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# A monotone convergence theorem for a sequence of convex fuzzy sets on $\mathbb{R}^n$

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**Abstract**

In this paper, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preccurlyeq_K$ induced by a closed convex cone $K$ in $\mathbb{R}^n$. Our study is carried out by restricting the class of fuzzy sets into the subclass in which $\preccurlyeq_K$ becomes a partial order and a monotone convergence theorem is proved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class. These results are applied to obtain the limit theorem for a sequence of fuzzy sets defined by the dynamic fuzzy system with a monotone fuzzy relation.

*Keywords*: Pseudo-order, fuzzy max order, multidimensional fuzzy sets, monotone convergence theorem, determining class, dynamic fuzzy system.

## 1. Introduction and Notations

A convergence theorem for a sequence of fuzzy sets is mathematically interesting and applicable to sequential decision analysis in a fuzzy environment. In fact, the limiting behavior of fuzzy states of dynamic fuzzy system or sequential fuzzy decision process have been studied by developing a suitable convergence theorem of a sequence of fuzzy sets. (cf. [4, 5, 6, 14, 15, 16, 17]) Also, the theory of metric space of fuzzy sets has been developed by many authors (cf. [2, 9, 13]), in which several convergence theorems of fuzzy sets are given. On the other hand, in multiple criteria decision making, the rewards from dynamic system are described in terms of fuzzy sets and the model is often optimized under some order or pseudo order relation among fuzzy sets. In this case, it is more important to study the convergence theorem related to fuzzy order relation.

Recently, Kurano et al [7] have introduced a pseudo order $\preccurlyeq_K$ in the class of fuzzy sets on an $n$-dimensional Euclidian space $\mathbb{R}^n$, which is natural extension of fuzzy max order (cf. [3], [11]) in fuzzy numbers on $\mathbb{R}$ and induced by a closed convex cone $K$ in $\mathbb{R}^n$. For a lattice-structure of the fuzzy max order, see [1], [19]. Here, we study the convergence of a sequence of fuzzy sets on $\mathbb{R}^n$ which is monotone w.r.t. a pseudo order $\preccurlyeq_K$. Our study is done by restricting the class of fuzzy sets into the subclass in which $\preccurlyeq_K$ becomes a partial order and a monotone convergence theorem is proved. This restricted subclass of fuzzy sets is created and characterized in the concept of a determining class. These results are applied to obtain the limit theorem for a sequence of fuzzy states defined by the dynamic fuzzy system with a monotone fuzzy relation.

In the remainder of this section, we will give some notations and basic concepts of fuzzy sets and review a vector ordering of $\mathbb{R}^n$ by a convex cone. In Section 2, a pseudo order of fuzzy sets on $\mathbb{R}^n$ is reviewed referring to Kurano et al [7] and the related new results are given. In Section 3, we introduce a concept of determining class and give a convergence theorem for a sequence of convex compact subclass $\mathbb{R}^n$. In Section 4, these results are applied to obtain a monotone convergence theorem for fuzzy sets on $\mathbb{R}^n$. In Section 5, we consider the limit of a sequence of fuzzy states defined by the monotone dynamic fuzzy system.

We write fuzzy sets on $\mathbb{R}^n$ by their membership functions $\widetilde{s} : \mathbb{R}^n \to [0,1]$ (see Novák [10] and Zadeh [18]). The $\alpha$-cut ($\alpha \in [0,1]$) of the fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ is defined as

$$\widetilde{s}_\alpha := \{x \in \mathbb{R}^n \mid \widetilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \widetilde{s}_0 := \mathrm{cl}\{x \in \mathbb{R}^n \mid \widetilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\widetilde{s}$ is called convex if

$$\widetilde{s}(\lambda x + (1-\lambda)y) \geq \widetilde{s}(x) \wedge \widetilde{s}(y) \quad x, y \in \mathbb{R}^n, \ \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\widetilde{s}$ is convex if and only if the $\alpha$-cut $\widetilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets whose membership functions $\widetilde{s} : \mathbb{R}^n \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^n} \widetilde{s}(x) = 1$) and have a compact support. In the one-dimensional case $n = 1$, $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$, and $\mathcal{C}_r(\mathbb{R}^n)$ be the set of all rectangles in $\mathbb{R}^n$. For $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we have $\widetilde{s}_\alpha \in \mathcal{C}(\mathbb{R}^n)$ ($\alpha \in [0,1]$). We write a rectangle in $\mathcal{C}_r(\mathbb{R}^n)$ by

$$[x, y] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n]$$

for $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ with $x_i \leq y_i$ ($i = 1, 2, \cdots, n$). For the case of $n = 1$, $\mathcal{C}(\mathbb{R}) = \mathcal{C}_r(\mathbb{R})$ and it denotes the set of all bounded closed intervals. When $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ satisfies $\widetilde{s}_\alpha \in \mathcal{C}_r(\mathbb{R}^n)$ for all $\alpha \in [0,1]$, $\widetilde{s}$ is called a rectangle-type. We denote by $\mathcal{F}_r(\mathbb{R}^n)$ the set of all rectangle-type fuzzy sets on $\mathbb{R}^n$. Obviously $\mathcal{F}_r(\mathbb{R}) = \mathcal{F}(\mathbb{R})$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$(\widetilde{s} + \widetilde{r})(x) := \sup_{\substack{x_1, x_2 \in \mathbb{R}^n \\ x_1 + x_2 = x}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\}, \tag{1.1}$$

$$(\lambda \widetilde{s})(x) := \begin{cases} \widetilde{s}(x/\lambda) & \text{if } \lambda > 0 \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^n), \tag{1.2}$$

where $\mathbf{1}_{\{\cdot\}}(\cdot)$ is an indicator.

By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$, the following holds immediately.

$$(\widetilde{s} + \widetilde{r})_\alpha := \widetilde{s}_\alpha + \widetilde{r}_\alpha \quad \text{and} \quad (\lambda \widetilde{s})_\alpha = \lambda \widetilde{s}_\alpha \quad (\alpha \in [0,1]). \tag{1.3}$$

We need a representative theorem (cf. [7, 10]).

**The representative theorem:**

(i) *For any* $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}(x) = \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{\widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}^n$.

(ii) *Conversely, for a family of subsets* $\{D_\alpha \in \mathcal{C}(\mathbb{R}^n) \mid 0 \le \alpha \le 1\}$ *with* $D_\alpha \subset D_{\alpha'}$ *for* $\alpha' \le \alpha$ *and* $\cap_{\alpha'<\alpha} D_{\alpha'} = D_\alpha$, *if we set* $\widetilde{s}(x) := \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{D_\alpha}(x)\}, \ x \in \mathbb{R}^n$ *then* $\widetilde{s}$ *belongs to* $\mathcal{F}(\mathbb{R}^n)$ *and satisfies* $\widetilde{s}_\alpha = D_\alpha, \ \alpha \in [0,1]$.

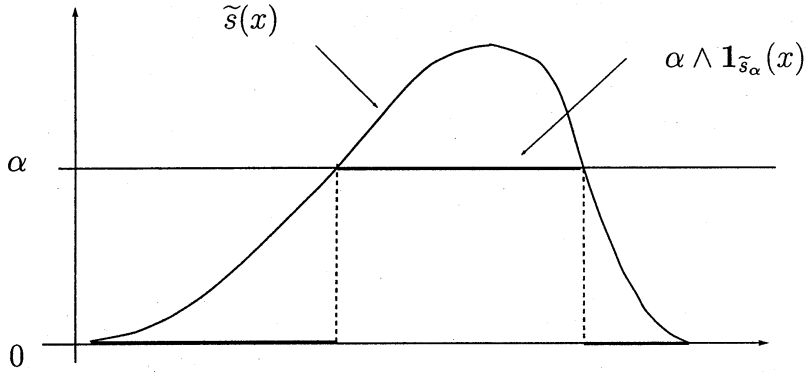


Figure 3: $\widetilde{s}(x) = \sup_{\alpha\in[0,1]} \{\alpha \wedge \mathbf{1}_{\widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}^n$

## 2. A Pseudo-Order on $\mathcal{F}(\mathbb{R}^n)$

In this section, we review a pseudo order introduced by [7] and give a related result necessary in the sequel. Henceforth we assume that the convex cone $K \subset \mathbb{R}^n$ is given. A pseudo order $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ is defined, whose idea is based on a set-relation treated in [8], as follows.

*For* $A, B \in \mathcal{C}(\mathbb{R}^n)$, $A \preccurlyeq_K B$ *means the following* (C.a) *and* (C.b) :

(C.a) *For any* $x \in A$, *there exists* $y \in B$ *such that* $x \preccurlyeq_K y$.

(C.b) *For any* $y \in B$, *there exists* $x \in A$ *such that* $x \preccurlyeq_K y$.

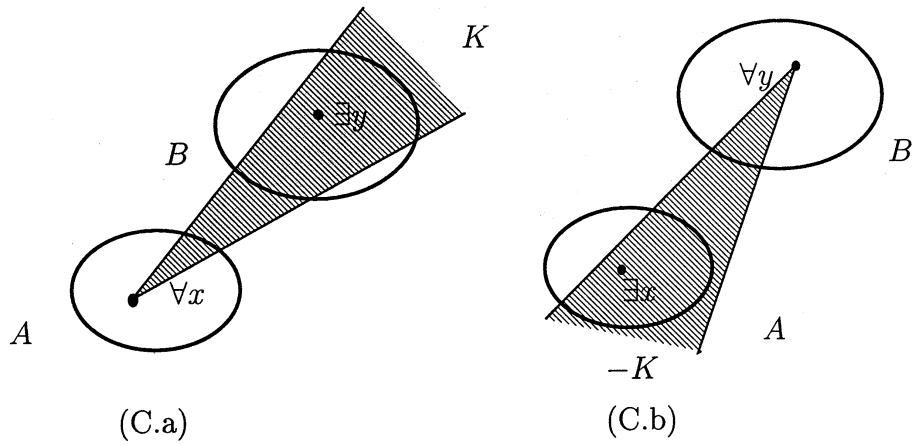


Figure 4: The binary relation $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^2)$

When $K = \mathbb{R}^n_+$, the relation $\preccurlyeq_K$ on $\mathcal{C}(\mathbb{R}^n)$ will be written simply by $\preccurlyeq_n$ with some abuse of notation and for $[x, y]$, $[x', y'] \in \mathcal{C}_r(\mathbb{R}^n)$, $[x, y] \preccurlyeq_n [x', y']$ means $x \preccurlyeq_n x'$ and $y \preccurlyeq_n y'$. Note that $\preccurlyeq_n$ on $\mathcal{C}(\mathbb{R}^n)$ is partial order.

Using a pseudo order $\preccurlyeq_K$ on $\mathbb{R}^n$, a pseudo order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ is defined as follows.

*For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ means the following* (F.a) *and* (F.b)*:*

(F.a) *For any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \leq \widetilde{r}(y)$.*

(F.b) *For any $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $x \preccurlyeq_K y$ and $\widetilde{s}(x) \geq \widetilde{r}(y)$.*

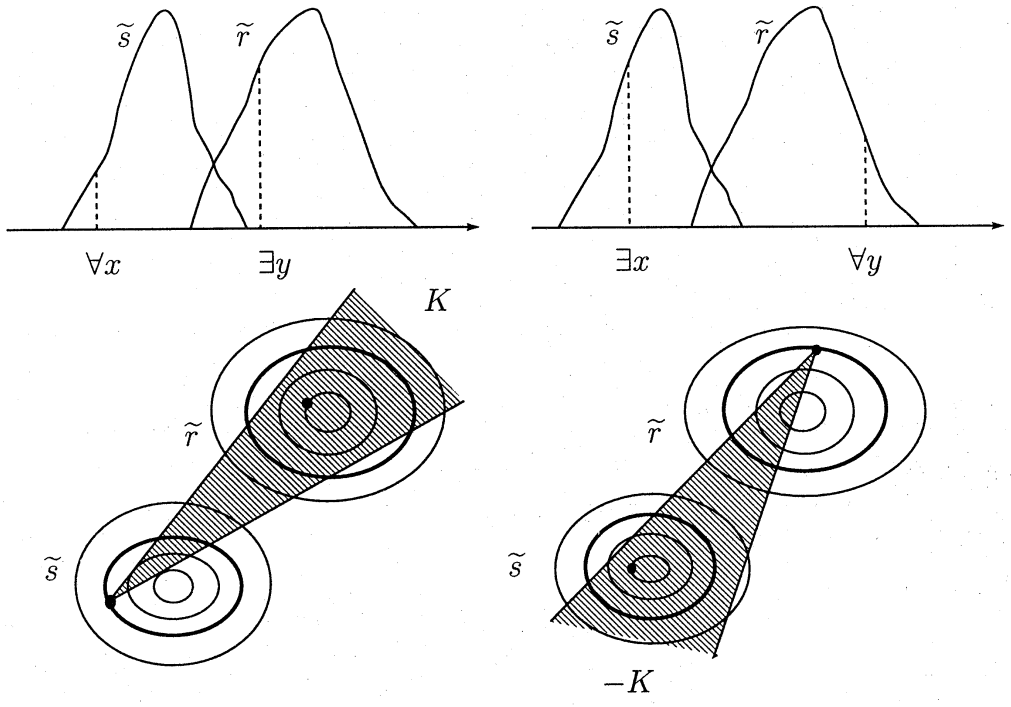


Figure 5: The binary relation $\widetilde{s} \preccurlyeq_K \widetilde{r}$ on $\mathcal{F}(\mathbb{R})$ and $\mathcal{F}(\mathbb{R}^2)$

The following lemma says the correspondence between the pseudo order on $\mathcal{F}(\mathbb{R}^n)$ and the pseudo order on $\mathcal{C}(\mathbb{R}^n)$ for the $\alpha$-cuts.

**Lemma 2.1**([7]). *Let $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$. $\widetilde{s} \preccurlyeq_K \widetilde{r}$ on $\mathcal{F}(\mathbb{R}^n)$ if and only if $\widetilde{s}_\alpha \preccurlyeq_K \widetilde{r}_\alpha$ on $\mathcal{C}(\mathbb{R}^n)$ for all $\alpha \in (0, 1]$.*

Define the dual cone of a cone $K$ by

$$K^+ := \{a \in \mathbb{R}^n \mid a \cdot x \geq 0 \text{ for all } x \in K\},$$

where $x \cdot y$ denotes the inner product on $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. For a subset $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we define

$$a \cdot A := \{a \cdot x \mid x \in A\} \ (\subset \mathbb{R}). \tag{2.1}$$

The equation (2.1) means the projection of $A$ on the extended line of the vector $a$ if $a \cdot a = 1$. It is trivial that $a \cdot A \in \mathcal{C}(\mathbb{R})$ if $A \in \mathcal{C}(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$.

**Lemma 2.2**([7]). *Let $A, B \in \mathcal{C}(\mathbb{R}^n)$. $A \preccurlyeq_K B$ on $\mathcal{C}(\mathbb{R}^n)$ if and only if $a \cdot A \preccurlyeq_1 a \cdot B$ on $\mathcal{C}(\mathbb{R})$ for all $a \in K^+$.*

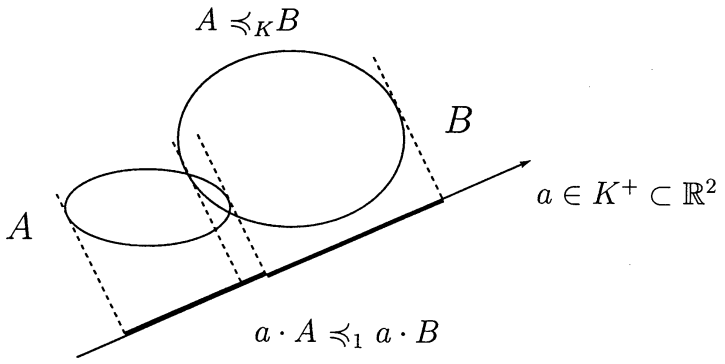


Figure 6: The image of Lemma 2.2

For $a \in \mathbb{R}^n$ and $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, we define a fuzzy number $a \cdot \widetilde{s} \in \mathcal{F}(\mathbb{R})$ by

$$a \cdot \widetilde{s}(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, \mathbf{1}_{a \cdot \widetilde{s}_\alpha}(x)\}, \quad x \in \mathbb{R}. \tag{2.2}$$

The following theorem gives the correspondence between the pseudo-order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^n)$ and the fuzzy max order $\preccurlyeq_1$ on $\mathcal{F}(\mathbb{R})$.

**Lemma 2.3**([7]). *For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ if and only if $a \cdot \widetilde{s} \preccurlyeq_1 a \cdot \widetilde{r}$ for all $a \in K^+$.*

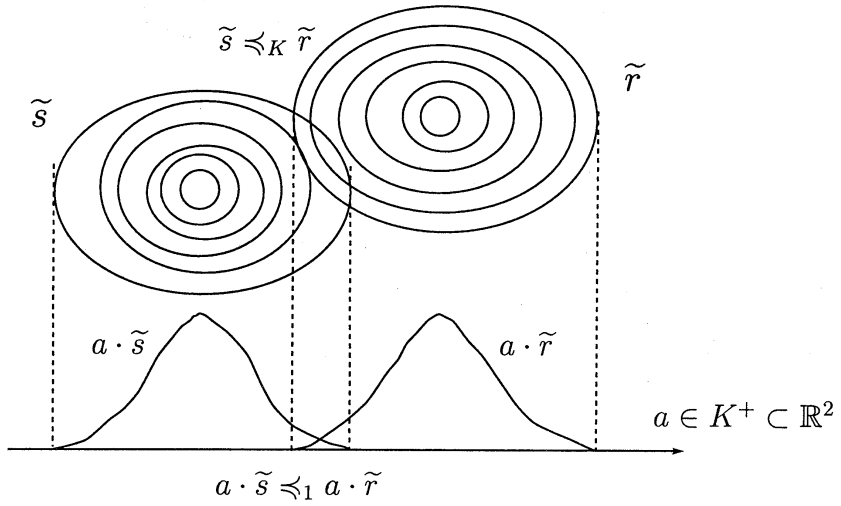


Figure 7: The image of Lemma 2.3

A closed cone $K$ is said to be acute (cf. [12]) if there exists an $a \in \mathbb{R}^n$ such that $a \cdot x > 0$ for all $x \in K$ with $x \neq 0$.

We have the following lemma.

**Lemma 2.4.** *Let $K$ be a closed, acute convex cone and $x_0, y_0 \in \mathbb{R}^n$ with $x_0 \preccurlyeq_K y_0$. Then, $(x_0 + K) \cap (y_0 - K)$ is nonempty and bounded.*

Let $\rho$ be the Hausdorff metric on $\mathcal{C}(\mathbb{R}^n)$, that is, for $A, B \in \mathcal{C}(\mathbb{R}^n)$, $\rho(A,B) = \max_{a\in A} d(a,B) \vee \max_{b\in B} d(b,A)$, where $d$ is a metric in $\mathbb{R}^n$ and $d(x,Y) = \min_{y\in Y} d(x,y)$ for $x \in \mathbb{R}^n$ and $Y \in \mathcal{C}(\mathbb{R}^n)$. It is well-known that $(\mathcal{C}(\mathbb{R}^n), \rho)$ is a complete metric space. A sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^n)$ converges to $D \in \mathcal{C}(\mathbb{R}^n)$ w.r.t. $\rho$ if $\rho(D_\ell, D) \to 0$ as $\ell \to \infty$.

**Definition**(Convergence of fuzzy set, [17]).
*For $\{\widetilde{s}_\ell\}_{\ell=1}^{\infty} \subset \mathcal{F}(\mathbb{R}^n)$ and $\widetilde{r} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}_\ell$ converges to $\widetilde{r}$ w.r.t. $\rho$ if $\rho(\widetilde{s}_{\ell,\alpha}, \widetilde{r}_\alpha) \to 0$ as $\ell \to \infty$ except at most countable $\alpha \in [0,1]$.*

## 3. Sequences in $\mathcal{C}(\mathbb{R}^n)$

In this section, restricting $\mathcal{C}(\mathbb{R}^n)$ into the subclass by use of the concept of determining class, we prove the monotone convergence theorem.

Let $K$ be a convex cone. The sequence $\{D_\ell\}_{\ell=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preccurlyeq_K$ if there exists $F, D \in \mathcal{C}(\mathbb{R}^n)$ such that $F \preccurlyeq_K D_\ell \preccurlyeq_K D$ for all $\ell \geq 1$ and said to be monotone w.r.t. $\preccurlyeq_K$ if $D_1 \preccurlyeq_K D_2 \preccurlyeq_K \cdots$.

Let $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we say that $A$ is a determining class for $\mathfrak{L}$ if $a \cdot D = a \cdot F$ for all $a \in A$ and $D, F \in \mathfrak{L}$ implies $D = F$. For example, the set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ in $\mathbb{R}^n$ is a determining class for $\mathcal{C}_r(\mathbb{R}^n)$. Also, by the separation theorem, $\mathbb{R}^n$ is a determining class for $\mathcal{C}(\mathbb{R}^n)$.

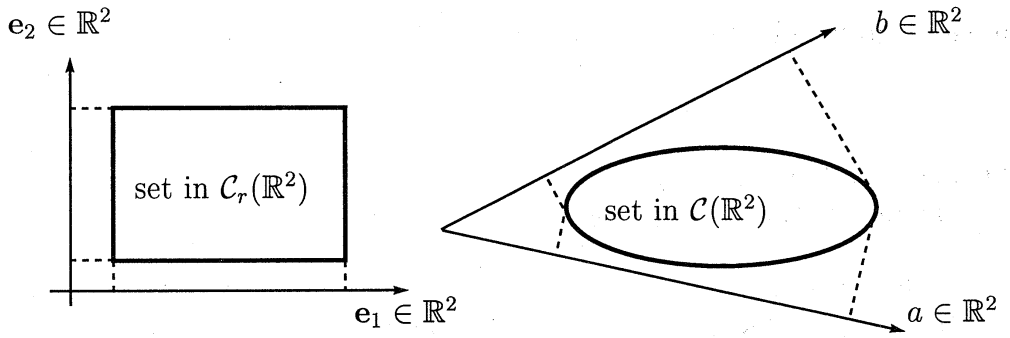


Figure 9: The example of determining class

**Theorem 3.1.** *Let $K$ be a closed convex cone of $\mathbb{R}^n$. Suppose that $K^+$ is a determining class for $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$. Then, the pseudo order $\preccurlyeq_K$ becomes a partial order in the restricted class $\mathfrak{L}$.*

As a simple application of Theorem 3.1, we have the following.

**Corollary 3.1.** *Let $K$ be a closed convex cone of $\mathbb{R}^n$ and $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ closed. Suppose that $K^+$ is a determining class for $\mathfrak{L}$. Then, any sequence $\{D_l\} \subset \mathfrak{L}$ which is monotone w.r.t. $\preccurlyeq_K$ and satisfies $D_l \subset X$ $(l \geq 1)$ for some compact subset $X$ of $\mathbb{R}^n$ converges w.r.t. $\rho$.*

In order to continue a further discussion, we need the acuteness of the ordering cone $K$.

We have the following.

**Lemma 3.1.** *Let $K$ be a closed, acute convex cone and $D, F, G \in \mathcal{C}(\mathbb{R}^n)$ with $D \preccurlyeq_K F \preccurlyeq_K G$. Let*

$$X := \bigcup_{\substack{x \preccurlyeq_K y \\ x \in D, y \in G}} (x + K) \cap (y - K). \tag{3.1}$$

*Then, it holds that $F \subset X$ and $X$ is bounded.*

**Theorem 3.2.** *Let $K$ be a closed, acute convex cone of $\mathbb{R}^n$ and $\mathfrak{L} \subset \mathcal{C}(\mathbb{R}^n)$ closed. Suppose that $K^+$ is a determining class for $\mathfrak{L}$. Then, any sequence $\{D_l\}_{l=1}^{\infty} \subset \mathfrak{L}$ which is bounded and monotone w.r.t. $\preccurlyeq_K$ converges w.r.t. $\rho$.*

As applications of Theorem 3.2, we have the following Corollaries.

**Corollary 3.3.** *Any sequence in $\mathcal{C}_r(\mathbb{R}^n)$ with monotonicity and boundedness w.r.t. $\preccurlyeq_n$ converges w.r.t. $\rho$.*

For any $D \in \mathcal{C}(\mathbb{R}^n)$ and $\varepsilon > 0$, the $\varepsilon$-closed neighborhood of $D$ will be denoted by

$$S_\varepsilon(D) := \{x \in \mathbb{R}^n \mid d(x, D) \leq \varepsilon\}, \tag{3.2}$$

which is a compact convex subset of $\mathbb{R}^n$. Note that

$$S_\varepsilon(D) = D + \varepsilon U_0, \tag{3.3}$$

where $U_0$ is the closed unit ball (cf. [2]).

The following lemma is useful in the sequel.

**Lemma 3.2.** *The following* (i) *to* (iii) *hold.*

(i) *For any $D, F \in \mathcal{C}(\mathbb{R}^n)$, if $S_{\delta_1}(D) \subset S_{\delta_2}(F)$ for some $\delta_1, \delta_2 \geq 0$, then $S_{\delta_1+\varepsilon}(D) \subset S_{\delta_2+\varepsilon}(F)$ for any $\varepsilon \geq 0$.*

(ii) *For any $D \in \mathcal{C}(\mathbb{R}^n)$ and $\lambda > 0$, $S_\varepsilon(\lambda D) = \lambda S_{\varepsilon/\lambda}(D)$.*

(iii) *For any sequence $\{D_l\} \subset \mathcal{C}(\mathbb{R}^n)$ and $D \in \mathcal{C}(\mathbb{R}^n)$, if $D_l \to D$ as $l \to \infty$, then $S_\delta(D_l) \to S_\delta(D)$ as $l \to \infty$ $(\delta \geq 0)$.*

For any closed convex cone $K \subset \mathbb{R}^n$, let $\mathfrak{L}(K^+)$ be the set of all $D \in \mathcal{C}(\mathbb{R}^n)$ satisfying that for any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ with $x_0 \notin S_\varepsilon(D)$ there exists $a \in K^+$ $(a \neq 0)$ such that

$$a \cdot y \geq a \cdot x_0 \quad \text{for all } \ y \in S_\varepsilon(D).$$

The properties of $\mathfrak{L}(K^+)$ are stated in the following lemma.

**Lemma 3.3.** *The following* (i) *to* (iii) *hold.*

(i) *$K^+$ is a determining class for $\mathfrak{L}(K^+)$.*

(ii) *$\mathfrak{L}(K^+)$ is closed w.r.t. $\rho$.*

(iii) *For any* $D \in \mathfrak{L}(K^+)$, $\lambda D + \mu D \in \mathfrak{L}(K^+)$ $(\lambda, \mu \geq 0)$.

Noting that $K^+ = \mathbb{R}^2_+$ when $K = \mathbb{R}^2_+$ in $\mathbb{R}^2$, the sets included in $\mathfrak{L}(\mathbb{R}^2_+)$ are illustrated in Figure 10.

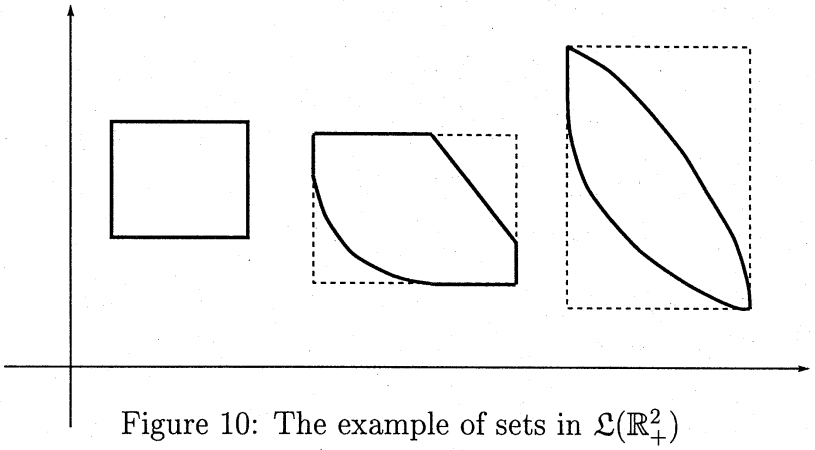

Figure 10: The example of sets in $\mathfrak{L}(\mathbb{R}^2_+)$

**Theorem 3.3.** *Let $K$ be a closed, acute convex cone of $\mathbb{R}^n$. Then, any sequence $\{D_l\}_{l=1}^{\infty} \subset \mathfrak{L}(K^+)$ which is bounded and monotone w.r.t. $\preccurlyeq_K$ converges w.r.t. $\rho$.*

## 4. Sequences in $\mathcal{F}(\mathbb{R}^n)$

In this section, the monotone convergence theorem for a sequence in $\mathcal{F}(\mathbb{R}^n)$ is given.

Let $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$. Then we call $A$ a determining class for $\widetilde{\mathfrak{L}}$ if $a \cdot \widetilde{s} = a \cdot \widetilde{r}$ for all $a \in A$ and $\widetilde{s}, \widetilde{r} \in \widetilde{\mathfrak{L}}$ implies $\widetilde{s} = \widetilde{r}$.

A natural extension of Theorem 3.1 to fuzzy sets will be given in the following theorem.

**Theorem 4.1.** *Let $K$ be a closed convex cone of $\mathbb{R}^n$ and $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$. Suppose that $K^+$ is a determining class for $\widetilde{\mathfrak{L}}$. Then, a pseudo order $\preccurlyeq_K$ is a partial order in $\widetilde{\mathfrak{L}}$.*

Let $K$ be a convex cone. The sequence $\{\widetilde{s}_l\} \subset \mathcal{F}(\mathbb{R}^n)$ is said to be bounded w.r.t. $\preccurlyeq_K$ if there exists $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R}^n)$ such that $\widetilde{u} \preccurlyeq_K \widetilde{s}_l \preccurlyeq_K \widetilde{v}$ for all $l \geq 1$ and said to be monotone w.r.t. $\preccurlyeq_K$ if $\widetilde{s}_1 \preccurlyeq_K \widetilde{s}_2 \preccurlyeq_K \cdots$.

In order to obtain the convergence theorem, we need the concept of directionality given in [17]. Denote the surface of the unit ball by $U := \{x \in \mathbb{R}^n \mid\mid\mid x \mid\mid = 1\}$. Let $V \subset U$. Then, for $D, D' \in \mathcal{C}(\mathbb{R}^n)$ with $D \subset D'$, we call $D'$ $V$-directional to $D$ (written by $D' \supset_V D$) if there exists a real $\lambda > 0, y \in D$ and $z \in D'$ such that
(i) $d(z, y) = \rho(D', D)$ and (ii) $z - y = \lambda v$ for some $v \in V$.

**Definition** ($V$-directional). *Let $V \subset \mathbb{R}^n$. For $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$, $\widetilde{s}$ is called $V$-directional if $\widetilde{s}_\alpha \supset_V \widetilde{s}_{\alpha'}$ for $0 \leq \alpha \leq \alpha' \leq 1$.*

**Corollary 4.1.** *Let $K$ be a closed convex cone of $\mathbb{R}^n$ and $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ closed. Suppose that $K^+$ is a determining class for $\widetilde{\mathfrak{L}}$. Let a sequence $\{\widetilde{s}_l\} \subset \mathcal{F}(\mathbb{R}^n)$ be satisfied that*

(a) *$\{\widetilde{s}_l\}$ is bounded and monotone w.r.t. $\preccurlyeq_K$,*

(b) *each* $\widetilde{s}_l$ *is* $V$*-directional for a finite set* $V \subset \mathbb{R}^n$ *and*

(c) *there exists a compact subset* $D$ *of* $\mathbb{R}^n$ *such that* $\widetilde{s}_{l0} \subset D$ *for all* $l \geq 1$, *where* $\widetilde{s}_{l0}$ *is the support or the* 0*-cut of* $\widetilde{s}_l$.

*Then the sequence* $\{\widetilde{s}_l\}$ *converges w.r.t.* $\rho$.

The following monotone convergence theorem is thought of as an extension of Theorem 3.2 to fuzzy sets.

**Theorem 4.2.** *Let* $K$ *be a closed, acute convex cone of* $\mathbb{R}^n$ *and* $\widetilde{\mathfrak{L}} \subset \mathcal{F}(\mathbb{R}^n)$ *closed. Suppose that* $K^+$ *is a determining class for* $\widetilde{\mathfrak{L}}$ *closed. Then, any sequence* $\{\widetilde{s}_l\}_{l=1}^{\infty} \subset \widetilde{\mathfrak{L}}$ *which satisfies (a) and (b) in Corollary 4.1 converges w.r.t.* $\rho$.

Now, for any closed convex cone $K$, we define $\widetilde{\mathfrak{L}}(K^+)$ by

$$\widetilde{\mathfrak{L}}(K^+) := \{\widetilde{s} \in \mathcal{F}(\mathbb{R}^n) \mid \widetilde{s}_\alpha \in \mathfrak{L}(K^+) \text{ for all } \alpha \in [0,1]\}.$$

The previous Lemma 3.3 is extended to that for $\mathcal{F}(\mathbb{R}^n)$ in the following lemma.

**Lemma 4.1.** *The following* (i) *to* (iii) *hold.*

(i) $K^+$ *is a determining class for* $\widetilde{\mathfrak{L}}(K^+)$.

(ii) $\widetilde{\mathfrak{L}}(K^+)$ *is closed w.r.t. the convergence defined in Section 2.*

(iii) *For any* $\widetilde{s} \in \widetilde{\mathfrak{L}}(K^+)$, $\lambda\widetilde{s} + \mu\widetilde{s} \in \widetilde{\mathfrak{L}}(K^+)$ $(\lambda, \mu \geq 0)$.

We have the following.

**Theorem 4.3.** *Let* $K^+$ *be a closed, acute convex cone of* $\mathbb{R}^n$. *Then, any sequence* $\{\widetilde{s}_l\}_{l=1}^{\infty} \subset \widetilde{\mathfrak{L}}(K^+)$ *which satisfies* (a) *and* (b) *in Corollary 4.1 converges.*

## 5. Applications to Monotone Dynamic Fuzzy Systems

In this section, as an application of the results obtained in the preceding section, we consider a limit theorem for a sequence of fuzzy states defined by the dynamic fuzzy system (cf. [5, 6, 14, 15, 16, 17]) with a monotone fuzzy relation.

Let $\widetilde{q} : \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ be a continuous fuzzy relation such that $\widetilde{q}(x,\cdot) \in \mathcal{F}(\mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and $\widetilde{q}(\cdot,\cdot)$ is convex, that is,

$$\widetilde{q}(\lambda x^1 + (1-\lambda)x^2, \lambda y^1 + (1-\lambda)y^2) \geq \widetilde{q}(x^1, y^1) \wedge \widetilde{q}(x^2, y^2) \tag{5.1}$$

for any $x^1, x^2, y^1, y^2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$. From this fuzzy relation $\widetilde{q}$, we define $\widetilde{q} : \mathcal{F}(\mathbb{R}^n) \to$ {the set of fuzzy sets on $\mathbb{R}^n$} as follows.

$$\widetilde{q}(\widetilde{u})(y) := \sup_{x \in \mathbb{R}^n} \{\widetilde{u}(x) \wedge \widetilde{q}(x,y)\}, \ \in \mathbb{R}^n, \tag{5.2}$$

where $a \wedge b = \min\{a, b\}$. Also, for any $\alpha \in [0,1]$, $\widetilde{q}_\alpha : \mathcal{C}(\mathbb{R}^n) \to 2^{\mathbb{R}^n}$ will be defined by

$$\widetilde{q}_\alpha(D) := \begin{cases} \{\, y \mid \widetilde{q}(x,y) \geq \alpha \text{ for some } x \in D\}, & \text{for } \alpha > 0,\ D \in \mathcal{C}(\mathbb{R}^n) \\ \mathrm{cl}\{\, y \mid \widetilde{q}(x,y) > 0 \text{ for some } x \in D\}, & \text{for } \alpha = 0,\ D \in \mathcal{C}(\mathbb{R}^n), \end{cases} \tag{5.3}$$

where cl denotes the closure of a set and $2^{\mathbb{R}^n}$ the set of all closed subsets of $\mathbb{R}^n$. For simplicity, we put $\widetilde{q}(x) := \widetilde{q}(\{x\})$ for $x \in \mathbb{R}^n$.

The following facts are well-known (cf. [4, 5, 17]).

**Lemma 5.1** *The following* (i) *to* (iii) *hold.*

(i) $\widetilde{q}_\alpha(D) \in \mathcal{C}(\mathbb{R}^n)$ *for any* $D \in \mathcal{C}(\mathbb{R}^n)$ *and* $\widetilde{q}_\alpha(\cdot)$ *is continuous in* $\mathcal{C}(\mathbb{R}^n)$ *for each* $\alpha \in (0, 1]$.

(ii) $\widetilde{q}(\widetilde{u}) \in \mathcal{F}(\mathbb{R}^n)$ *for any* $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$.

(iii) $\widetilde{q}(\widetilde{u})_\alpha = \widetilde{q}_\alpha(\widetilde{u})_\alpha$ *for any* $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$ *and* $\alpha \in [0, 1]$, *where* $\widetilde{q}(\widetilde{u})_\alpha$ *is the* $\alpha$*-cut of* $\widetilde{q}(\widetilde{u})$.

The sequence of fuzzy states, $\{\widetilde{s}_t\}_{t=1}^{\infty} \subset \mathcal{F}(\mathbb{R}^n)$, for the dynamic system with fuzzy transition $\widetilde{q}$ is defined as follows.

$$\widetilde{s}_{t+1} = \widetilde{q}(\widetilde{s}_t) \quad (t \geq 1), \tag{5.4}$$

where $\widetilde{s}_1 \in \mathcal{F}(\mathbb{R}^n)$ is the initial fuzzy state.

The problem in this section is to consider a convergence of the sequence $\{\widetilde{s}_t\}_{t=1}^{\infty}$ defined by (5.4), so that we derive the monotone property of the fuzzy relation $\widetilde{q}$ w.r.t. the pseudo order $\preccurlyeq_K$ defined by the ordering cone $K$ in $\mathbb{R}^n$.

**Definition** ($\preccurlyeq_K$-monotone). *The fuzzy relation* $\widetilde{q}$ *is called* $\preccurlyeq_K$*-monotone if* $x^1 \preccurlyeq_K x^2$ $(x^1, x^2 \in \mathbb{R}^n)$ *means* $\widetilde{q}(x^1, \cdot) \preccurlyeq_K \widetilde{q}(x^2, \cdot)$.

**Remark.** Yoshida et al [17] has introduced a monotone property concerning the fuzzy relation $\widetilde{q}$ whose definition is as follows: $\widetilde{q}_\alpha(y) \subset \widetilde{q}_\alpha(x) + \ell(x, y)$ for $x, y \in \mathbb{R}^n$, where $\ell(x, y) := \{\gamma(y - x) \mid \gamma \geq 0\}$. Obviously, if $\widetilde{q}$ is monotone in the sense of [17], then $\widetilde{q}$ is $\preccurlyeq_n$-monotone, but the converse is not necessarily true.

The following lemma is useful for our further discussion.

**Lemma 5.2.** *Suppose that* $\widetilde{q}$ *is* $\preccurlyeq_K$*-monotone. Then, for any* $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R}^n)$ *with* $\widetilde{u} \preccurlyeq_K \widetilde{v}$, *it holds that* $\widetilde{q}(\widetilde{u}) \preccurlyeq_K \widetilde{q}(\widetilde{v})$.

**Assumption A.** *The following* (i) *to* (iii) *hold.*

(i) *The ordering cone* $K$ *is a closed, acute convex one in* $\mathbb{R}^n$.

(ii) *The fuzzy relation* $\widetilde{q}$ *is* $\preccurlyeq_K$*-monotone.*

(iii) *There exists a finite subset* $V \subset U$ *such that, for any* $D, D' \in \mathcal{C}(\mathbb{R}^n)$ $(D' \supset D)$, *if* $D' \supset_V D$ *then* $\widetilde{q}_{\alpha'}(D') \supset_V \widetilde{q}_\alpha(D)$ *for all* $\alpha, \alpha'$ $(0 \leq \alpha' \leq \alpha \leq 1)$.

For any given $\widetilde{u} \in \mathcal{F}(\mathbb{R}^n)$, putting $\widetilde{s}_1 := \widetilde{u}$, we define the sequence $\{\widetilde{s}_t\}_{t=1}^{\infty}$ by (5.4). Then, we have the following.

**Theorem 5.1.** *In addition to Assumption A, suppose that the following* (iv) *to* (vi) *hold.*

(iv) $\widetilde{u} \in \widetilde{\mathfrak{L}}(K^+)$ *and* $\widetilde{u} \preccurlyeq_K \widetilde{q}(\widetilde{u})$.

(v) $\widetilde{u}_{\alpha'} \supset_V \widetilde{q}_\alpha$ *for all* $\alpha, \alpha'$ $(0 \leq \alpha' \leq \alpha \leq 1)$, *where* $V$ *is as in Assumption* A(iii).

(vi) $\{\widetilde{s}_t\} \subset \widetilde{\mathfrak{L}}(K^+)$ *and bounded from above.*

*Then, the sequence* $\{\widetilde{s}_t\}$ *converges and the limit* $\widetilde{s} := \lim_{t\to\infty} \widetilde{s}_t$ *satisfies the following fuzzy relational equation:*

$$\widetilde{s} = \widetilde{q}(\widetilde{s}). \tag{5.5}$$

**Theorem 5.2.** *In addition to Assumption A, suppose that the following* (iv'), (v) *and* (vi') *hold.*

(iv') $\widetilde{u} \in \widetilde{\mathfrak{L}}(K^+)$ *with* $\widetilde{u}_0 \subset K$ *and* $\widetilde{q}(\widetilde{u}) \preccurlyeq_K \widetilde{u}$.

(v) $\widetilde{u}_{\alpha'} \supset_V \widetilde{q}_\alpha$ *for all* $\alpha, \alpha'$ $(0 \le \alpha' \le \alpha \le 1)$, *where* $V$ *is as in Assumption* A(iii).

(vi') $\{\widetilde{s}_t\} \subset \widetilde{\mathfrak{L}}(K^+)$.

*Then, the sequence* $\{\widetilde{s}_t\}$ *converges and the limit* $\widetilde{s} := \lim_{t\to\infty} \widetilde{s}_t$ *satisfies the fuzzy relational equation* (5.5).

As an example of $\preccurlyeq_K$-monotone fuzzy relation, we put the fuzzy relation $\widetilde{q}$ by

$$\widetilde{q}(x, y) := \widetilde{r}(y) + \beta \mathbf{1}_{\{x\}} \quad (x, y \in \mathbb{R}^n), \tag{5.6}$$

where $\widetilde{r} \in \widetilde{\mathfrak{L}}(K^+)$ with $\widetilde{r}_{\alpha'} \supset_V \widetilde{r}_\alpha$ for some finite set $V \subset U$ and $\alpha, \alpha'$ $(0 \le \alpha' \le \alpha \le 1)$ and $0 < \beta < 1$.

Obviously, Assumption A is satisfies for $\widetilde{q}$ of (5.6). Also, we observe from Lemma 4.1 that the assumptions (iv) to (vi) in Theorem 5.1 hold for $\widetilde{u} = \widetilde{r}$. So that by Theorem 5.1, the sequence $\{\widetilde{s}_t\}$ defined by (5.4) with $\widetilde{s}_1 = \widetilde{r}$ converges.

**Remark.** Note that the fuzzy relation $\widetilde{q}$ of (5.6) satisfies the contraction property introduced in [4]. Thus, we see that the limit $\widetilde{s} = \lim_{t\to\infty} \widetilde{s}_t$ is a unique solution of the fuzzy relational equation (5.5) and given by $\widetilde{s} = (1-\beta)^{-1}\widetilde{r}$.

**Example.** We give a one-dimensional numerical example whose fuzzy relation $\widetilde{q}$ is given by

$$\widetilde{q}(x, y) = (1 - 2|y - (3 - x^{-2})|) \vee 0 \quad (x > 0).$$

For $\alpha \in [0, 1]$, it holds that by (5.3)

$$\widetilde{q}_\alpha(x) = [3 - (1-\alpha)2^{-1} - x^{-2}, 3 + (1-\alpha)2^{-1} - x^{-2}].$$

This is illustrated in Figure 2. So, we observe that $\widetilde{q}$ is $\preccurlyeq_1$-monotone in $(0, \infty) \times (0, \infty)$, also that $\mathbf{1}_{\{1\}} \preccurlyeq_1 \widetilde{q}(\mathbf{1}_{\{1\}})$ and $\widetilde{q}(x, \cdot) \preccurlyeq_1 \mathbf{1}_{\{7/2\}}(x)$.

Applying Theorem 5.1, the sequence $\{\widetilde{s}_t(x)\}$ defined by (5.4) with $\widetilde{s}_1(x) = \mathbf{1}_{\{1\}}(x)$ converges. The convergence is shown in Figure 2 and 3 with the limit $\widetilde{s}(x) = \lim_{t\to\infty} \widetilde{s}_t(x)$, where the $\alpha$-cut $\widetilde{s}_\alpha$ of the limit $\widetilde{s}(x)$ for $\alpha = 0$ and $\alpha = 1$ are $\min \widetilde{s}_0 = 2.313099034$, $\max \widetilde{s}_0 = 3.414213562$ and $\widetilde{s}_1 = 2.879385242$.

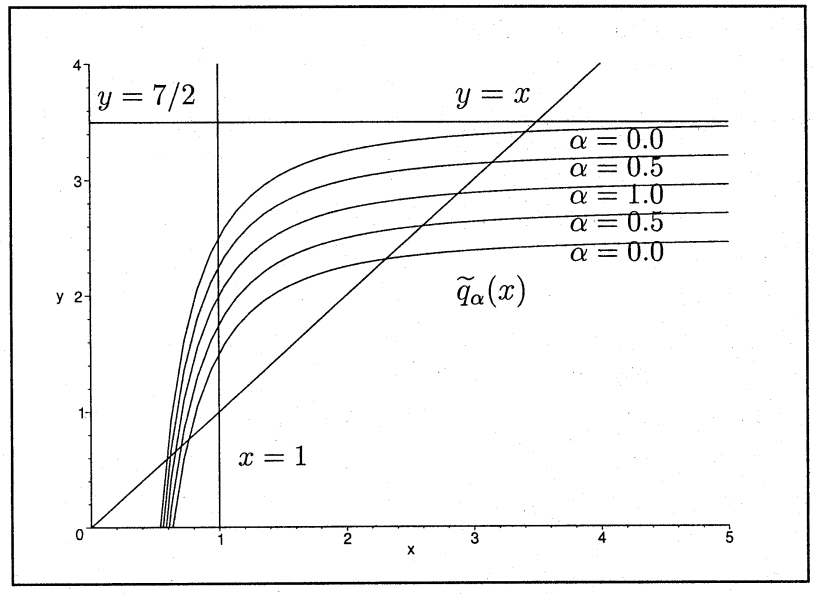


Figure 11: $\widetilde{q}_\alpha(x)$ and the limit $\widetilde{s}(x)$ of $\{\widetilde{s}_t(x)\}$

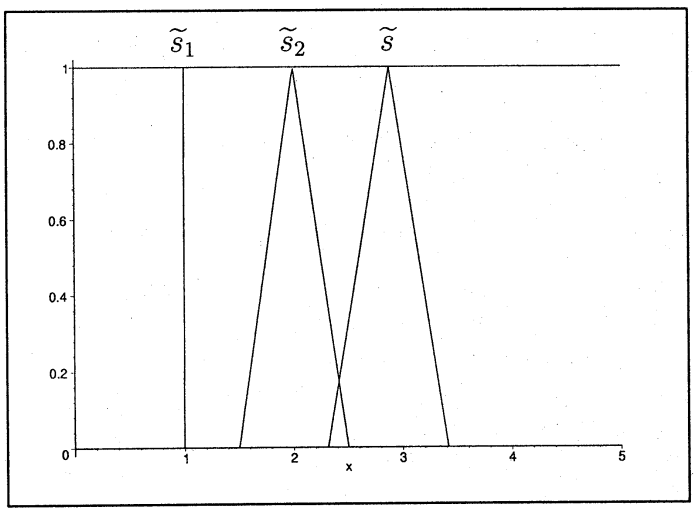


Figure 12: The sequence $\{\widetilde{s}_t(x)\}$

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