

Theorem 4.4 *The same conditions of Corollary 4.1 hold. Then* $D_0^+(f) = \{(x_0, t) \in D_0(f) : t > 0\}$ *is a relatively open subset in* $\mathcal{F}_{\mathbf{b}}^n \times \mathbf{R}$ *and the mapping* x_f *is continuous on* $D_0(f)$.

Proof. Let $(x_0^*, t^*) \in D_0(f)$. There exists $r > 0$ such that $J = [0, t^* + r] \subset J(x_0^*)$. Since the set $B_J = \{x_f(x_0^*, t) : t \in J\}$ is compact, there exists $\delta > 0$ such that $B_J(\delta) = \{\xi \in \mathcal{F}_{\mathbf{b}}^n : dist(\xi, B_J) \leq \delta\} \subset V$, so that f satisfies the locally Lipschitz condition with the constant $L > 0$. We shall prove the existence and uniqueness of solutions for the integral equation $x(t) = x_0 + \int_0^t f(x(s))ds$ for $x_0 \in \mathcal{F}_{\mathbf{b}}^n$ satisfying $\| x_0 - x_0^* \| \leq \rho/2$ and $t \in J$, where $\rho = \delta e^{-2L(t^*+r)}$. Denote a norm in $C(J, X^n)$ by $\| x \|_L = \max\{\| x(t) \| e^{-2Lt} : t \in J\}$ and $x^0(t) = x_f(x_0^*, t)$. Let $S_\rho = \{x \in C(J, \mathcal{F}_{\mathbf{b}}^n) : \| x - x^0 \|_L \leq \rho\}$ which is a closed subset in the Banach space $C(J, X^n)$. Then we have $x(t) \in B_J(\delta) \subset V$ for $x \in S_\rho, t \in J$ and there exists $f(x(t))$ on J. Define a mapping $T_{x_0} : S_\rho \to C(J, \mathcal{F}_{\mathbf{b}}^n)$ such that $(T_{x_0}(x))(t) = x_0 + \int_0^t f(x(s))ds$ for $t \in J$. Since $\| (T_{x_0}(x))(t) - x^0(t) \| \leq \rho e^{2Lt}/2$, $T_{x_0}(x) \in S_\rho$ and T_{x_0} is a contraction mapping, because

$$\| (T_{x_0}(x_1))(t) - (T_{x_0}(x_2))(t) \| \leq \frac{e^{2Lt}}{2} \| x_1 - x_2 \|_L,$$

so that $\| T_{x_0}(x_1) - T_{x_0}(x_2) \|_L \leq \frac{1}{2} \| x_1 - x_2 \|_L$ for $x_i \in S_\rho, i = 1, 2$. Thus there exists a unique solution of the integral equation as well as $(x_0, t) \in D_0(f)$ for $x_0 \in \mathcal{F}_{\mathbf{b}}^n$ satisfying $\| x_0 - x_0^* \| \leq \rho/2$ and $t \in J$. Therefore $D_0^+(f) \subset \mathcal{F}_{\mathbf{b}}^n \times \mathbf{R}$ is a relatively open subset.

We have for $\| x_0 - x_0^* \| \leq \rho/2, t \in J$,

$$\begin{aligned} &\| x_f(x_0, t) - x_f(x_0^*, t^*) \| \\ &\leq \| x_f(x_0, t) - x_f(x_0^*, t) \| + \| x_f(x_0^*, t) - x_f(x_0^*, t^*) \| . \end{aligned}$$

Since $x_f(x_0, \cdot), x_f(x_0^*, \cdot) \in S_\rho$ are fixed points of $T_{x_0}, T_{x_0^*}$, respectively, it follows that

$$\begin{aligned} &\| x_f(x_0, t) - x_f(x_0^*, t) \| \\ &\leq \| x_0 - x_0^* \| + \| \int_0^t (f(x_f(x_0, s)) - f(x_f(x_0^*, s)))ds \| \\ &\leq \| x_0 - x_0^* \| + L \int_0^t \| x_f(x_0, s) - x_f(x_0^*, s) \| ds \end{aligned}$$

for $t \in J$. From the Gronwall's Lemma (e.g., [2]), we get

$$\begin{aligned} \| x_f(x_0,t) - x_f(x_0^*,t) \| &\leq \| x_0 - x_0^* \| e^{L\int_0^t ds} \\ &\leq \| x_0 - x_0^* \| e^{L(t^*+r)}. \end{aligned}$$

Thus $x_f(x_0,t)$ is continuous in $(x_0,t) \in D_0(f)$.
Q.E.D.

Condition (L) For any $p = (t_0, x_0) \in V$ there exists a relative neighborhood $V(p) \subset V$ and a number $L_p > 0$ such that

$$\| f(t_1,x_1) - f(t_2,x_2) \| \leq L_p \| (t_1,x_1) - (t_2,x_2) \|$$

for $(t_1,x_1),(t_2,x_2) \in V(p)$.

It is said that $y : J \to \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n$ is differentiable at $t \in J$ if

$$y(t+h) = y(t) + \zeta h + o(h)$$

as $h \to 0$, where $\zeta \in \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n$ and $o(h)/h \to 0$, denoted $\zeta = y'(t)$.

Theorem 4.5 *Consider Problem ((4.1),(4.2)). Let $f : V \to \mathcal{F}_{\mathbf{b}}^n$ satisfy Condition (L), where V is a relatively open subset in $\mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n$. Then $D^+(f) = \{(t_0,x_0,t) \in D(f) : t > t_0\}$ is a relatively open subset in $\mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n \times \mathbf{R}$ and the mapping x_f is continuous on $D(f)$.*

Proof. Let x be the solution of ((4.1),(4.2)) defined on $J = [t_0, T)$, where $T > t_0$. We denote mapping $y = (y_1,y_2) : J \to \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n$ such that $y_1(t) = t, y_2(t) = x(t)$ and mapping $g : V \to \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n$ such that $g(\eta) = (1, f(\eta))$, where $\eta \in V$. Then $y = (y_1,y_2)$ satisfies $y' = g(y(t))$ for $t \in J$ and $y(t_0) = (t_0,x_0)$. Conversely if y satisfies the above equation and initial condition, then $x = y_2$ is the solution of ((4.1),(4.2)).

Denote the solution of $y' = g(y), y(\tau) = (t_0,x_0)$ with the maximal interval, which satisfies $y = (y_1,y_2)$ such that

$$y_1(\tau,t_0,x_0,t) = t_0 - \tau + t \tag{4.5}$$
$$y_2(t_0,t_0,x_0,t) = x_f(t_0,x_0,t). \tag{4.6}$$

Since $y(\tau,t_0,x_0,t) = y(0,t_0,x_0,t-\tau)$, we have

$$x_f(t_0,x_0,t) = y_2(0,t_0,x_0,t-t_0). \tag{4.7}$$

The function $y(0,t_0,x_0,t)$ exists on $D_0(g)$ so that x_f exists on

$$D(f) = \{(t_0,x_0,t) \in \mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n \times \mathbf{R} : (t_0,x_0,t-t_0) \in D_0(g)\}.$$

Denote an into mapping Φ on $\mathbf{R} \times \mathcal{F}_{\mathbf{b}}^n \times \mathbf{R}$ such that $\Phi(t_0,x_0,t) = (t_0,x_0,t-t_0)$. Then it follows that

$$D(f) = \Phi^{-1}(D_0(g)), \quad D^+(f) = \Phi^{-1}(D_0^+(g)). \tag{4.8}$$

Since $D_0^+(g)$ is a relatively open subset in $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}$ and Φ is continuous, $D^+(f)$ is relatively open in $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}$. From Theorem 4.4, (4.7) and (4.8), x_f is continuous on $D(f)$.
Q.E.D.

In the following Corollary we assume that $f : U \to \mathcal{F}_{\mathrm{b}}^n$ satisfies some properties corresponding to the smoothness in the sense of Fréchet, where U is a open subset in $\mathbf{R} \times X^n$ such that $U \cap (\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n) \neq \emptyset$.

Property (P). It follows that there exists the product $T(y)\Delta \in \mathcal{F}_{\mathrm{b}}^n$such that

$$f(y + \Delta) = f(y) + T(y)\Delta + o(\| \Delta \|)$$

as $\| \Delta \| \to 0$, where $y+\Delta \in U$ and the above product means the one of extension principle. Denote the derivative $f'(y) = T(y)$. Suppose that f' is continuous on U.

Corollary 4.2 *Let f in (4.1) satisfy Property (P). Then $D^+(f)$ is a relatively open subset in $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}$ and x_f is continuous on $D(f)$.*

Proof. It can be easily seen that $\| f(y + \Delta) - f(y) \| \leq \| \Delta \| \sup_{0 \leq \alpha \leq 1} \| f'(y + \alpha\Delta) \|$. Since f satisfy Condition (L), the conclusion holds by Theorem 4.5. **Q.E.D.**

Theorem 4.6 *Consider Problem ((4.3)). Let $f : V \to \mathcal{F}_{\mathrm{b}}$ satisfy the locally Lipschitz condition and be continuous, where $V \subset \mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n$ is a relatively open subset. Then for $(t_0, \xi_1, \xi_2, \cdots, \xi_n) \in V$ there exists one and only one solution x_f of (4.3) on the maximal interval. Moreover if f satisfies Condition (L), then the set $D^+(f) = \{(t_0, \xi_1, \xi_2, \cdots, \xi_n, t) \in D(f) : t > t_0\}$ is a relatively open subset in $\mathbf{R} \times \mathcal{F}_{\mathrm{b}}^n \times \mathbf{R}$. and the mapping x_f is continuous on $D(f)$.*

Proof. It can be proved in the similar way as the proof of Theorem 4.5.

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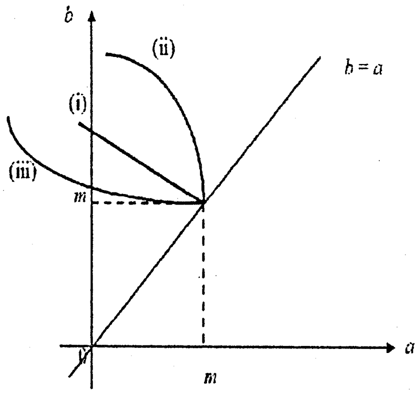

Figure 1: Fuzzy numbers $x = (a, b)$ in the following cases(i)-(iii).
(i) $b - m = c_1(m - a), \quad c_1 > 0;$
(ii) $(b - m)^2 = c_2(m - a), \quad c_2 > 0;$
(iii) $b - m = c_3(a - m)^2, \quad c_3 > 0.$