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Omni-Lie Algebras

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1 Introduction

We introduce on the space $\mathcal{E}_n = \mathfrak{gl}(n,\mathbb{R}) \times \mathbb{R}^n$ the antisymmetric bracket operation

$$[\![(A_1,v_1),(A_2,v_2)]\!] = ([A_1,A_2], \tfrac{1}{2}(A_1v_2 - A_2v_1)). \tag{1}$$

Without the factor of $\frac{1}{2}$, this would be the semidirect product Lie algebra for the usual action of $\mathfrak{gl}(n,\mathbb{R})$ on $\mathbb{R}^n$. With the factor of $\frac{1}{2}$, the bracket does not satisfy the Jacobi identity. Nevertheless, it does satisfy the Jacobi identity on many subspaces which are closed under the bracket. In fact, we will see that any Lie algebra structure on $\mathbb{R}^n$ is realized on such a subspace.

If B is any bilinear operation on $\mathbb{R}^n$, we define the adjoint operator $\mathrm{ad}_B : \mathbb{R}^n \to \mathfrak{gl}(n,\mathbb{R})$ by $\mathrm{ad}_B(v)(w) = B(v,w)$, and we denote by $\mathcal{F}_B \subset \mathcal{E}_n$ the graph of this operator. A simple calculation shows:

Proposition 1.1 *If B is a skew symmetric bilinear operation on $\mathbb{R}^n$, then B satisfies the Jacobi identity if and only if $\mathcal{F}_B$ is closed under the bracket operation (1) on $\mathcal{E}_n$. When this condition is satisfied, the restriction to $\mathcal{F}_B$ of the natural projection from $\mathcal{E}_n$ to $\mathbb{R}^n$ is an isomorphism between the restricted $\mathcal{E}_n$ bracket and the operation B.*

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The skew symmetry of an operation B can also be seen as a property of the graph $\mathcal{F}_B$. We introduce a symmetric bilinear form on $\mathcal{E}_n$ with values in $\mathbb{R}^n$:

$$\langle (A_1, v_1), (A_2, v_2) \rangle = \tfrac{1}{2}(A_1 v_2 + A_2 v_1). \tag{2}$$

The operation B is skew symmetric if and only if $\mathcal{F}_B$ is isotropic for this bilinear form. When this is the case, $\mathcal{F}_B$ is actually a maximal isotropic subspace.

As we mentioned above, the bracket operation $[\![\ ,\ ]\!]$ does not satisfy the Jacobi identity. In fact, we have a simple formula for the jacobiator

$$J(e_1, e_2, e_3) = [\![[\![e_1, e_2]\!], e_3]\!] + \text{c.p.}, \tag{3}$$

where the e_j are elements of $\mathcal{E}_n$ and "c.p." means to add the two terms obtained by cyclic permutation of the three indices. In terms of the "Cartan 3-form"

$$T(e_1, e_2, e_3) = \tfrac{1}{3}\langle [\![e_1, e_2]\!], e_3 \rangle + \text{c.p.}, \tag{4}$$

the jacobiator is given by:

$$J(e_1, e_2, e_3) = (0, T(e_1, e_2, e_3)). \tag{5}$$

Proposition 1.1 follows immediately from Equation 5 and the fact that the graph $\mathcal{F}_B$ is *maximal* isotropic.

We define a D-structure on $\mathbb{R}^n$ to be any maximal isotropic subspace of $\mathcal{E}_n$ which is closed under the bracket operation. By (5), any D-structure is a Lie algebra with the restricted bracket. Among the D-structures are not only the n-dimensional Lie algebras, realized on the graphs of their adjoint representations, but also the "horizontal" subspace $\mathfrak{gl}(n, \mathbb{R}) \oplus \{0\}$. Unfortunately, this is the only graph of a mapping from $\mathfrak{gl}(n, \mathbb{R})$ to $\mathbb{R}^n$ which is isotropic in $\mathcal{E}_n$. The reader is invited to find other D-structures.

We come now to the principal question raised by the construction above. All n-dimensional Lie algebras can be embedded in the space $\mathcal{E}_n$, which has a bracket operation which does not quite satisfy the Jacobi identity. We will refer to $\mathcal{E}_n$ as an **omni-Lie algebra**. *Is there a global object corresponding to $\mathcal{E}_n$, obtained by some kind of "integration", which would serve as an "omni-Lie group"?*

Remark. In response to the posting of a preprint version of this work, Michael Kinyon made some suggestions which have led to the resolution of some (but not all!) of the problems raised here. Details will appear in [5].

2 Courant algebroids

Although our presentation of the omni-Lie algebras in the previous section was self contained, in fact we came to this construction by linearizing at a point the following bracket on the sections of $E = TM \oplus T^*M$, where M is a differentiable manifold. It was introduced by T. Courant [3].

$$[\![(\xi_1, \theta_1), (\xi_2, \theta_2)]\!] = ([\xi_1, \xi_2], \mathcal{L}_{\xi_1}\theta_2 - \mathcal{L}_{\xi_2}\theta_1 - \tfrac{1}{2}d(i_{\xi_1}\theta_2 - i_{\xi_2}\theta_1)), \tag{6}$$

where $\mathcal{L}_\xi$ and i_ξ are the operations of Lie derivative and interior product by the vector field ξ.

Courant introduced his bracket to unify the treatment of Poisson structures (bivector fields π on M for which the bracket $\{f, g\} = \pi(df, dg)$ on $C^\infty(M)$ satisfies the Jacobi identity) and presymplectic structures (2-forms on M which are closed). Each bivector field π or 2-form ω on M gives rise to a graph which is a subbundle F of E whose fibres are maximal isotropic with respect the the (indefinite) inner product

$$\langle (\xi_1, \theta_1), (\xi_2, \theta_2) \rangle = \tfrac{1}{2}(\theta_2(\xi_1) + \theta_1(\xi_2)). \tag{7}$$

The space of sections of the subbundle F is closed under the **Courant bracket** (6) if and only if π [resp. ω] is a Poisson [resp. presymplectic] structure. Courant defined a **Dirac structure** on M to be any maximal isotropic subbundle $F \subset TM \oplus T^*M$ whose sections are closed under the Courant bracket. Dirac structures include not only Poisson and presymplectic structures, but also foliations on M. If B is an integrable subbundle of TM, $B^\perp$ its annihilator in T^*M, then $B \oplus B^\perp$ is a Dirac structure.

Any Dirac structure F on M is a **Lie algebroid**. This means that: (1) the sections of M have a Lie algebra structure $[\ ,\ ]$ (over $\mathbb{R}$); (2) there is a bundle map $\rho : F \to TM$, called the **anchor**, which induces a Lie algebra homomorphism from sections of F to vector fields on M; (3) for any sections e_1 and e_2 of F and a function f on M,

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1) \cdot f)e_2. \tag{8}$$

For a Dirac structure, the bracket and anchor are the restriction to F of the Courant bracket and the projection on the first summand of $TM \oplus T^*M$. When the Dirac structure is a Poisson structure π, the subbundle F may be identified with T^*M by projection onto the second summand of E, and the resulting Lie algebroid structure on T^*M is the infinitesimal object corresponding to (local) symplectic groupoids [2][13] for the Poisson manifold (M, π).

Thus, the bundle E carries a structure which does not quite satisfy the axioms of a Lie algebroid, since its bracket does not satisfy the Jacobi identity, but it contains many subbundles on which the restricted bracket *is* a Lie algebroid structure; these include the Lie algebroids of the symplectic groupoids of all the Poisson structures on M. Since Lie algebroids are the infinitesimal objects for (local) Lie groupoids, it is natural to ask whether there is a global, groupoid-like object corresponding to E which contains all these groupoids.

The properties of Courant's bracket were the basis for the definition of a **Courant algebroid** by Liu, Xu, and the author [7]. This object is defined to be a vector bundle E over a manifold M carrying a field of inner products (i.e. nondegenerate symmetric bilinear forms) along the fibres, an antisymmetric bracket operation $[\![\, , \,]\!]$ on its space of sections, and a bundle map $\rho : E \to TM$ such that the Jacobi identity and the Leibniz rule (8) are satisfied modulo terms which are differentials, in a certain sense, of terms involving the inner products. In addition, the bilinear form itself satisfies some conditions, one of which is a modified version of "adjoint invariance". (The precise axioms can be found in algebraic form in Section 3 below.) When M is a point, all the error terms vanish, and a Courant algebroid is just a Lie algebra with an adjoint-invariant inner product. In this case, the corresponding global object is clearly a Lie group with a bi-invariant (possibly indefinite) metric.

In any Courant algebroid, one may define the Dirac structures to be the maximal isotropic subbundles whose sections are closed under the bracket; since the anomalies vanish on Dirac structures, they are Lie algebroids. When M is a point, a pair of complementary Dirac structures is a Manin triple corresponding to a Lie bialgebra, and conversely the direct sum of a Lie bialgebra and its dual is a Courant algebroid over a point. This is the double of the Lie bialgebra, and the global object is the double of a Poisson Lie group corresponding to the Lie bialgebra. Similarly, there is a notion of Lie bialgebroid due to Mackenzie and Xu [10]. The double of a Lie bialgebroid is a Courant algebroid [7]; what is missing is the double of the Poisson groupoid corresponding to the Lie bialgebroid.

Some progress has been made in relating Courant algebroids to other algebraic structures. In [12], it is shown that Courant algebroids can be considered as strongly homotopy Lie algebras. In Roytenberg's thesis [11], an approach to Lie algebroids in terms of homological vector fields on supermanifolds was developed to describe arbitrary Courant algebroids. The thesis also develops the idea, suggested by some calculations in [7] and observations by Y. Kosmann-Schwarzbach and P. Severa, that a non-antisymmetric version of the bracket on a Courant algebra is an example of the Leibniz algebras introduced by Loday [8] and called Loday algebras in [6], where Kosmann-Schwarzbach shows how Loday brackets can be obtained as so-called derived brackets from Poisson brackets and derivations. It is this version of the Courant bracket which plays the central role in Roytenberg's work.

Although P. Severa has observed that a class of Courant algebroids obtained by deforming Courant's original example may be seen as the infinitesimal objects corresponding to gerbes (see [1] for a discussion of gerbes), there is still no satisfactory description of the groupoid-like object corresponding to a general Courant algebroid, nor of the group-like object corresponding to the sections of a Courant algebroid. In the hope of clarifying the situation, we may try to "linearize" Courant's original example at a point of $M = \mathbb{R}^n$. The result, as we shall explain next, is the omni-Lie algebra $\mathcal{E}_n$ of Section 1.

3 C-Algebras

There is an algebraic version of the notion of Lie algebroid, in which the role of a vector bundle over a manifold is played by the algebraic analogue of its space of sections, namely a module over a commutative algebra. This concept goes under various names, including "Lie-Rinehart algebra" and $(R, \mathcal{A})$ Lie algebra; we refer the reader to [9] for an extensive list of them. In this section, we introduce an analogous algebraic version of Courant algebroids.

We begin with a commutative ground ring R and a commutative R-algebra $\mathcal{A}$ (not necessarily unital). Next, we consider an $\mathcal{A}$-module $\mathcal{E}$ with a homomorphism ρ from $\mathcal{E}$ to the $\mathcal{A}$-module of derivations of $\mathcal{A}$. For any $f \in A$, we define its $\mathcal{E}$ differential $d_{\mathcal{E}} f$ in the dual $\mathcal{A}$ module $\mathcal{E}^*$ by $d_{\mathcal{E}}f(e) = \rho(e)f$. Now suppose that $\mathcal{E}$ also carries a symmetric $\mathcal{A}$-bilinear form $\langle \ , \ \rangle$ which is **weakly nondegenerate** in the sense that the associated homomorphism β from $\mathcal{E}$ to the dual $\mathcal{A}$-module $\mathcal{E}^*$ is injective. If $d_{\mathcal{E}}f$ is in the image of β for a given $f \in \mathcal{A}$, we define the **gradient** Df to be the well-defined element $\frac{1}{2}\beta^{-1}d_{\mathcal{E}}f$ of $\mathcal{E}$; i.e. $\langle Df, e\rangle = \frac{1}{2}\rho(e)f$ for all $e \in \mathcal{E}$. (The factor of $\frac{1}{2}$ is irritating, but it has to go somewhere; we have chosen to put it here to match the conventions in [7].)

Definition 3.1 *With the definitions and notation above, an $(R, \mathcal{A})$ C-***algebra** *is an $\mathcal{A}$-module $\mathcal{E}$ carrying a nondegenerate $\mathcal{A}$-valued symmetric bilinear form $\langle \ , \ \rangle$, an anti-symmetric R-bilinear operation $[\![\ , \ ]\!]$, and an $\mathcal{A}$-module homomorphism ρ with values in the R-derivations of $\mathcal{A}$ such that the following properties are satisfied:*

0. The gradient Df is defined for all $f \in \mathcal{A}$;

1. For any e_1, e_2, e_3 in $\mathcal{E}$, $[\![[\![e_1, e_2]\!], e_3]\!] + c.p. = DT(e_1, e_2, e_3)$;

2. for any e_1, e_2 in $\mathcal{E}$, $\rho[\![e_1, e_2]\!] = [\rho e_1, \rho e_2]$;

3. for any e_1, e_2 in $\mathcal{E}$ and f in $\mathcal{A}$, $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2 - \langle e_1, e_2\rangle Df$;

4. $\rho \circ D = 0$, i.e., for any f, g in $\mathcal{A}$ $\langle Df, Dg\rangle = 0$;

5. for any e, h_1, h_2 in $\mathcal{E}$, $\rho(e)\langle h_1, h_2\rangle = \langle [\![e, h_1]\!] + D\langle e, h_1\rangle, h_2\rangle + \langle h_1, [\![e, h_2]\!] + D\langle e, h_2\rangle\rangle$,

where $T(e_1, e_2, e_3)$ is the element of $\mathcal{A}$ defined by:

$$T(e_1, e_2, e_3) = \tfrac{1}{3}\langle [\![e_1, e_2]\!], e_3\rangle + c.p., \tag{9}$$

When $R = \mathbb{R}$ and $\mathcal{A}$ is the algebra of smooth functions on a manifold M, an $(R, \mathcal{A})$ C-algebra is just the space of sections of a Courant algebroid. On the other hand, to to see the omni-Lie algebras of Section 1 as $(R, \mathcal{A})$ C-algebras, we let R be $\mathbb{R}$ and $\mathcal{A}$ be $\mathbb{R}^n$ with the multiplication in which all products are zero. Geometrically, we should think of the latter as the algebra of functions which are defined on an infinitesimal neighborhood of the origin in the dual space of $\mathbb{R}^n$ and which vanish at the origin. (One could adjoin the constant functions at the cost of complicating the example slightly.) $\mathcal{E}_n$ is $\mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n$ with the module structure in which all scalar products are zero. We identify the derivations of $\mathcal{A}$

with $\mathfrak{gl}(n,\mathbb{R})$, so that ρ can be defined as projection on the first summand of $\mathfrak{gl}(n,\mathbb{R})\oplus\mathbb{R}^n$. The inner product and bracket are given by (2) and (1). The gradient D is then given for all v by $Dv=(0,v)$. The axioms of an $(R,\mathcal{A})$ C-algebra may be checked directly, or as a consequence of identities satisfied in the original Courant algebroid.

It is clear that isotropic subalgebras of $(R,\mathcal{A})$ C-algebras are Lie algebras. The fact that *all* n-dimensional Lie algebras arise this way in the omni-Lie algebra $\mathfrak{gl}(n,\mathbb{R})\oplus\mathbb{R}^n$ can be seen as a consequence of the fact that, among the Dirac structures on the dual space of $\mathbb{R}^n$ are the Lie-Poisson structures attached to all Lie algebra structures on $\mathbb{R}^n$.

Finally, we note that Evens and Lu [4] have recently shown that the variety of maximal isotropic subalgebras in the double of a Lie bialgebra carries a natural Poisson structure. It would be interesting to see whether their work extends to our more general setting.

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