数理解析研究所講究録
1174 巻 2000 年 13-22

Some conditions for reservation on optimal stopping problem with infinite-period reservation

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Abstract

In Saito [8] it is revealed that in an optimal stopping problem where any offer is allowed to be reserved in exchange for some cost and each reservation is valid indefinitely after the contract, any reserved offer should not be recalled except at the deadline of the planning horizon. It fails to account, however, that what condition makes the action of reservation economically meaningful. On account of this we shall explore the condition in this paper.

1. Introduction

This paper presents some additional results to the author [8], dealing with optimal stopping problem with infinite-period reservation. In order to understand the so-called optimal stopping problem, consider the following example:

Suppose that you would like to dispose of a house by a certain date in the future (the deadline) and that you have just found a buyer. The offer of the buyer seems pretty good but you reject it on account that you feel you could find a better deal within the periods remaining. If you continually repeated such rejections and reached the deadline, you would face a dire situation where the offer of the buyer just found is too low while you may have no choice but to sell to him. However, if a too low selling price is accepted early in the time frame because of the fear of the dire situation, there is every chance that better buyers who may have appeared later would be missed. To avoid these two extremes, you should have a rule to guide to the best action on a daily basis.

To sum up, optimal stopping problem is a problem of when and how to make a decision to maximize the expected reward in the situation where an offer has to be taken among ones appearing subsequently and randomly up to the deadline. We easily notice that the question whether or not a rejected offer will become available again in the future dominates the structure of the optimal decision rule. In view of this point, almost all models investigated so far are classified into three types: Without recall type([2][4]), in which every offer once rejected is supposed to disappear instantly and become unavailable forever, thus at any point in time, the offer found recently is only the offer available then. With recall type([7][10]), in which every rejected offer is supposed to stay there forever, thus at any point in time, all the offers having appeared up to then remain available. With uncertain recall type([1][3]), in which the future availability of a rejected offer is supposed to be determined by a certain probability, thus at any point in time, it is uncertain whether an offer available then remains available at the next point in time.

What to be emphasized here is that, in these three types, the future availability of an offer appearing in the past is determined independently of the will of the decision maker. In other words, little attention has been given to the question of what happens if the decision maker himself can control the availability, or reserve an opportunity, in exchange for a certain compensation. Rose [5][6] and the author [8][9] tried to answer this question. In Rose [6] each offer is allowed to be reserved for k periods in return for a cost bk with a given $b > 0$ where only one offer can be reserved at any point in time and it is prohibited to renew the reservation of an offer at the time of its maturity. A similar model is treated in Rose [5] where $k = 1$ and renewals are permitted. In the author [8][9] any offer is allowed to be reserved if some cost is payed. The term of validity of reservation is not considered in [8] and is restricted to k periods in [9].

As stated earlier, in this paper we shall research the model which is the same as that dealt with in [8]. A major result obtained in [8] is that you should not recall any reserved offer prior to the deadline. Optimal behavior against reserved offers was revealed, however, the condition for reservation was not treated. So, in this paper, we will try to reveal the condition which makes action of reservation economically meaningful.

The model and the optimal equation is formulated in sections 2 and 3, respectively. In section 4, we introduce some mathematical results obtained in Saito [8] and show some other results. The main topic of the paper, the condition for reservation, is dealt with in section 5.

2. Model

Consider a person who periodically searches for offers with the intention to accept one of them up to the deadline. The searcher can find one offer at each time as long as a *search cost* $s > 0$ was paid at the previous time, however, he does not know in advance what offer will come up. The only information available is that values of subsequent offers w, w', w'', $\cdots$ are i.i.d. random variables following a distribution function F such that $F(w) = 0$ for $w < a$, $0 < F(w) < 1$ for $a \leq w < b$, and $F(w) = 1$ for $b \leq w$ with $0 \leq a < b < \infty$. Let the expectation of values of offers be denoted by μ.

After inspecting the offer just drawn, the *current offer*, he must decide his next action. As for the current offer, he is allowed not only to accept it or pass it up but also to reserve it. Reserving an offer with value w, simply called *offer* w later on, gives him a right to recall and accept it at any time in the future, but requires him to pay a *reserving cost* $r(w)$, which is a positive, continuous, and nondecreasing function of w. On the other hand, he can never return to any of non-reserved offers in the future. Note here that although any reserved offer is recallable at each time, the reserved offer to be recalled is only the most lucrative one. Let such reserved offer be called *leading offer*.

Then, the actions which can be taken at each time except for the deadline are classified into the following four: accepting current offer and stopping the search (**AS**), reserving current offer and continuing the search (**RC**), passing up current offer and stopping the search by accepting leading offer (**PS**), and passing up current offer and continuing the search (**PC**), where **AS**, **RC**, **PS**, and **PC** represent the four decisions, respectively. Of course, at the deadline, only decisions **AS** and **PS** are permitted.

In the model, the value of time is considered by a discount factor β such as $0 < \beta \leq 1$, that is, the present value of q monetary units obtained at the next time is given by βq monetary units.

The objective here is to find an optimal decision rule that guides him to which action should be taken at each decision point so as to maximize the total expected discounted present net profit obtainable in the process ahead, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search by accepting an offer.

Consequently, at each time, leading offer is the offer with the highest value of all offers reserved up to the previous time.

Finally, for later discussion, we let

$$\alpha \equiv -s + \beta\mu > a. \tag{2.1}$$

By noting that α represents the expected discounted net profit attainable from one more search and that a is the worst value of offers, we can understand that if $\alpha \leq a$, no one is willing to engage in the search process. Not only intuitively but also theoretically we find that the relationship $\alpha \leq a$ reduces the the optimal decision rule to a trivial form as follows: "as soon as you start the search, you should stop it by accepting an offer you first draw" (see p.98 in Saito [8]).

3. Optimal Equation and Optimal Decision Rule

Let points in time t, simply referred to as *time* t later on, be equally spaced and numbered backward from the deadline $t = 0$, thus t also represents the number of periods remaining.

Let $u_t(w, x)$ denote the maximum total expected present discounted net profit attainable by starting the search for offers from time t with having current offer w and leading offer x. Then,

$$u_t(w, x) = \max\{w, \ -r(w) - s + \beta v_{t-1}(\max\{w, x\}), \ x, \ -s + \beta v_{t-1}(x)\}, \quad t \geq 0,$$

where $v_t(x) = \int_a^b u_t(w, x)\mathrm{d}F(w)$ for $t \geq 0$ and $v_{-1}(x) = -\infty$.

However, we easily obtain that, for any w, x, and $t \geq 0$,

$$\begin{aligned} &\max\{w, \ -r(w) - s + \beta v_{t-1}(\max\{w, x\}), \ x, \ -s + \beta v_{t-1}(x)\} \\ &= \max\{w, \ -r(w) - s + \beta v_{t-1}(w), \ x, \ -s + \beta v_{t-1}(x)\}. \end{aligned} \tag{3.1}$$

Furthermore, the left-hand side of (3.1) is maximized by the first term in its braces if and only if the right-hand side of (3.1) is maximized by the first term in its braces, and this relationship holds also with respect to the second, third, and fourth terms, respectively. These results are interpreted intuitively as that we have no need to reserve an offer if its value is less than that of leading offer because every reserved offer is recallable at any time in the future.

Owing to the facts above we redefine $u_t(w, x)$ as

$$u_t(w, x) = \max \left\{ \begin{array}{lcl} \mathtt{AS} & : & w, \\ \mathtt{RC} & : & -r(w) - s + \beta v_{t-1}(w), \\ \mathtt{PS} & : & x, \\ \mathtt{PC} & : & -s + \beta v_{t-1}(x) \end{array} \right\}, \quad t \geq 0, \tag{3.2}$$

and $v_t(x)$, by using the redefined $u_t(w, x)$, as

$$v_t(x) = \int_a^b u_t(w, x)\mathrm{d}F(w), \quad t \geq 0; \qquad v_{-1}(x) = -\infty. \tag{3.3}$$

Then, $u_0(w, x) = \max\{w, x\}$, which indicates decisions RC and PC are prohibited at the deadline.

Let us define two functions $z_t^o(x)$ and $z_t^r(w)$ as follows:

$$z_t^o(x) = \max\{x, \ -s + \beta v_{t-1}(x)\}, \qquad t \geq 0, \tag{3.4}$$

$$z_t^r(w) = \max\{w, \ -r(w) - s + \beta v_{t-1}(w)\}, \quad t \geq 0. \tag{3.5}$$

Clearly $z_0^o(x) = x$ and $z_0^r(w) = w$. Since $z_t^o(x)$ consists of the third and fourth terms in braces of (3.2), it represents the total expected present discounted net profit attainable by passing up current offer w at time t and following the optimal decision rule after that time. Similarly, $z_t^r(w)$ can be interpreted as the total expected present discounted net profit attainable by not passing up current offer w, that is, either accepting or reserving it at time t and following the optimal decision rule after that time.

Accordingly, the set of offers to be either accepted or reserved at time t can be expressed by

$$W_t(x) \equiv \{w \mid z_t^o(x) \leq z_t^r(w)\} \subseteq \mathbb{R}, \quad t \geq 0. \tag{3.6}$$

All these things make it clear that

$$v_t(x) = \int_a^b \max\{z_t^r(w), z_t^o(x)\}\mathrm{d}F(w) \tag{3.7}$$

$$= \int_{W_t(x)} z_t^r(w)\mathrm{d}F(w) + \int_{W_t(x)^c} z_t^o(x)\mathrm{d}F(w). \tag{3.8}$$

Let θ_t denote the number of x which equates the two terms in braces of (3.4) and let λ_t denote the number of w which equates the two terms in braces of (3.5), if they exist. As their definition, θ_t is a point of indifference between accepting leading offer x (decision PS) and continuing the search (decision PC), and λ_t is a point of indifference between accepting current offer w (decision AS) and reserving it (decision RC).

From what has seen above, the optimal decision rule can be generally described as follows:

Optimal decision rule: Suppose that you are at time t with leading offer x and have just drawn an offer w. Then the optimal choices are:

(a) if $w \in W_t(x)$, then:

- if $\lambda_t < w$, AS (accept current offer w and stop the search),
- if $w \leq \lambda_t$, RC (reserve current offer w and continue the search);

(b) if $w \notin W_t(x)$, then:

- if $\theta_t < x$, PS (pass up current offer w and stop the search by accepting leading offer x),
- if $x \leq \theta_t$, PC (pass up current offer w and continue the search).

4. Fundamental Properties of Optimal Decision Rule

In this section, we first pick up some properties with respect to $v_t(x)$, $z_t^o(x)$, $z_t^r(w)$, θ_t, λ_t, and $W_t(x)$ from Saito [8], and then show some other properties of them.

First of all, let θ denote a root of equation

$$\beta \int_a^b \max\{w, x\} \mathrm{d}F(w) - x - s = 0$$

where it can be shown that θ exists uniquely with $\alpha \leq \theta < b$ (see Saito [8]).

Lemma 4.1 *For* $t \geq 0$:

(a) $v_t(x)$ *is continuous, convex, and nondecreasing in* x; *and nondecreasing in* t.

(b) $\beta v_t(x) - x$ *is strictly decreasing in* x.

Proof: See Lemma 4.1 in Saito [8]. ∎

Lemma 4.2 *For* $t \geq 1$:

(a) θ_t *exists uniquely with* $\alpha \leq \theta_t < b$ *and satisfies* $\theta_t = \theta$.

(b) λ_t *exists uniquely with* $\lambda_t < \theta$ *and is nondecreasing in* t.

Proof: See Lemma 5.1 and Theorem 5.3 in Saito [8]. ∎

The lemma above indicates $\lambda_t < \theta_t = \theta$ for $t \geq 1$, and the relationship tells us that, at each time except for the deadline, any offer to be reserved has value under θ, thus there comes no leading offer with value over θ. Therefore, we conclude that no reserved offer should be recalled except at the deadline.

In the model it is allowed to recall the leading offer at any time over the whole planning horizon, and a search cost $s > 0$ must be spent in order to proceed the search. So, it seems natural to think that recalling the leading offer in the middle of the search can become an optimal decision. The result above, however, tells that such an action is not optimal at all. Consequently, we conclude that the aim of reserving an offer is only to avoid any dire situation which may be awaiting at the deadline.

By virtue of Lemma 4.2(a), we use θ instead of θ_t from now on.

Lemma 4.3 *For $t \geq 1$:*

$$(a)\ \ z_t^o(x) = \begin{cases} -s + \beta v_{t-1}(x), & x < \theta, \\ -s + \beta v_{t-1}(x) = x, & x = \theta, \\ x, & \theta < x. \end{cases}$$

$$(b)\ \ z_t^r(w) = \begin{cases} -r(w) - s + \beta v_{t-1}(w), & w < \lambda_t, \\ -r(w) - s + \beta v_{t-1}(w) = w, & w = \lambda_t, \\ w, & \lambda_t < w. \end{cases}$$

Proof: See Corollary 5.2 in Saito [8]. ∎

Lemma 4.4 *For $t \geq 0$:*

(a) $W_t(x_1) \supseteq W_t(x_2)$ *for any* x_1 *and* x_2 *such that* $x_1 < x_2$.

(b) $W_t(x) \supseteq W_{t+1}(x)$ *for any* x.

Proof: See Theorem 5.4 in Saito [8]. ∎

Lemma 4.5 $v_t(x) - v_{t-1}(x)$ *is nonincreasing in* x *for* $t \geq 1$.

Proof: We shall first show the assertion for $\theta \leq x$. Choose x so that $\theta \leq x$. Then, due to Lemma 4.3(a) we have $z_t^o(x) = x$.

If $w \leq \lambda_t$, then $w < \theta$ by Lemma 4.2(b), thus it follows from Lemmas 4.3(b,a) and 4.1(a) that

$$z_t^r(w) = -r(w) - s + \beta v_{t-1}(w) \leq -r(w) - s + \beta v_{t-1}(\theta) = -r(w) + \theta < \theta. \tag{4.1}$$

If $\lambda_t < w$, Lemma 4.3(b) indicates $z_t^r(w) = w$. From this and (4.1) we obtain $z_t^r(w) \leq \theta \leq x$ for any w and x while $w \leq \theta \leq x$.

Therefore, it follows from (3.7) that, for $\theta \leq x$ and $t \geq 1$,

$$v_t(x) = \int_a^b \max\{z_t^r(w), x\} \mathrm{d}F(w) = \int_a^\theta x \mathrm{d}F(w) + \int_\theta^b \max\{w, x\} \mathrm{d}F(w) = \int_a^b \max\{w, x\} \mathrm{d}F(w).$$

From this and $v_0(x) = \int_a^b \max\{w, x\} \mathrm{d}F(w)$, we conclude that $v_t(x) - v_{t-1}(x) = 0$ holds for $\theta \leq x$, thus the assertion holds true for $\theta \leq x$.

Next, in order to prove the assertion for $x \leq \theta$, choose x^1 and x^2 so that $x^1 < x^2 \leq \theta$ and let $W_1 = W_t(x^1)$ and $W_2 = W_t(x^2)$. Then, from (3.8) we get

$$v_t(x^1) = \int_{W_1} z_t^r(w) \mathrm{d}F(w) + \int_{W_1^c} z_t^o(x^1) \mathrm{d}F(w), \tag{4.2}$$

$$v_t(x^2) = \int_{W_2} z_t^r(w) \mathrm{d}F(w) + \int_{W_2^c} z_t^o(x^2) \mathrm{d}F(w). \tag{4.3}$$

Since $W_1 \supseteq W_2$ due to Lemma 4.4(a), we have $W_1 = W_2 \cup (W_1 \cap W_2^c)$ and $W_2^c = W_1^c \cup (W_1 \cap W_2^c)$. Hence, due to (4.2) and (4.3) we obtain

$$v_t(x^2) - v_t(x^1) = \int_{W_1^c} \left(z_t^o(x^2) - z_t^o(x^1)\right) \mathrm{d}F(w) + \int_{W_1 \cap W_2^c} \left(z_t^o(x^2) - z_t^r(w)\right) \mathrm{d}F(w). \tag{4.4}$$

From (3.6) we have $z_t^o(x^1) \leq z_t^r(w)$ for any $w \in W_1$, thus

$$z_t^o(x^2) - z_t^r(w) \leq z_t^o(x^2) - z_t^o(x^1), \quad w \in W_1 \cap W_2^c \ (\subseteq W_1)\,. \tag{4.5}$$

It follows from Lemma 4.3(a) that if $x^1 < x^2 \leq \theta$, then

$$z_t^o(x^2) - z_t^o(x^1) = -s + \beta v_{t-1}(x^2) + s - \beta v_{t-1}(x^1) = \beta \left(v_{t-1}(x^2) - v_{t-1}(x^1)\right). \tag{4.6}$$

Hence, by using (4.4) to (4.6) we arrive at

$$\begin{aligned} v_t(x^2) - v_t(x^1) &\leq \int_{W_1^c} \left(z_t^o(x^2) - z_t^o(x^1)\right) \mathrm{d}F(w) + \int_{W_1 \cap W_2^c} \left(z_t^o(x^2) - z_t^o(x^1)\right) \mathrm{d}F(w) \\ &= \left(z_t^o(x^2) - z_t^o(x^1)\right) \int_{W_2^c} \mathrm{d}F(w) \\ &= \beta \left(v_{t-1}(x^2) - v_{t-1}(x^1)\right) \int_{W_2^c} \mathrm{d}F(w) \\ &\leq v_{t-1}(x^2) - v_{t-1}(x^1), \end{aligned}$$

from which $v_t(x^1) - v_{t-1}(x^1) \geq v_t(x^2) - v_{t-1}(x^2)$.

We have confirmed the assertion as for $x \leq \theta$, thus the proof is completed. ∎

Lemma 4.6 *For any c such that $a \leq c \leq b$,*

$$\int_a^b \max\{\xi, c\} \mathrm{d}F(\xi) = \mu + \int_a^c F(\xi) \mathrm{d}\xi.$$

Proof: For any c with $a \leq c \leq b$ we have

$$\int_a^b \max\{\xi, c\} \mathrm{d}F(\xi) = \int_a^c c \mathrm{d}F(\xi) + \int_c^b \xi \mathrm{d}F(\xi). \tag{4.7}$$

As for the first integral in the right-hand side of (4.7), since $F(a) = 0$,

$$\int_a^c c \mathrm{d}F(\xi) = c \int_a^c \mathrm{d}F(\xi) = c\left(F(c) - F(a)\right) = cF(c). \tag{4.8}$$

As for the second, since F is bounded, integration by parts can be applied and it follows from $F(b) = 1$ that

$$\int_c^b \xi \mathrm{d}F(\xi) = b - cF(c) - \int_c^b F(\xi) \mathrm{d}\xi. \tag{4.9}$$

Substitute (4.8) and (4.9) into (4.7) and remember $F(\xi) = 0$ for $\xi < a$ and $F(\xi) = 1$ for $b \leq \xi$. Then we obtain

$$\begin{aligned} \int_a^b \max\{\xi, c\} \mathrm{d}F(\xi) &= b - \int_c^b F(\xi) \mathrm{d}\xi \\ &= \int_0^b \mathrm{d}\xi - \int_0^b F(\xi) \mathrm{d}\xi + \int_0^c F(\xi) \mathrm{d}\xi \\ &= \int_0^\infty \left(1 - F(\xi)\right) \mathrm{d}\xi + \int_0^a F(\xi) \mathrm{d}\xi + \int_a^c F(\xi) \mathrm{d}\xi \\ &= \mu + \int_a^c F(\xi) \mathrm{d}\xi. \quad ∎ \end{aligned}$$

Now, let us define a new function $\psi(x)$ as

$$\psi(x) = \alpha + \beta \int_a^x F(\xi)\mathrm{d}\xi, \tag{4.10}$$

and let

$$\psi^{(1)}(x) = \psi(x); \qquad \psi^{(k)}(x) = \psi\left(\psi^{(k-1)}(x)\right), \quad k \geq 2. \tag{4.11}$$

It can be easily shown that $\psi^{(t)}(x)$ is nondecreasing in x and t.

5. Reserving region

Define, for all x,

$$R_t(x) = \{w \mid w \in W_t(x),\ w \leq \lambda_t\} \subseteq \mathbb{R}, \quad t \geq 1, \tag{5.1}$$

and

$$R_t = \{(x, w) \mid x \in \mathbb{R},\ w \in R_t(x)\} \subseteq \mathbb{R}^2, \quad t \geq 1. \tag{5.2}$$

We find that $R_t(x)$ represents the range of value of current offer which should be reserved at time t with leading offer x. Thus R_t means the set of pairs of leading offer x and corresponding current offer w to be reserved at time t. Let us call R_t *reserving region* of time t.

In the case where we can search for offers indefinitely, it is confirmed that there is no reserving region at every time, that is, reservation has no economical meanings (see Theorem 7.1 in Saito [8]).

If the period of planning horizon we have is finite, as we mentioned in the comment after Lemma 4.2, reservations are done to prepare for the deadline at which a dire situation may be awaiting. This makes us to surmise that reserving region appears and becomes larger with the lapse of time. There is some case, however, that reserving region disappears at time 1, although it exists at time 2 or 3. For example, in the case where $\beta = 0.97$, $s = 0.005$, $a = 0.2$, $b = 1.2$, $F(w) = 9(w - 0.2)$ for $0.2 \leq w < 0.3$ and $(w + 7.8)/9$ for $0.3 \leq w \leq 1.2$, $r(w) = 0.04$ for $0.2 \leq w < 0.48$, $0.5w - 0.2$ for $0.48 \leq w < 0.72$, and 0.16 for $0.72 \leq w \leq 1.2$, there is no reserving region at time 1 but it exists from time 2 to 14. Therefore, it is not enough to consider conditions of reservation only for time 1.

Now, we shall reveal conditions of reserving cost function $r(w)$ not to induce us for any reservation.

Lemma 5.1 *$-\, s + \beta v_0(x) = \psi^{(1)}(x)$ for any x; and for $t \geq 2$, if $R_i = \phi$ for $i \leq t-1$, then $-\, s + \beta v_{t-1}(x) = \psi^{(t)}(x)$ for $x \leq \theta$.*

Proof: Due to (3.2), (3.3), and Lemma 4.6, we obtain $v_0(x) = \int_a^b \max\{\xi, x\}\mathrm{d}F(\xi) = \mu + \int_a^x F(\xi)\mathrm{d}\xi$ for any x, implying

$$-\, s + \beta v_0(x) = -\, s + \beta \left(\mu + \int_a^x F(\xi)\mathrm{d}\xi\right) = \psi^{(1)}(x). \tag{5.3}$$

Suppose that the assertion holds true for a certain $t \geq 2$, that is, $R_i = \phi$ for $i \leq t-1$ and then

$$-\, s + \beta v_{t-1}(x) = \psi^{(t)}(x), \quad x \leq \theta. \tag{5.4}$$

Note here that (5.3) indicates that (5.4) holds even for $t = 1$.

For the assertion with respect to $t+1$, suppose further $R_t = \phi$, thus we have $R_i = \phi$ for $i \leq t$. Since $R_t = \phi$ means no reservation at time t, it follows from (3.2) that $R_t = \phi$ if and only if, for any w and x,

$$-\, r(w) - s + \beta v_{t-1}(w) < \max\{w,\ x,\ -\, s + \beta v_{t-1}(x)\} = u_t(w, x). \tag{5.5}$$

For $x \leq \theta$, due to (3.4) and Lemma 4.3(a) we have $x \leq -s + \beta v_{t-1}(x)$, thus (5.5) and (5.4) yield $u_t(w, x) = \max\{w, \ -s + \beta v_{t-1}(x)\} = \max\{w, \ \psi^{(t)}(x)\}$, from which and Lemma 4.6 we obtain $v_t(x) = \int_a^b \max\{\xi, \ \psi^{(t)}(x)\} \mathrm{d}F(\xi) = \mu + \int_a^{\psi^{(t)}(x)} F(\xi)\mathrm{d}\xi$ for $x \leq \theta$. Hence, it follows that if $R_i = \phi$ for $i \leq t$, then

$$-s + \beta v_t(x) = \alpha + \beta \int_a^{\psi^{(t)}(x)} F(\xi)\mathrm{d}\xi = \psi\left(\psi^{(t)}(x)\right) = \psi^{(t+1)}(x), \quad x \leq \theta,$$

which shows that the assertion also holds for $t+1$. By induction on t, the proof is completed. ∎

Theorem 5.2 *For $t \geq 1$:*

(a) *$R_t = \phi$ if and only if $\beta\left(v_{t-1}(w) - v_{t-1}(a)\right) < r(w)$ for $w \leq -s + \beta v_{t-1}(a)$.*

(b) *$R_1 = \phi$ if and only if $\psi^{(1)}(w) - \psi^{(1)}(a) < r(w)$ for any $w \in [a, \psi^{(1)}(a)]$; and if $R_i = \phi$ for $i \leq t-1$, then $R_t = \phi$ if and only if $\psi^{(t)}(w) - \psi^{(t)}(a) < r(w)$ for any $w \in [\psi^{(t-1)}(a), \psi^{(t)}(a)]$.*

Proof: (a) We should note first that $R_t = \phi$ and $R_t(a) = \phi$ are equivalent, because it follows from Lemma 4.4(a) that if $R_t(a) = \phi$, then $R_t(x) = \phi$ for any $x \geq a$.

Note that $z_t^o(a) = -s + \beta v_{t-1}(a)$ because $a < \theta$ holds from assumption $a < \alpha$, Lemmas 4.2(a) and 4.3(a), and that $z_t^r(w) = -r(w) - s + \beta v_{t-1}(w)$ for $w \leq \lambda_t$ due to Lemma 4.3(b). Furthermore, Lemma 4.3(b) implies that $w \leq \lambda_t$ if and only if $w \leq -r(w) - s + \beta v_{t-1}(w)$. Then we have

$$\begin{aligned} R_t(a) &= \{w \mid z_t^o(a) \leq z_t^r(w), \ w \leq \lambda_t\} \\ &= \{w \mid -s + \beta v_{t-1}(a) \leq -r(w) - s + \beta v_{t-1}(w), \ w \leq -r(w) - s + \beta v_{t-1}(w)\}. \\ &= \{w \mid \max\{-s + \beta v_{t-1}(a), \ w\} \leq -r(w) - s + \beta v_{t-1}(w)\}. \end{aligned} \tag{5.6}$$

Now, define

$$\begin{aligned} h_t(w) &= -s + \beta v_{t-1}(w) - \max\{-s + \beta v_{t-1}(a), \ w\} \\ &= \begin{cases} \beta\left(v_{t-1}(w) - v_{t-1}(a)\right), & w \leq -s + \beta v_{t-1}(a), \\ -s + \beta v_{t-1}(w) - w, & -s + \beta v_{t-1}(a) < w. \end{cases} \end{aligned} \tag{5.7}$$

Note that $h_t(w)$ is continuous in w because $v_{t-1}(w)$ is continuous in w and $\beta\left(v_{t-1}(w) - v_{t-1}(a)\right) = -s + \beta v_{t-1}(w) - w$ holds at $w = -s + \beta v_{t-1}(a)$.

By comparing (5.6) and (5.7) we immediately find that $R_t(a) = \phi$ if and only if $h_t(w) < r(w)$ for all w. It follows from Lemma 4.1(a,b) and continuity of $h_t(w)$ that $h_t(w)$ is maximized at $w = -s + \beta v_{t-1}(a)$. Therefore, $h_t(w) < r(w)$ holds for all w if and only if $\beta\left(v_{t-1}(w) - v_{t-1}(a)\right) < r(w)$ holds for $w \leq -s + \beta v_{t-1}(a)$. We have thus confirmed the assertion.

(b) By using assertion (a) and Lemma 5.1 we immediately find that the former part of the assertion holds true.

If $R_{t-1} = \phi$, then it follows from (a) and Lemma 4.5 that, for $w \in [a, \ -s + \beta v_{t-2}(a)]$,

$$\beta\left(v_{t-1}(w) - v_{t-1}(a)\right) \leq \beta\left(v_{t-2}(w) - v_{t-2}(a)\right) < r(w).$$

From this and (a) we conclude that if $R_{t-1} = \phi$ is supposed, it is enough for $R_t = \phi$ that

$$\beta\left(v_{t-1}(w) - v_{t-1}(a)\right) < r(w), \quad w \in [-s + \beta v_{t-2}(a), \ -s + \beta v_{t-1}(a)]. \tag{5.8}$$

Since Lemmas 4.1(a), 4.2(a), and 4.3(a) provide $-s + \beta v_{t-1}(a) \leq -s + \beta v_{t-1}(\theta) = \theta$, any w such as $w \leq -s + \beta v_{t-1}(a)$ satisfies $w \leq \theta$. Consequently, since $R_i = \phi$ is assumed for $i \leq t-1$, we can apply Lemma 5.1 for $w \leq -s + \beta v_{t-1}(a) \ (\leq \theta)$ and thus rewrite (5.8) as

$$\psi^{(t)}(w) - \psi^{(t)}(a) < r(w), \quad w \in [\psi^{(t-1)}(a), \psi^{(t)}(a)].$$

We have thus confirmed the assertion. ∎

Corollary 5.3 *If $\beta\int_a^w F(\xi)\mathrm{d}\xi < r(w)$ for $w \le \alpha$ and if $\psi^{(2t)}(a) - \psi^{(t)}(a) \le \beta\int_a^\alpha F(\xi)\mathrm{d}\xi$ for $t \ge 2$, then there is no reserving region at every time.*

Proof: Due to (4.10) and Theorem 5.2(b) we get $\beta\int_a^w F(\xi)\mathrm{d}\xi < r(w)$ for $w \le \alpha$ if and only if $R_1 = \phi$.

Given any $t \ge 2$, it follows from (4.10) and (4.11) that, for any $w \ge a$,

$$\begin{aligned}\psi^{(t)}(w) - \psi^{(t)}(a) &= \psi\left(\psi^{(t-1)}(w)\right) - \psi\left(\psi^{(t-1)}(a)\right) \\ &= \beta\int_{\psi^{(t-1)}(a)}^{\psi^{(t-1)}(w)} F(\xi)\mathrm{d}\xi \\ &\le \int_{\psi^{(t-1)}(a)}^{\psi^{(t-1)}(w)} \mathrm{d}\xi \;=\; \psi^{(t-1)}(w) - \psi^{(t-1)}(a),\end{aligned}$$

from which and (4.10) we obtain $\psi^{(t)}(w) - \psi^{(t)}(a) \le \psi^{(1)}(w) - \psi^{(1)}(a) = \beta\int_a^w F(\xi)\mathrm{d}\xi$ for $w \ge a$. Hence, it follows from Theorem 5.2(b) that if $R_1 = \phi$, then

$$\psi^{(t)}(w) - \psi^{(t)}(a) \le \beta\int_a^w F(\xi)\mathrm{d}\xi < r(w), \qquad w \le \alpha\ (= \psi^{(1)}(a)). \tag{5.9}$$

Given any $t \ge 2$, since $\psi^{(t)}(w)$ is nondecreasing in w and $\psi^{(t)}\left(\psi^{(t)}(a)\right) = \psi^{(2t)}(a)$, we get

$$\psi^{(t)}(w) - \psi^{(t)}(a) \le \psi^{(2t)}(a) - \psi^{(t)}(a), \qquad w \le \psi^{(t)}(a). \tag{5.10}$$

By remembering that $r(w)$ is nondecreasing in w, we find that if $R_1 = \phi$ and $\psi^{(2t)}(a) - \psi^{(t)}(a) \le \beta\int_a^\alpha F(\xi)\mathrm{d}\xi$, then from (5.9), $\psi^{(t)}(w) - \psi^{(t)}(a) < r(w)$ for $w \le \alpha$, and from (5.10) and (5.9), $\psi^{(t)}(w) - \psi^{(t)}(a) \le \psi^{(2t)}(a) - \psi^{(t)}(a) \le \beta\int_a^\alpha F(\xi)\mathrm{d}\xi < r(\alpha) \le r(w)$ for $\alpha \le w \le \psi^{(t)}(a)$, thus $\psi^{(t)}(w) - \psi^{(t)}(a) < r(w)$ for $w \le \psi^{(t)}(a)$.

Comparing this and Theorem 5.2(b) we conclude that the assertion holds true. ∎

According to several hundreds of numerical experiments the author made, both uniform distribution and triangular distribution always lead to $\psi^{(2t)}(a) - \psi^{(t)}(a) \le \beta\int_a^\alpha F(\xi)\mathrm{d}\xi$ for any a, b, s, and β with satisfying $a < \alpha$. As a result, it seems reasonable to conclude that in the cases where values of offers are considered to follow a uniform distribution or a triangular distribution, so as to investigate the economical efficiency of reservation, it suffices to check whether $\beta\int_a^w F(\xi)\mathrm{d}\xi < r(w)$ holds for $w \le \alpha$ or not.

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